Robert Fathauer
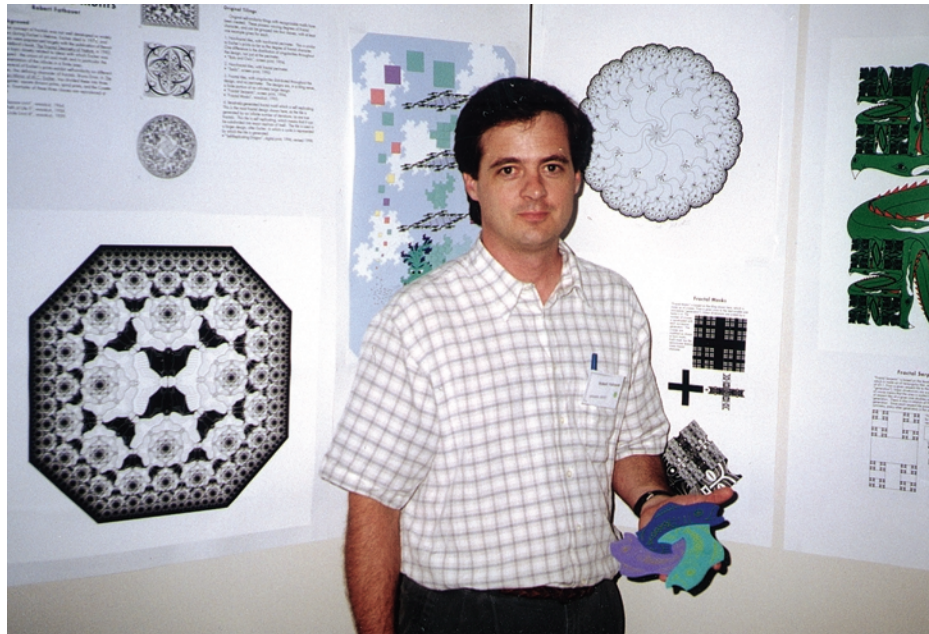
Extending Recognizable-Motif Tilings to Multiple-Solution Tilings and Fractal Tilings

Further Work

**"Extending Recognizable-motif Tilings to
Multiple-solution Tilings and Fractal Tilings "**

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Tiling of the plane is a theme with which M.C. Escher was preoccupied nearly his entire career as an artist, starting around 1920. He was particularly interested in tilings in which the individual tiles were recognizable motifs, and kept a notebook in which he enumerated 137 examples of this type of design [1]. Many of these were incorporated in finished woodcuts or lithographs. Notable examples include "Day and Night" (1938), "Reptiles" (1943), "Magic Mirror" (1946), "Circle Limit IV" (also known as "Heaven and Hell", 1960), and the "Metamorphosis" prints of 1937-1968.

Of Escher's 137 examples, all but the last are isohedral tilings. This means that a symmetry operation which maps a tile onto a congruent tile also maps the entire tiling onto itself. Escher's non-isohedral design, executed in 1971, was based on a wooden jigsaw puzzle given to him by the theoretical physicist Roger Penrose [1, pp. 318-319]. Penrose's and Escher's tiles were derived by modifying the edges of a 60°/120° rhombus. The four straight-line segments comprising the rhombus were each replaced by the same modified line segment, but in four different aspects, related to each other by rotations and glide reflections.

Another theme that intrigued Escher was the depiction of infinite repetition in a finite print. All of Escher's enumerated examples of recognizable-motif tilings can be continued to infinity; i.e., they tile the infinite Euclidean plane. A number of Escher's finished prints depict tiles that diminish to infinitesimally small size, for example, "Square Limit", "Smaller and Smaller", and "Path of Life I" and "Path of Life II". Other prints in this class employ hyperbolic geometry, where the infinite plane is represented in a finite circle. His four "Circle Limit" prints of 1958-1960 are in this class. Though the concept of fractals as such was not known to Escher (see next section), these prints possess one characteristic of fractals. They exhibit self-similarity near the edge of the circles.

Two mathematical advances that took place in the 1970's have interesting applications to tiling. Because Escher died in 1972, he was not able to employ these in his work. The first is the discovery by Roger Penrose of a set of two tiles that can tile the plane in an infinite number of ways, none of which are periodic [2]. The second is the development and formalization of fractal geometry, introduced by Benoit Mandelbrot [3]. In this article, I will show new Escher-like tilings using both Penrose tiles and fractals.

Recognizable-motif tilings based on Penrose and related tiles

There are three versions of the Penrose tiles, the first of which, known as P1, contains six different tiles. Penrose later succeeded in reducing the number of tiles to two. In the set P2,

these two tiles are commonly referred as "kites" and "darts" , while in the set P3, the tiles take the form of two rhombi. In all three sets, markings or distortions of the edges are necessary to indicate matching rules which force tilings constructed from the sets to be non-periodic. Tilings formed by all three sets are characterized by regions of local five-fold rotational symmetry.

Penrose was acquainted with Escher, so it was almost inevitable that he would try his hand at a recognizable-motif tiling based on his non-periodic tiles. He used the P2 kite and dart version of the tiles, and converted these to fat and skinny chickens [4]. I have also used the kites and darts as the basis for a recognizable-motif tiling, making a phoenix bird from the dart tile and a scorpion and diamondback rattlesnake combination from the kite tile (see Figure 1; a color print based on these tiles can be seen on page 21). Note that there are short edges and long edges in the kite and dart tiles, each type appearing four times. These two types of edges are modified independently to create recognizable motifs.

There are several characteristics that I feel a recognizable motif tiling should possess if it is to be as esthetically pleasing as possible. Escher touches on some of these in his writings. These are:

- The different tiles should be oriented in a way that makes sense. For example, if one of the tiles is the view of a creature from above, then other tiles in the tiling should also be viewed as from above. If side views are depicted, then gravity should be in the same direction for all of the tiles.
- The motifs for the different tiles should be complementary. For example, different types of sea life, or possibly opposites such as angels and devils.
- The different tiles should be commensurately scaled. That is, if a tiling is made up of one tile depicting a horse and another a man, then the two should be related in size in a similar way to real horses and men.

Note that the scorpion and rattlesnake tiles shown in Figure 1 violate the third rule, as a real scorpion is smaller than a rattlesnake. Penrose's tiling of chickens is likewise not completely satisfactory, as the chickens are shown in side view, but gravity points in different directions for different chickens. Violating the rules given above makes tilings seem less natural, less an extension of nature. Escher generally stuck to these rules, but there are some exceptions. These include his drawing number 72 (p. 174 in [1]), in which fish appear of comparable size to boats, and his drawing number 76 (p. 177 in [1]), in which birds appear of comparable size to horses.

One area Escher didn't explore was the design of recognizable-motif tiles that could be assembled in a number of different ways. The Penrose tiles are not the only tiles which fall in this category (see, e.g., related early work by Bøggild [5]), but they provide a convenient starting point for such explorations. As a departure from the Penrose tiles themselves, I examined alternative matching rules for the Penrose P3 rhombi, which have angles of 72°/108° and 36°/144°. In this case, a single line segment occurs eight times as the edges of the tiles, as shown in Figure 2, where a notch in the line indicates orientation. When the tiles are assembled, the notches must fit into one another.
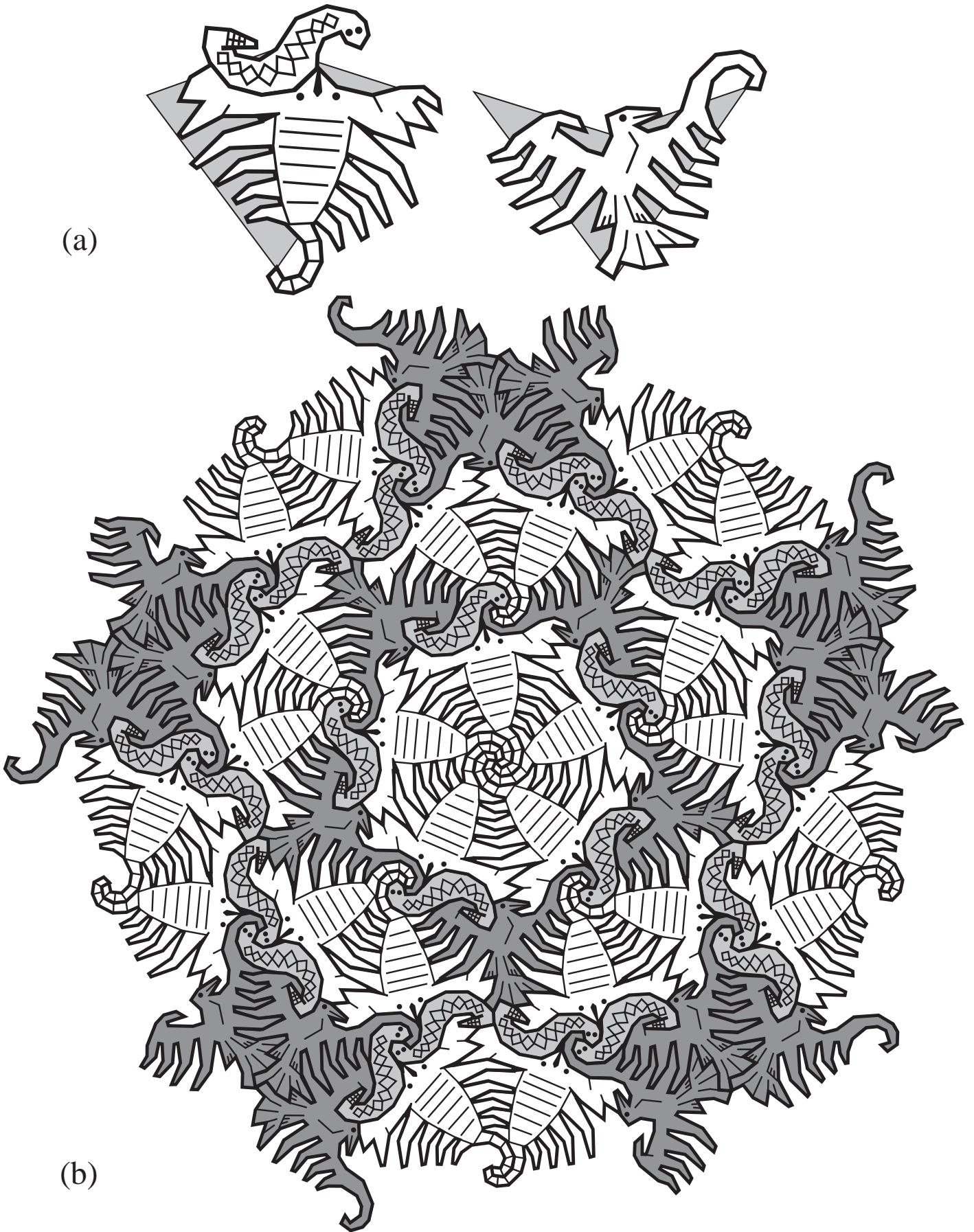
Figure 1. (a) Modification of the Penrose kites and darts to form scorpion, diamondback rattlesnake, and phoenix bird tiles. (b) A portion of a tiling based on these tiles.
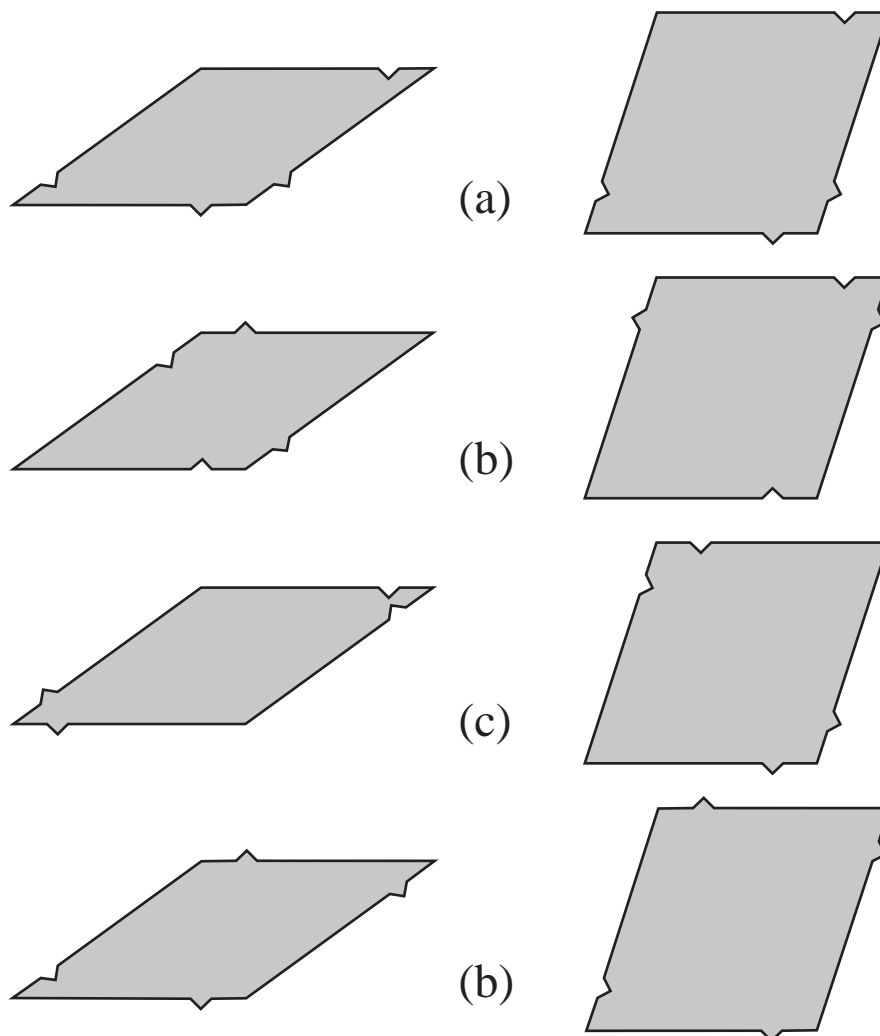
Figure 2. Four sets of matching rules for rhombi with angles 36°/144° and 72°/108°. (a) A combination that only tiles periodically (except for trivial variations). (b) A combination that only tiles non-periodically (the Penrose set P3). (c) A combination that tiles both periodically and non-periodically, used for "squids" and "rays". (d) A combination that doesn't tile at all.

If one examines a single rhombus, allowing only rotations of the edges, one finds 16 combinations, of which 10 yield distinct tiles. There are then 10x10 = 100 distinct pairs of rhombi with notched edges. If one also allows glide reflection of the edges, there are 4^8 = 65,536 combinations, not all of which yield distinct pairs of tiles, of course. Four of these pairs of tiles are shown in Figure 2, one of them being the Penrose set P3. Another of these pairs was used to create a set of tiles I call "squids" and "rays". Two tilings constructed using this set are shown in Figure 3. Note that the squid and ray tiles meet all three of the criteria listed above.

In addition to the Penrose tiles, with their characteristic five-fold (n=5) rotational symmetries, analogous sets of rhombic tiles can be constructed for other values of n [6]. I have also explored matching rules and recognizable-motif tiles for n=7 and n=12. The rhombi for n=7 are shown in Figure 4, along with insect-motif tiles based on these. The same line segment appears twelve times as edges of these tiles. The rhombi for n=12 are shown in Figure 5, along with two reptile-motif tiles for each rhombus. The same line segment appears 24 times as the edges of these six tiles. Two tilings constructed from these tiles are shown in Figure 6.

A different approach to realizing multiple-solution recognizable-motif tilings is to creatively combine geometric tiles. Penrose modified his set P1 by replacing straight line segments with arcs of circles as one means of enforcing matching rules [2]. By combining these tiles six different ways and adding internal details, the tiles shown in Figure 7 are obtained. The color plate on page 19 shows one of many possible tilings that can be constructed with this set of tiles.

It should be noted that John Osborn has independently done related work on recognizable-motif tilings with multiple solutions [7]. He has used Penrose tiles and other rhombi as templates. Penrose's chickens, two designs by Osborn, and several of the author's designs have been commercially produced as puzzles [8].

Recognizable-motif tilings based on fractals

Recognizing fractals as a distinct branch of geometry was significant for several reasons. It led to a new way of viewing the world around us, in which the fractal nature of coastlines, tree branches, and other objects has been recognized and characterized. In addition, the Mandelbrot set and related mathematical "beasts" have revealed visually intriguing constructs [3]. These are created by iteration, which is the repetitive calculation of a formula until it converges to a result. While Escher built some fractal character into his prints, he lacked the insight and vocabulary provided by Mandelbrot's work.

To clarify the degrees and ways in which fractals can be employed in recognizable-motif tilings, I would like to group these in four categories. The term "bounded tiling" as used here refers to a tiling which is fully contained in a finite area by means of the motifs becoming infinitesimally small at the boundary, while an "unbounded tiling" would cover the infinite Euclidean plane. A "singularity" is a point at which the motifs become infinitesimally small, so that any finite area containing that point contains an infinite number of tiles. Finally, a "self-replicating" tile is one which can be subdivided into smaller exact replicas of itself.
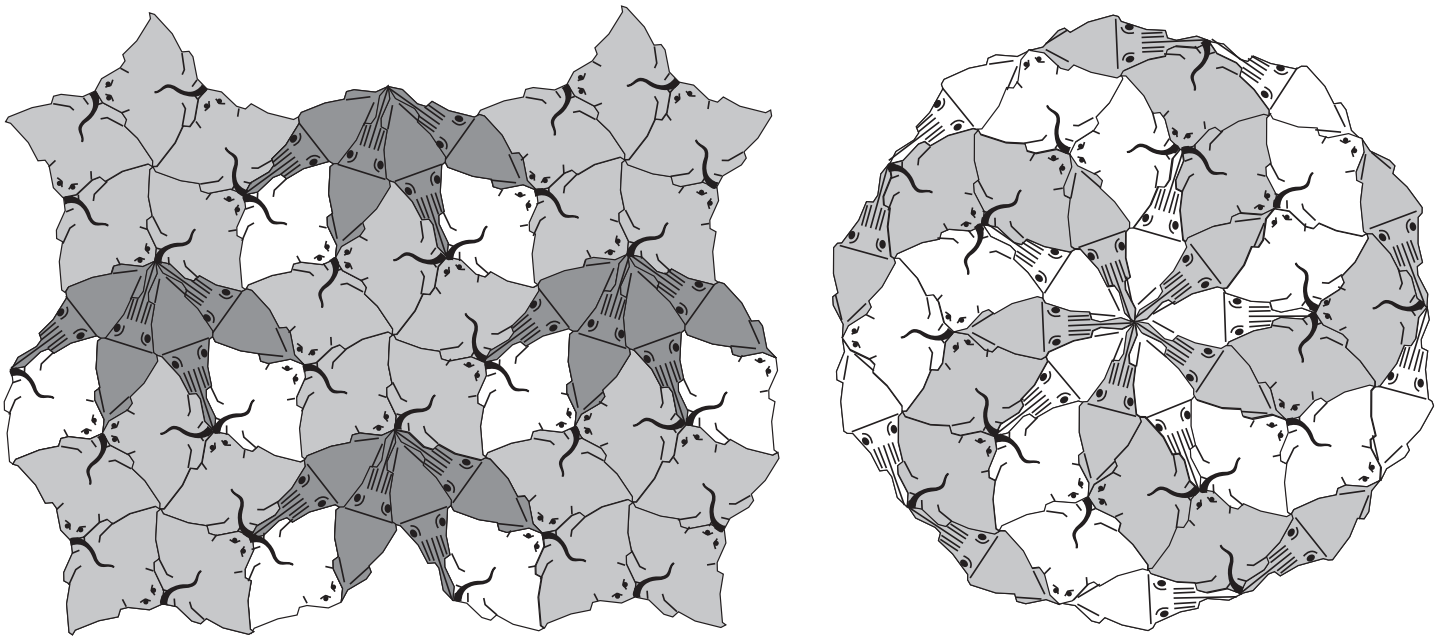
Figure 3. Two tilings constructed from the "squid" and "ray" tiles. The left one is periodic, while the right tiling, with five-fold rotational symmetry, is non-periodic.
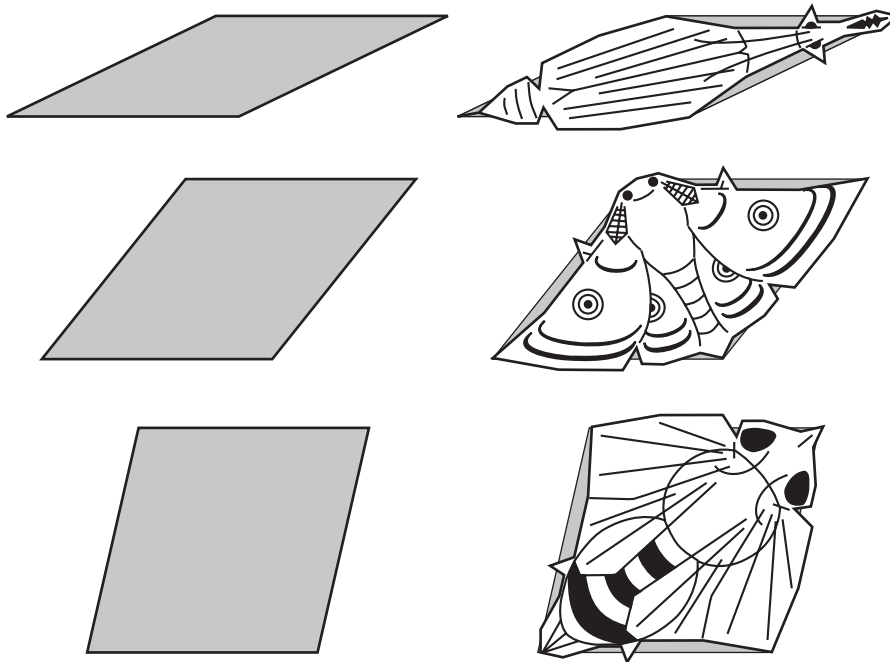

Figure 4. Rhombi and insect tiles for the n=7 analog to the Penrose tiles.
From top to bottom the tiles represent a beetle, a moth, and a bumblebee.
The angles of the rhombi are 25.7°/154.3°, 51.4°/128.6°, and 77.1°/102.9°.
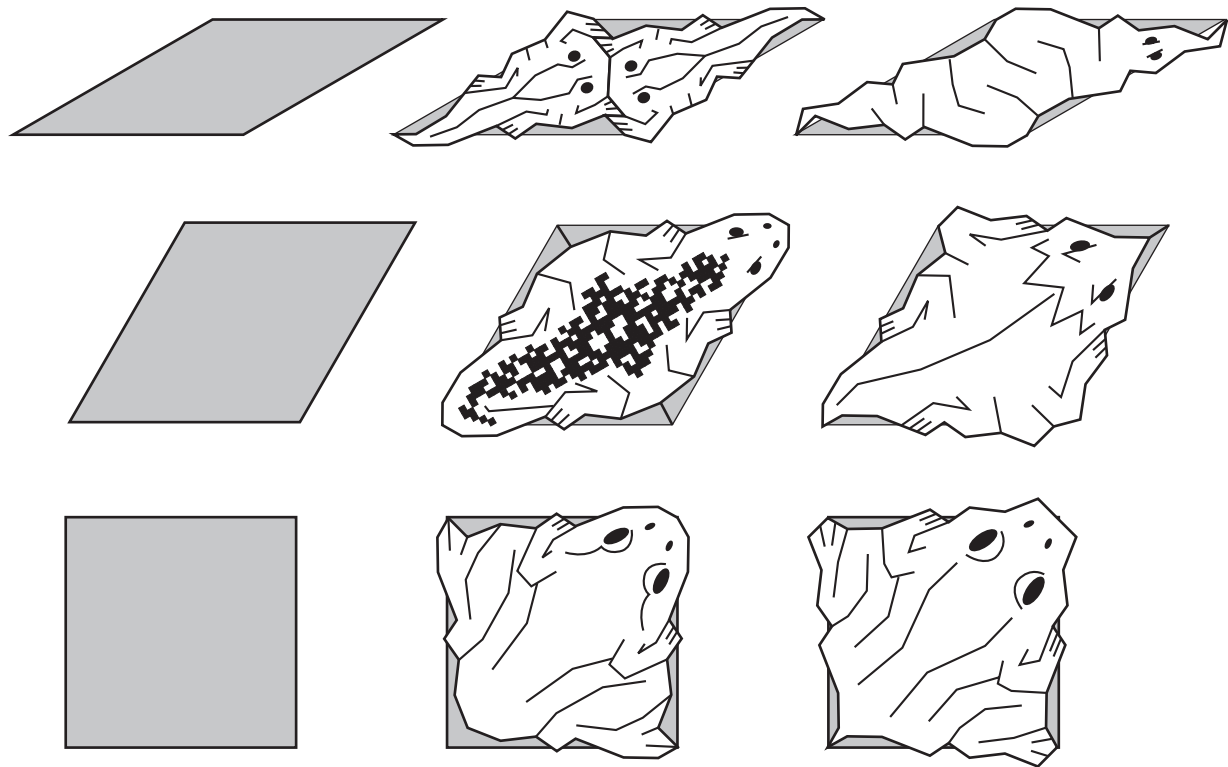
Figure 5. Rhombi and reptile tiles for the n=12 analog to the Penrose tiles. From top left to bottom right the tiles represent two tadpoles, a snake, a Gila monster, a horned lizard, a frog, and a toad. The angles of the rhombi are 30°/150°, 60°/120°, and 90°/90°.
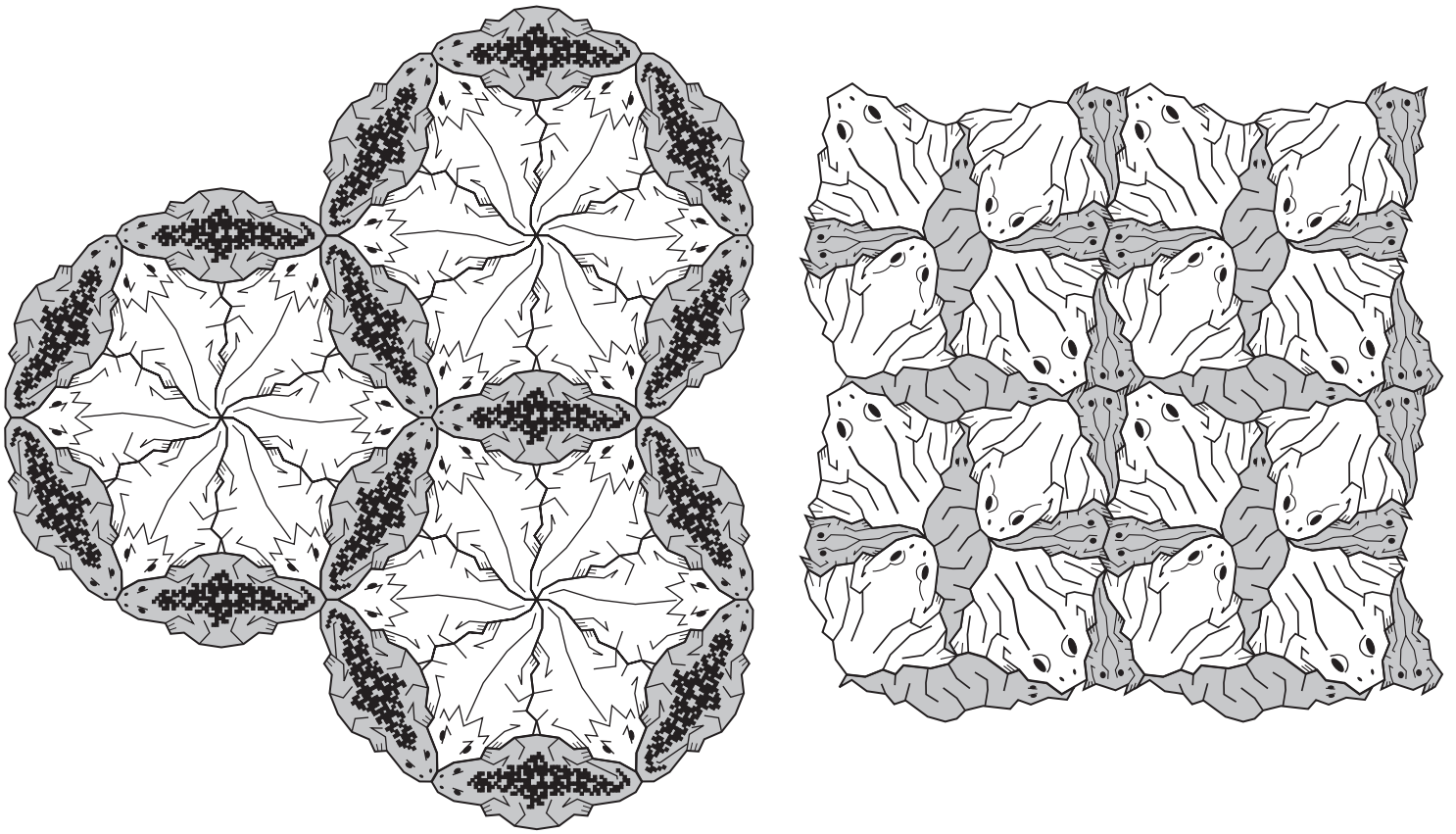
Figure 6. Two periodic tilings that can be constructed with the tiles of Figure 5.
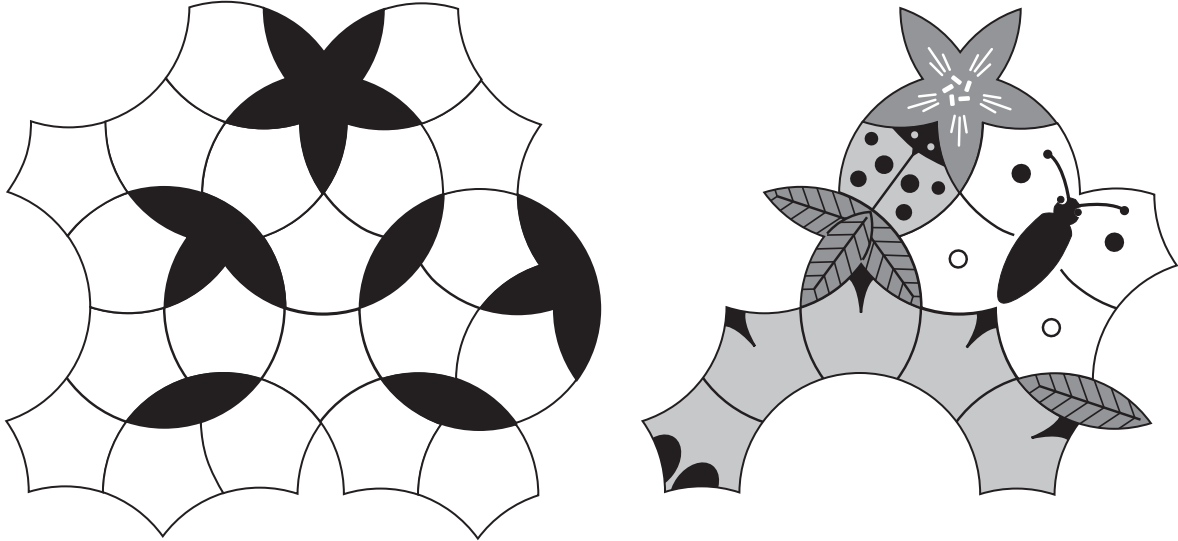
Figure 7. (a) A portion of a curvilinear version of the Penrose set P1. (b) Six tiles constructed from this set: a butterfly, a caterpillar, a ladybug, a flower, a single leaf, and a group of three leaves.

The four categories are:

(1) Bounded tilings with non-fractal tiles and non-fractal boundary.

(2) Bounded tilings with non-fractal tiles in which the boundary is fractal.

(3) Unbounded tilings with fractal tiles that are not self replicating.

(4) Unbounded tilings with fractal tiles that are self replicating.

Escher's "Square Limit" and "Circle Limit" woodcuts are all of type (1). My "Bats and Owls" design (Figure 8) is of also of this type. One distinction is that in "Bats and Owls" there are singularities in the interior of the tiling as well as on the boundary. Note that the boundary of this tiling is an octagon, but not a regular one. The ratio of the long side of the octagon to the short side is $\sqrt{2}:1$.

An example of a bounded tiling with non-fractal tiles and fractal boundary (type (2)) is seen in Figure 9. A quadrilateral tile is used as the basis for a seal motif. The overall tiling, which has ten-fold rotational symmetry, is made up of an infinite number of generations of successively smaller quadrilaterals. The boundary bears some similarity to a cross section of a head of cauliflower, which is often used as an example of a fractal-like structure occurring in nature.

Two examples of unbounded tilings with fractal tiles (type (3)) are seen in Figures 10 and 11. Both these designs have singularities distributed throughout the tiling, and both tilings if expanded would cover the infinite Euclidean plane. In Figure 10, a rectangle with long side to short side ratio of $\sqrt{3}:1$ is used to form a fractal tiling by repeated subdivision. With an infinite number of repeated subdivisions, the tiling possesses an infinite level of detail and can therefore be expanded to fill the Euclidean plane without loss of detail. The simplest distortion of this rectangle, for example taking triangular notches out of the top and bottom edges, automatically leads to an infinite number of triangular protrusions on the side edges as a consequence of the construction of the tiling. The color plate on page 20 shows a tiling constructed using the serpent tile of Figure 10.

Figure 11 is closely related in geometry to Figure 10. In this case, however, a cross is replicated and reduced in size to fill a square area. Only in the limit of an infinite number of steps is the square completely filled. Again, since there is an infinite level of detail, the crosses can then be expanded to fill the Euclidean plane. To form a tiling with recognizable motifs, the cross is replaced with a group of four primitive masks. Note that each mask is a fractal object, with an infinite number of rounded protrusions on the sides of the mask that result from making a single rounded notch in the top of the mask.

An example of a fractal tiling of type (4) is shown in Figure 12. The tile is generated iteratively according to an algorithm described in the mathematics literature [9,10]. With each iteration, the last shape is reflected about a vertical axis and replicated eight times, and these nine shapes are then placed in a particular arrangement. After an infinite number of iterations a fractal tile is obtained that is self replicating. That is, the tile can be subdivided into nine exact smaller replicas of itself which are reflected, or 9^2 exact even smaller replicas of itself which are not reflected, or 9^3 exact even smaller replicas of itself which are reflected, ... A print based on this tile can be seen on page 22.
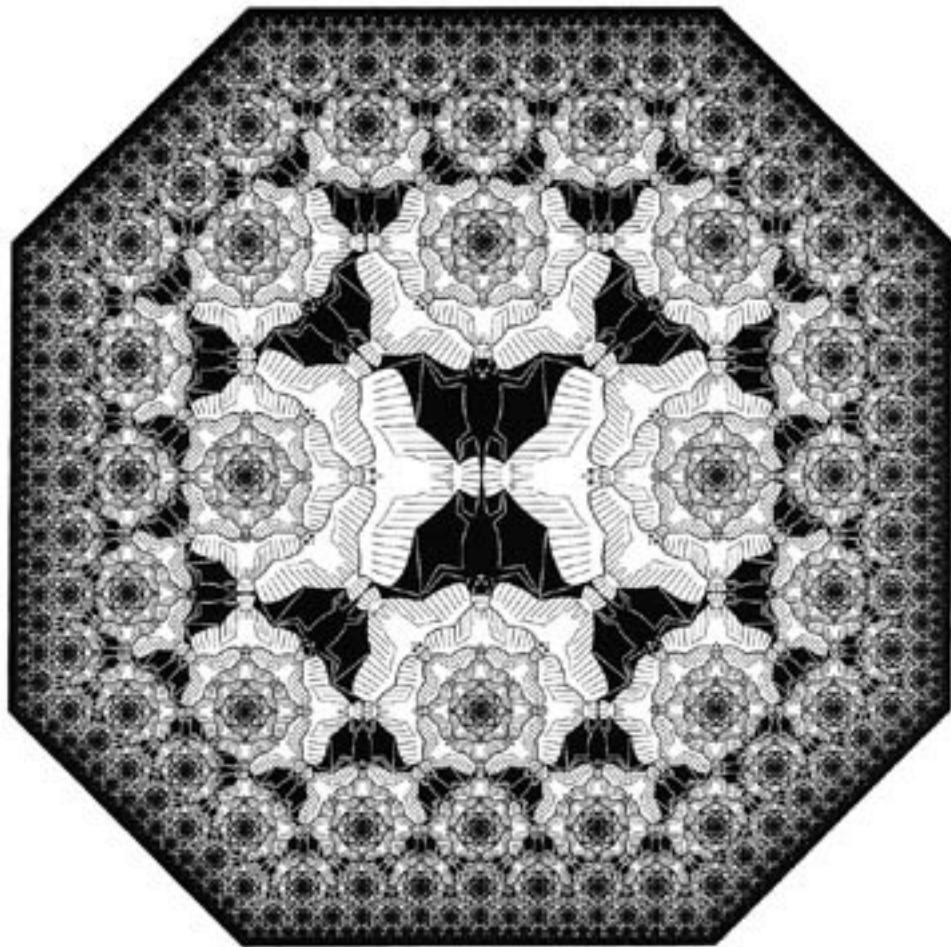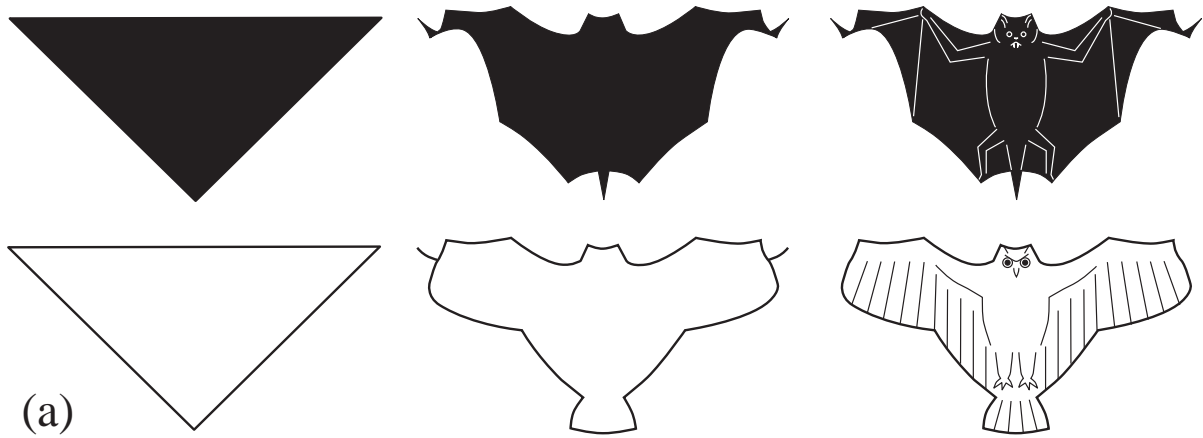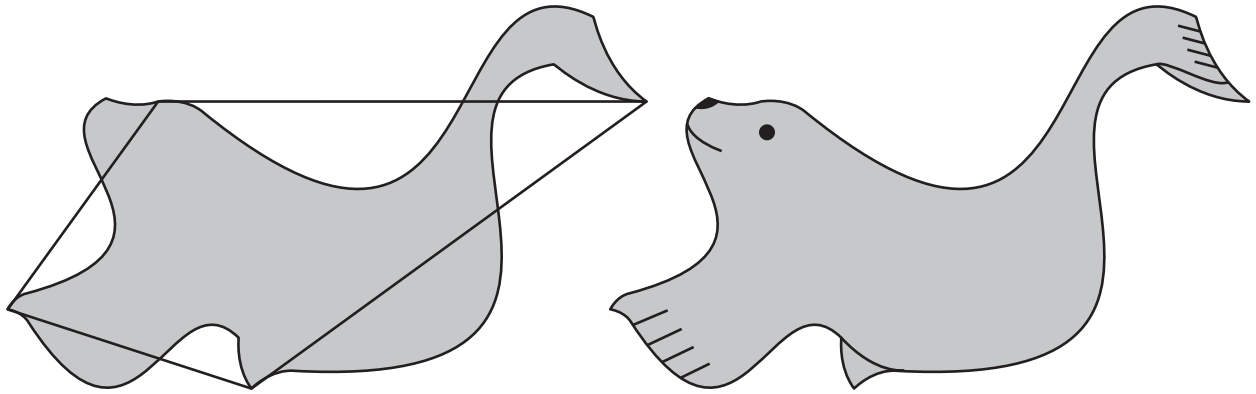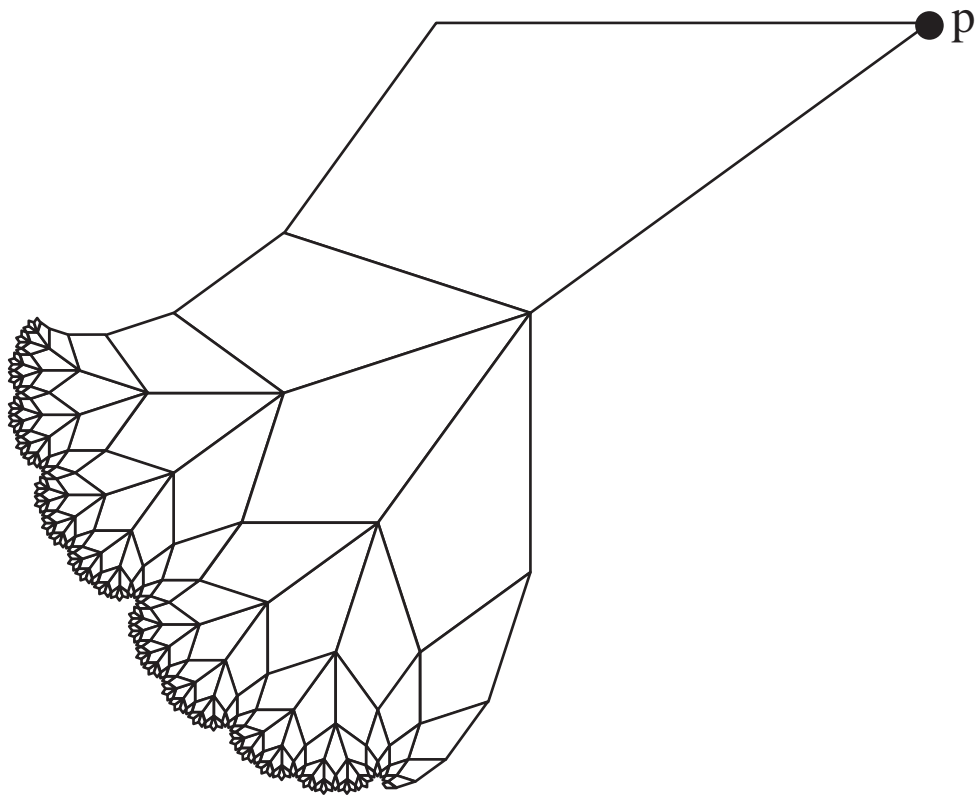
Figure 8. (a) Tiles on which "Bats and Owls" is based, and their relation to right triangles. (b) "Bats and Owls", a tiling executed as a limited-edition screen print in 1994.
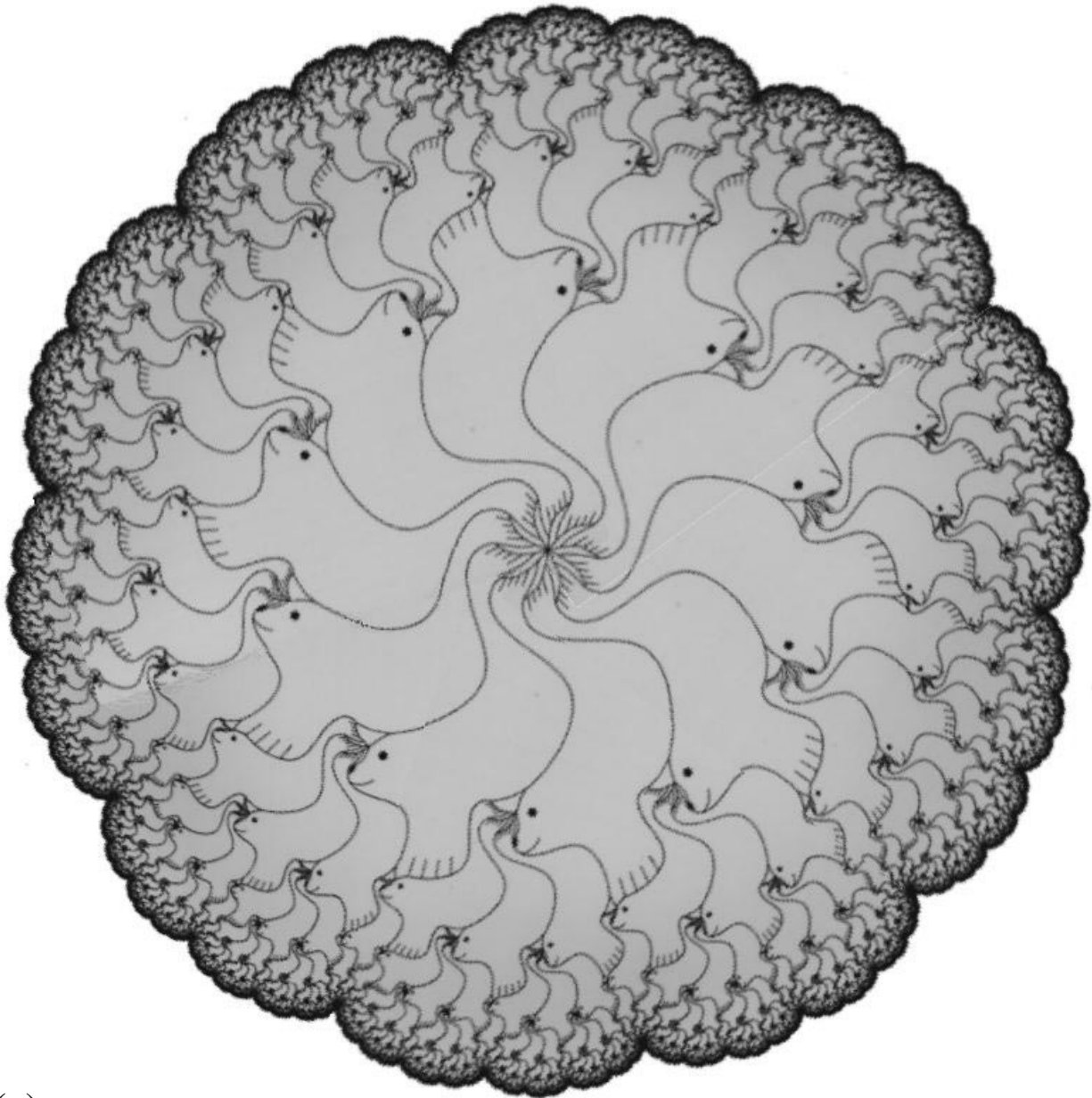
(a)

(b)

Figure 9. (a) Tile on which "Seals" is based, and its relation to a quadrilateral. (b) The first six generations of the quadrilateral tiles comprising one tenth of the overall tiling on which "Seals" is based. The full tiling is generated by replicating and rotating this group nine times by an angle of 36° about the point "p".
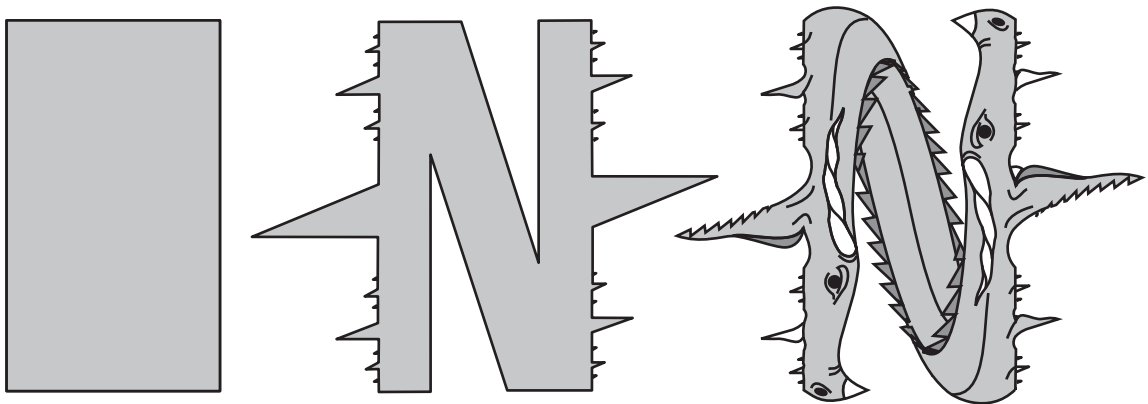
(c)

Figure 9. (c) "Seals", a tiling executed as a limited-edition screen print in 1993.
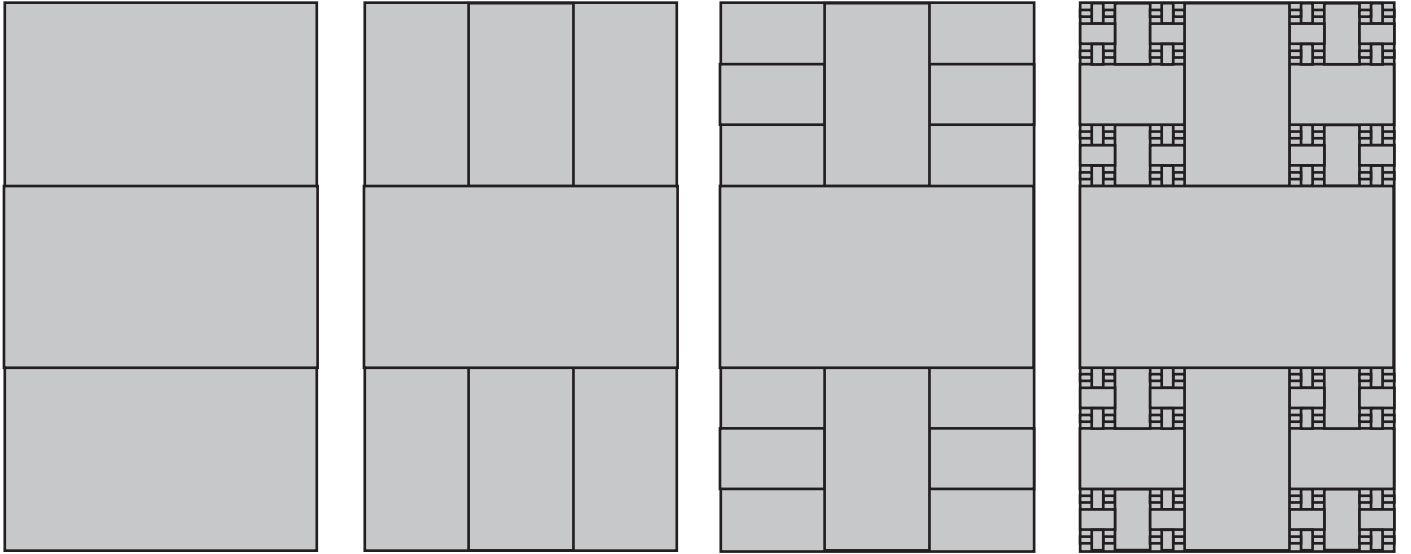
(a)

(b)

Figure 10. (a) The first, second, third, and seventh steps in generating an infinite fractal tiling from a rectangle with a long side to short side ratio of √3:1. (b) Tile on which "Fractal Serpents" is based, and its relation to one of the rectangular tiles in (a). Making the two long triangular notches in the top and bottom of the rectangle (middle figure in (b)) automatically creates the smaller triangular protrusions on the sides as a consequence of the construction of the tiling. This makes each individual tile a fractal object.
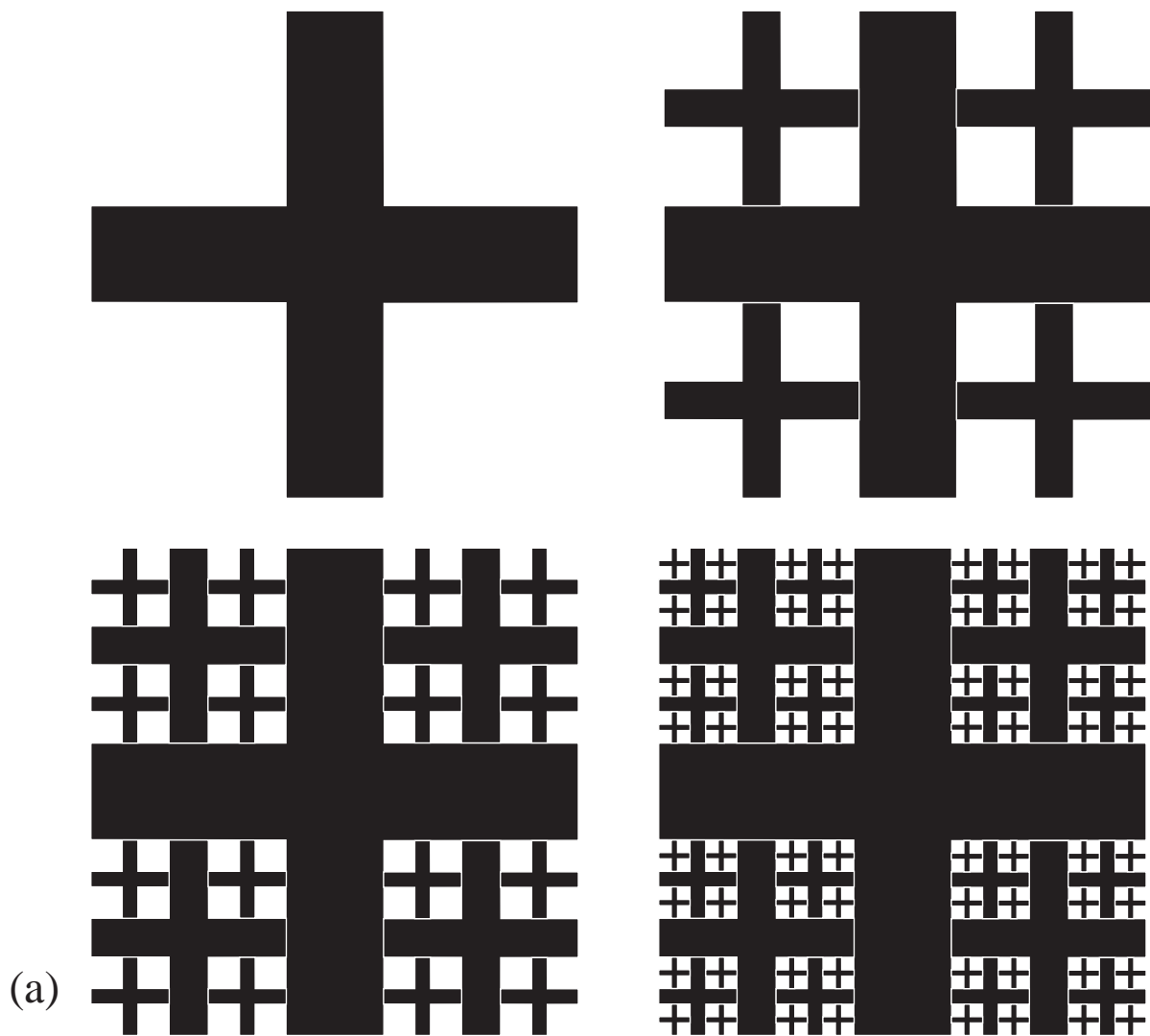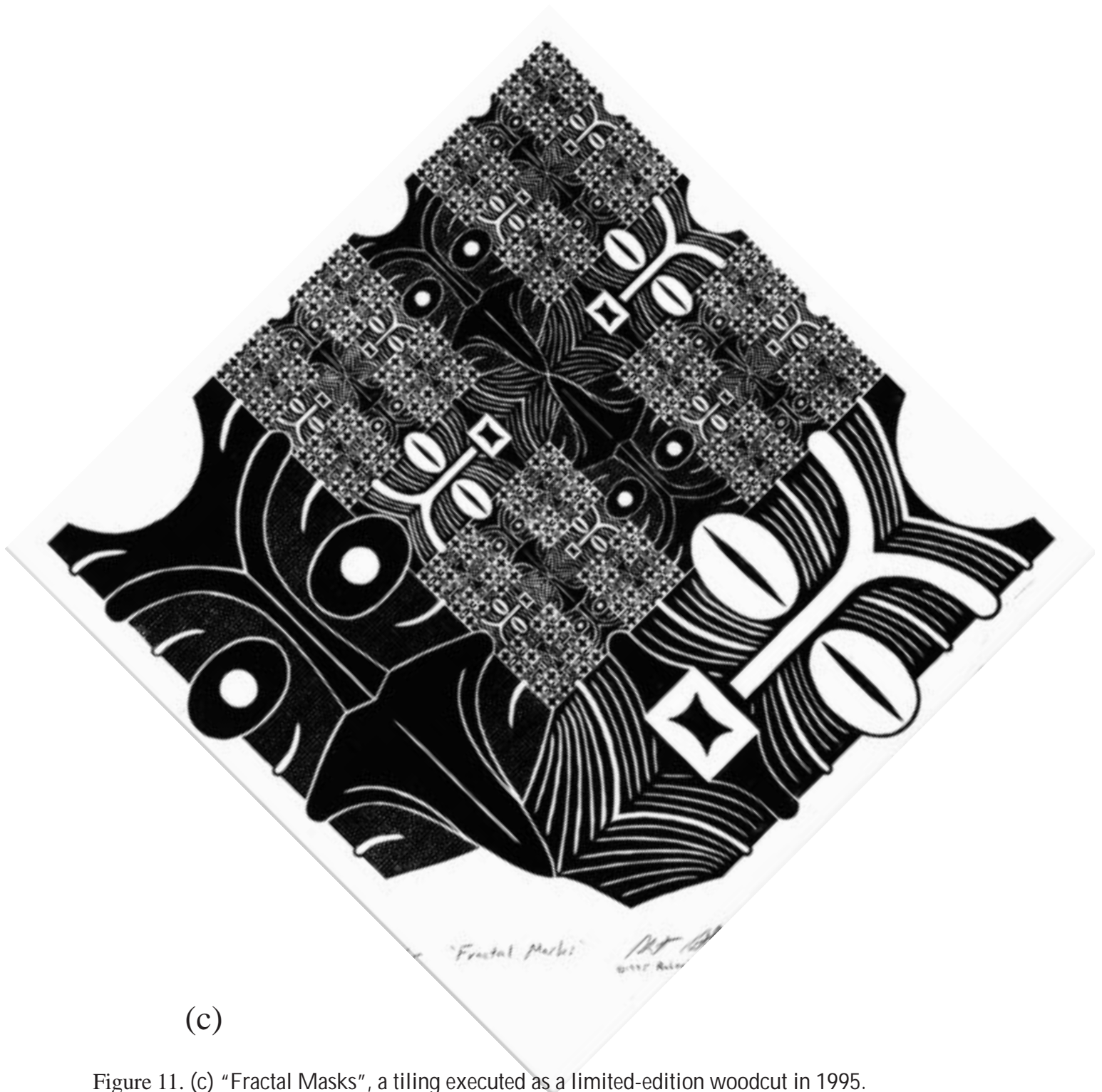
Figure 11. (a) The starting point and first, second, and third steps in generating an infinite fractal tiling from a cross. (b) Primitive mask motif based on the cross in (a). Making a rounded notch in the top of each mask automatically creates an infinite number of rounded protrusions along the sides of each mask.
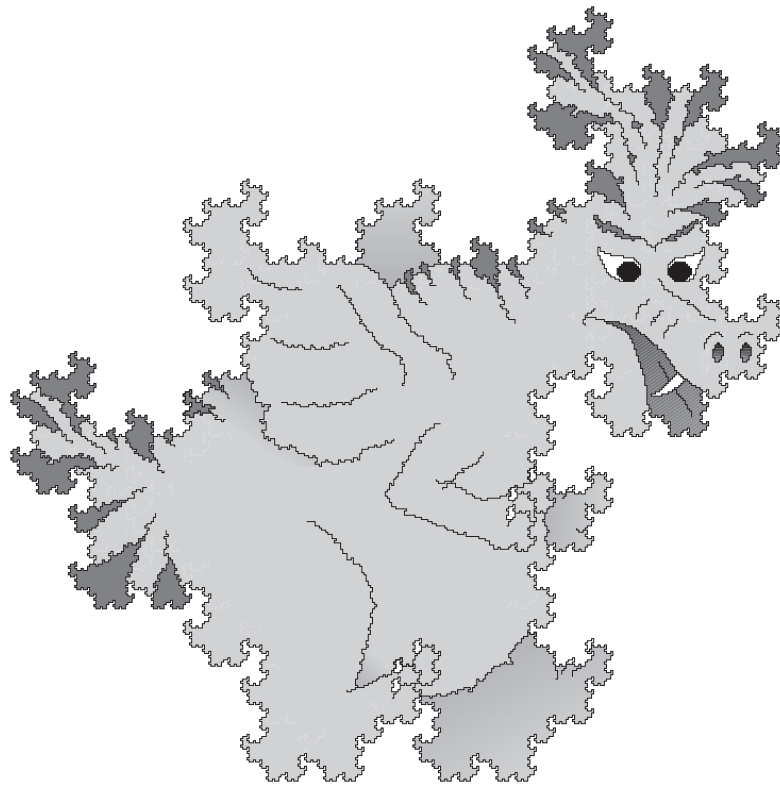
(c)

Figure 11. (c) "Fractal Masks", a tiling executed as a limited-edition woodcut in 1995.
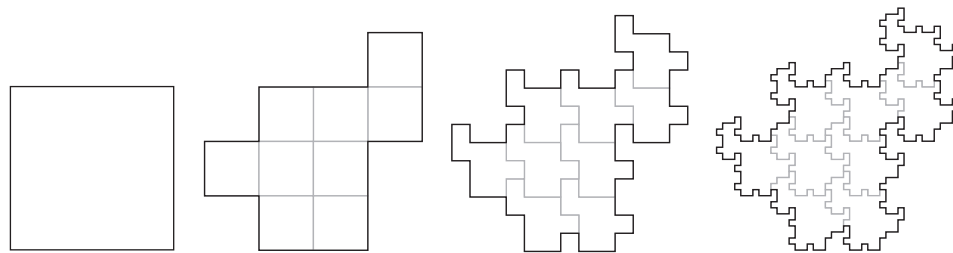
(a)

(b)

Figure 12. (a) The first three iterations used to generate a fractal self-replicating tile on which the design "Self-Replicating Dragon" is based (see page 22). (b) An approximation of the tile after five iterations, with interior details added to suggest a dragon motif.

Categories (3) and (4) may not have been fully satisfying to Escher, whose motivation for exploring fractal-like tilings was to fully contain an infinite tiling in a finite area. However, he surely would have enjoyed the complexity and fascinating forms found in fractal tiles, as well as the new way of looking at nature afforded by fractals. The four categories listed above are not the only possible types of tilings based on fractals. One can only speculate as to whether they are the sort of tilings that Escher might have made had he had access to the vocabulary of fractal geometry.

In conclusion, it has been shown that new mathematical discoveries unknown to Escher can be exploited to create recognizable-motif tilings that are quite distinct from Escher's work. This is particularly significant because of the thoroughness with which Escher explored the aspects of tiling which were known to him.
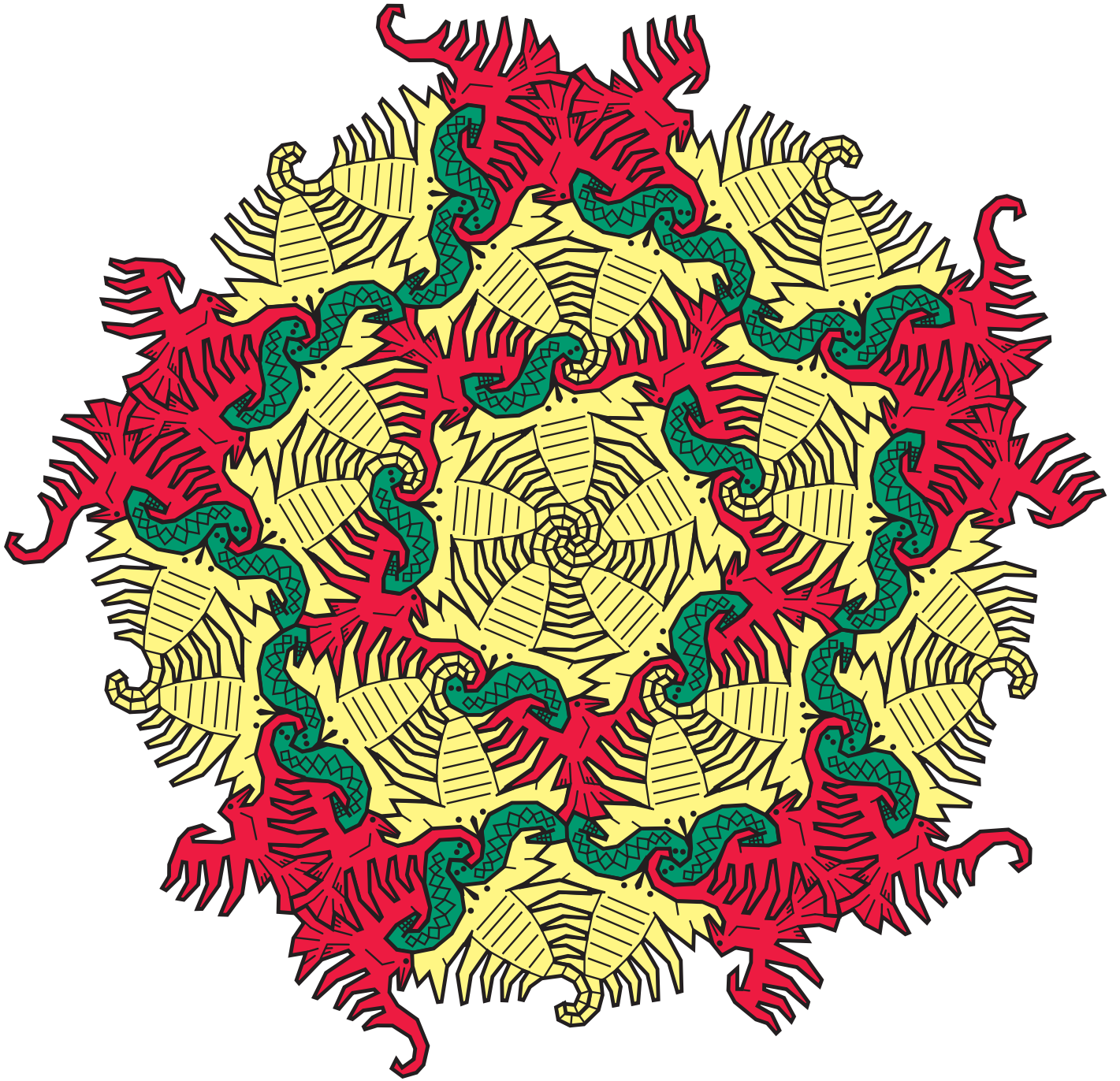
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[2] **Tilings and Patterns**, by Branko Grünbaum and G. C. Shephard (W.H. Freeman and Company, New York, 1987).

[3] **The Fractal Geometry of Nature**, by B. B. Mandelbrot (W.H. Freeman and Company, New York, 1982).

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[10] "Replicating Tessellations," by Andrew Vince, in **SIAM J. Discrete Mathematics,** Vol. 6, pp. 501-521 (1993).
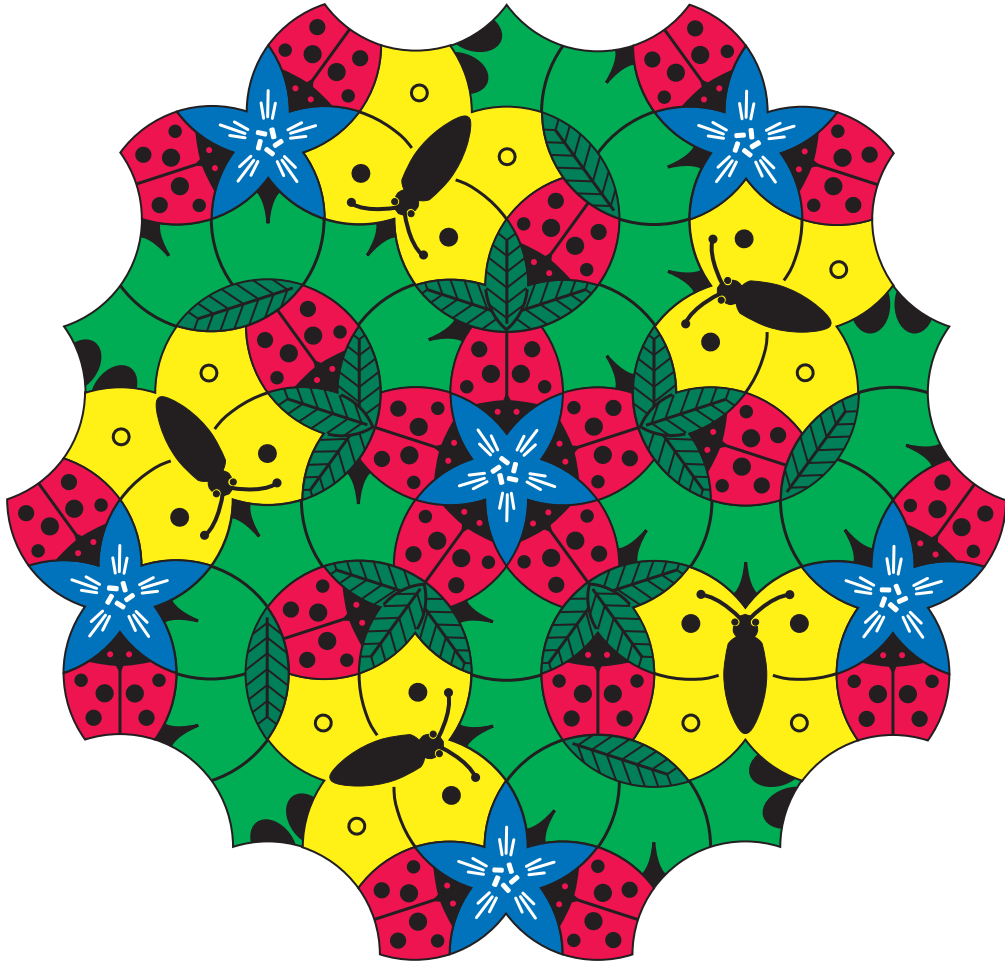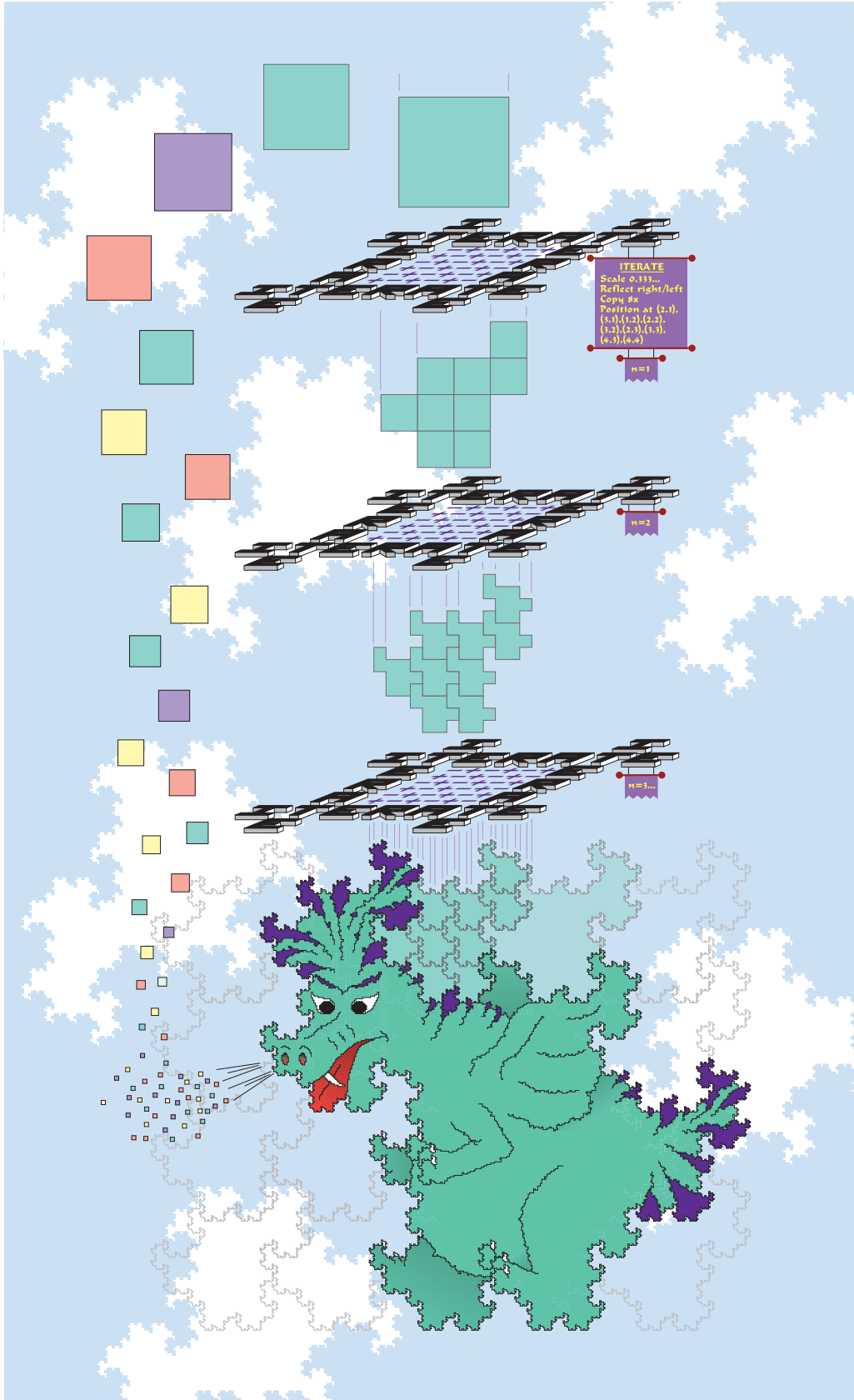
A portion of an infinite recognizable-motif tiling constructed using tiles derived from the Penrose set P1. Note the five-fold rotational symmetry characteristic of Penrose tilings.
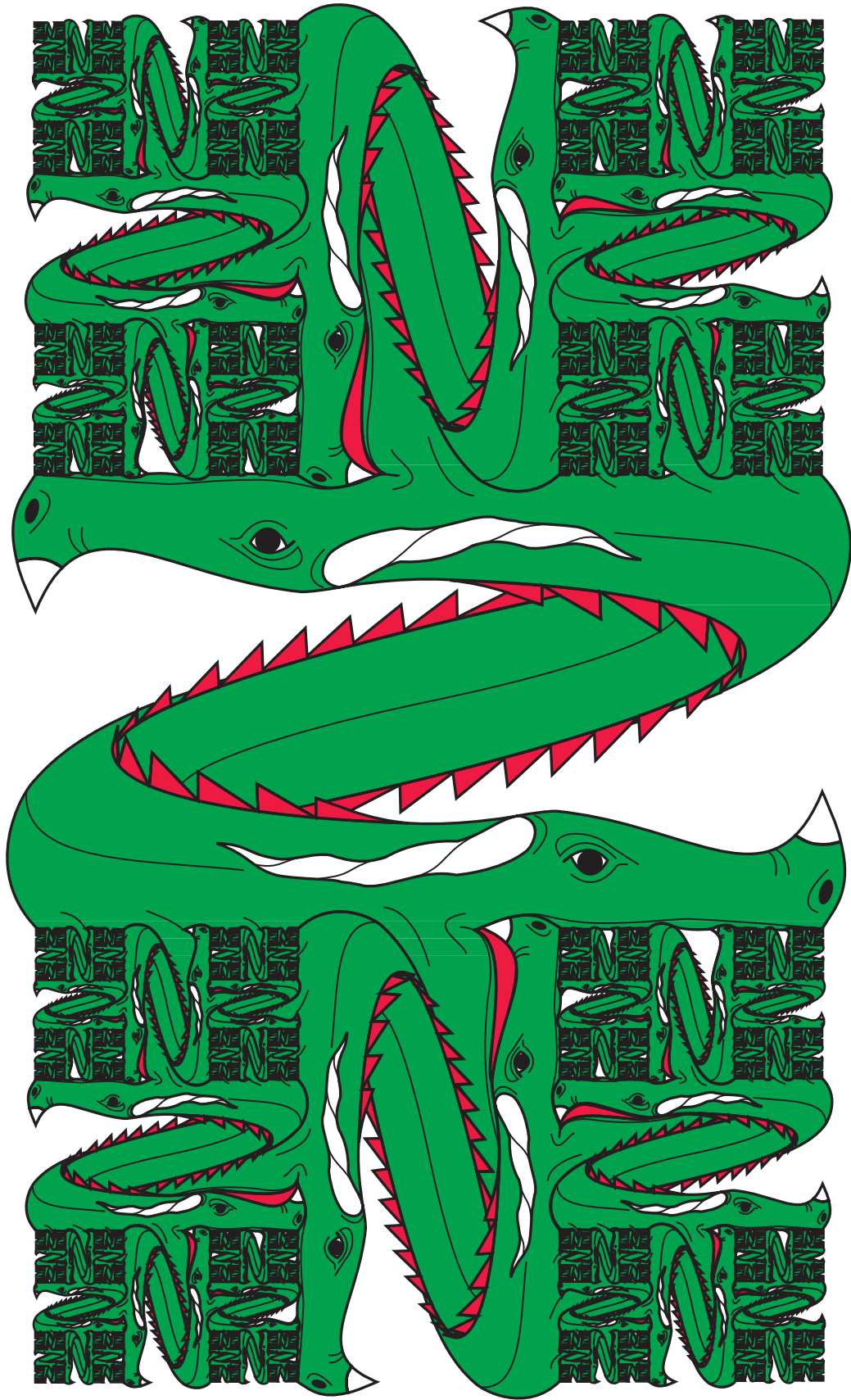
"Fractal Serpents", a limited-edition screen print which shows a portion of an infinite tiling in which each tile is fractal.

"Scorpion, Diamondback, and Phoenix," screen print by Robert Fathauer, 1993.

ITERATE
Scale 0.333...
Reflect right/left
Copy 8x
Position at (2,1),
(3,1),(1,2),(2,2),
(3,2),(2,3),(3,3),
(4,3),(4,4)

n=1

n=2

n=3...

"Self-Replicating Dragon", a digital artwork based on a fractal self-replicating tile. In this design, a closed cycle of the sort Escher employed in many of his prints is suggested. The dragon breaths square smoke particles which fall through machines which iteratively transform the squares particles into the dragon. For a background, white clouds which are the same tile, but inverted, are placed against a blue sky.

Grouped Groupers