Characterizing Simultaneous Embedding with Fixed Edges

J. Joseph Fowler[1], Michael Jünger[2], Stephen G. Kobourov[1], and Michael Schulz[2]

[1] University of Arizona, USA {jfowler,kobourov}@cs.arizona.edu *
[2] University of Cologne, Germany {mjuenger,schulz}@informatik.uni-koeln.de **

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Abstract. A set of planar graphs share a simultaneous embedding if they can be drawn on the same vertex set V in the plane without crossings between edges of the same graph. Fixed edges are common edges between graphs that share the same Jordan curve in the simultaneous drawings. While any number of planar graphs have a simultaneous embedding *without* fixed edges, determining which graphs always share a simultaneous embedding *with* fixed edges (SEFE) has been open.

We partially close this problem by giving a necessary condition to determine when pairs of graphs have a SEFE. As a direct application, we are able to determine for the set of planar graphs $\mathcal{P}$ and for the set of outerplanar graphs $\mathcal{O}$ (all vertices lie on an outerface), what the proper subsets of $\mathcal{P}$ and $\mathcal{O}$ are that always have a SEFE with all of $\mathcal{P}$ and $\mathcal{O}$, respectively. In both cases, we provide algorithms to compute the simultaneous drawings. Finally, we provide a polynomial time decision algorithm for deciding when a specific pair of outerplanar graphs has a SEFE. Whether two planar graphs have a SEFE can similarly be decided in polynomial time remains as an open problem.

1 Introduction

In many practical applications including the visualization of large graphs and very-large-scale integration (VLSI) of circuits on the same chip, edge crossings are undesirable. A single vertex set can suffice in which multiple edge sets correspond to different edge colors or circuit layers. While the union of any pair of edge sets may be nonplanar, a planar drawing of each layer may be possible, as crossings between edges of distinct edge sets is permitted. This corresponds to the problem of *simultaneous embedding* (SE) that generalizes the notion of planarity among multiple graphs.

Without restrictions on the types of edges used, this problem is trivial since any number of planar graphs can be drawn on the same fixed set of vertex locations [15]. However, difficulties arise once straight-line edges are required. Moving one vertex to reduce crossings in one layer can introduce additional crossings in other layers. This is the problem of *simultaneous geometric embedding* (SGE). If edge bends are allowed, then having common edges drawn the same way is important for mental map preservation. Such edges are called *fixed edges* leading to the problem of *simultaneous embedding with fixed edges* (SEFE). Since straight-line edges between a pair of vertices are also fixed, any graph that has a SGE also has a SEFE, but the converse is not true; see Fig. 1.

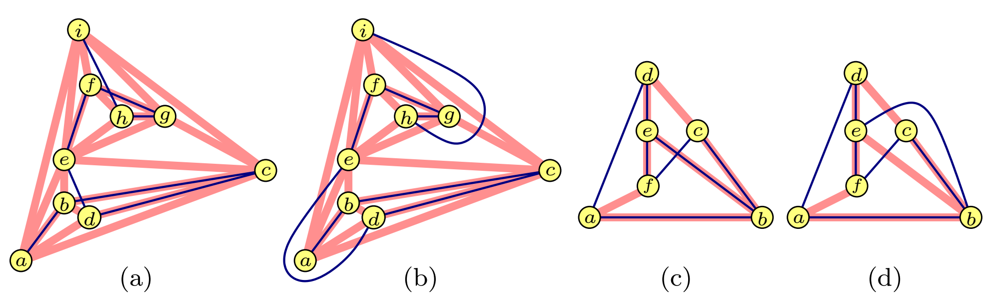

Fig. 1. The path and planar graph in (a) do not have a SGE with straight-line edges [2], but but have a SEFE in (b). The two outerplanar graphs in (c) do not have a SEFE, but have a SE in (d) if edge (b, e) is not fixed.

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Deciding if two graphs have a SGE is NP-hard [8], whereas, deciding if three graphs have a SEFE is NP-complete [11]. However, deciding if two graphs have a SEFE in polynomial-time remains open. In this paper we present a necessary condition in terms of forbidden subgraphs for whether pairs of graphs always have a SEFE. While this does not yet lead to a polynomial-time decision algorithm in the general case, it does in the more restricted case of pairs of outerplanar graphs. Additionally, we characterize which pairs of planar graphs and which pairs of outerplanar graphs have a SEFE and provide simultaneous drawing algorithms when possible.

1.1 Related Work

Relatively little is known regarding which graphs share a SGE. While it is known that any number of stars, two caterpillars (trees whose removal of leaf vertices is a path) and two cycles can always be done, whereas three paths [2] and two trees cannot [12], it is open as to whether a path and a tree always have a SGE. Moreover, which graphs always have a SGE with a path, a caterpillar, a tree, or a cycle remains unknown. The closest any of these questions has been answered is for a SGE with a path drawn monotonically. The set of graphs for which this is always possible was recently determined and characterized in terms of seven forbidden graphs [9]. Relaxing the straight-line edge condition slightly, three bends per edge suffice for pairs of planar graphs [7], whereas, simultaneous embedding an outerplanar graph with a straight-line path only requires one bend per edge [5].

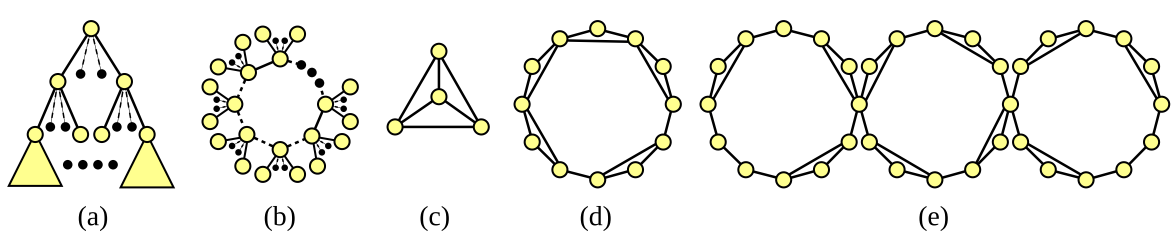
Little more is known regarding SEFE. While a planar graph and a tree can always be done, whereas two outerplanar graphs cannot [10], which (outer)planar graphs share a SEFE with the set of all (outer)planar graphs has been open. Planar graphs are characterized in terms of the forbidden graphs, K_5 and $K_{3,3}$ [14]. These form two minimum examples of nonplanarity. No similar description for SEFE in terms of forbidden graphs or minimum examples has been given until now.

Related to simultaneous embedding is the *thickness* of a graph G, the minimum number of planar subgraphs whose union is G. If vertices are co-located and straight-line edges are used as in SGE, the number of subgraphs is the *geometric thickness* of the graph. Using simultaneous embedding techniques, it was shown that graphs with degree at most 4 have geometric thickness 2 [6].

1.2 Our Contribution

Each of the subsequent sections is devoted to one of our three main contributions.

1. We show there exist three paths without a SEFE. On the other hand, while most pairs of graphs whose union forms a subdivided K_5 or $K_{3,3}$ share a SEFE, we provide 16 minimal forbidden pairs that do not. This gives a necessary condition for SEFE of two graphs.
2. Using this condition we show that the only graphs that always have a SEFE with any planar graph are either (i) forests, (ii) circular caterpillars (removal of all degree-1 vertices leaves a cycle), and (iii) subgraphs of K_4; see Fig. 2(a)–(c). We also show that the the only outerplanar graphs that always share a SEFE with any outerplanar graph either are (i) biconnected in which the endpoints of every chord are at a distance of two from each along the outerface (K_3-cycle) or (ii) have a cut vertex in which no two chords can be incident in the same biconnected component



(a) (b) (c) (d) (e)

Fig. 2. Forests in (a), circular caterpillars in (b), and subgraphs of K_4 in (c) have a SEFE with any planar graph. K_3-cycles as in (d) and outerplanar graphs composed of cubic K_3-cycles as in (e) have a SEFE with any outerplanar graph.

	SGE		SEFE				
	Path	Tree	Forest	Circular caterpillar	K_4 subgraph	K_3-cycle	each biconnected subgraph is a cubic K_3-cycle
Path	✓	?	✓	✓	✓	✓	✓
Caterpillar	✓	?	✓	✓	✓	✓	✓
Tree	?	✗	✓	✓	✓	✓	✓
Outerplanar	?	✗	✓	✓	✓	✓	✓
Planar	✗	✗	✓	✓	✓	✗	✗

Table 1. Old and new results for SGE and SEFE pairs. The last two rows for SEFE with shaded cells are new.

(each biconnected subgraph is a cubic K_3-cycle); see Fig. 2(d)–(e). Table 1 summarizes our results. In each case, we provide $O(n^2 \lg n)$ simultaneous drawing algorithms.

3. By comparing all possible subsets of vertices in a pair of outerplanar graphs against the two forbidden outerplanar pairs, we have a necessary condition for SEFE. We show that this condition also suffices, giving a $O(n^2)$ time decision algorithm for SEFE of two outerplanar graphs.

1.3 Preliminaries

Let P be a set of n distinct points in the xy-plane. A *planar drawing* of $G(V,E)$ consists of a bijection $\sigma : V \mapsto P$ with Jordan curves in the xy-plane connecting each pair of points $\sigma(u)$ and $\sigma(v)$ for each edge $(u,v) \in E$ where curves can only intersect at their endpoints. Let $\mathcal{G}$ be a set of planar graphs $\{G_1(V,E_1), G_2(V,E_2), \ldots, G_k(V,E_k)\}$. The set $\mathcal{G}$ has a *simultaneous embedding* if there exist planar drawings of $G_i(V,E_i)$ with the same bijection $\sigma : V \mapsto P$. If each edge is only composed of one straight-line segment, then $\mathcal{G}$ has a *simultaneous geometric embedding* (SGE). If every common edge in $\mathcal{G}$ connecting a pair of vertices uses the same Jordan curve, then $\mathcal{G}$ has a *simultaneous embedding with fixed edges* (SEFE).

Two vertices u and v are *adjacent* if $(u,v) \in E$. A vertex u and edge (v,w) are *incident*, if $u = v$ or $u = w$, and *nonincident* otherwise. Likewise, two edges e and f are incident if they share a common endpoint. The *degree* of a vertex v, denoted $deg(v)$, is the number of incident edges to v.

In a graph $G(V,E)$, *subdividing* an edge $(u,v) \in E$ replaces edge (u,v) with the pair of edges (u,w) and (w,v) in E by adding w to V. A *subdivision* of G is a graph obtained by performing a series of subdivisions of G. A graph $G(V,E)$ is *isomorphic* to a graph $\tilde{G}(\tilde{V},\tilde{E})$ if there exists a bijection $f : V \mapsto \tilde{V}$ such that $(u,v) \in E$ if and only if $\big(f(u), f(v)\big) \in \tilde{E}$. A graph $G(V,E)$ is *homeomorphic* to a graph $\tilde{G}(\tilde{V},\tilde{E})$ if a subdivision of G is isomorphic to a subdivision of $\tilde{G}$. The *induced subgraph* of G for the subset $V' \subseteq V$ is subgraph given by the edge set $E \cap (V' \times V')$.

2 Forbidden Simultaneous Embeddings with Fixed Edges

We start by stating the seminal theorem by Kuratowski [14] that characterizes all planar graphs.

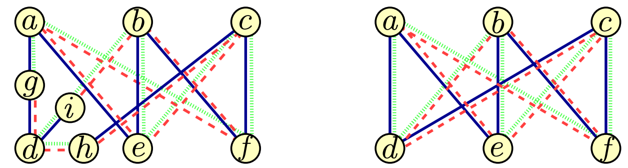
Theorem 1 (Kuratowski) *Every nonplanar graph has a subgraph homeomorphic to K_5 or $K_{3,3}$.*

2.1 Forbidden Triples of Paths and Cycles

The examples without a SGE in [2] and [1] for three paths and cycles, resp., also prevent a SEFE.

Theorem 2 *There exist three paths on 9 vertices and three cycles on 6 vertices without a SEFE.*

Proof. Consider the paths g–d–h–c–e–a–f–b–i, h–d–i–b–e–c–f–a–g, and i–d–g–a–e–b–f–c–h and cycles a–d–c–f–b–e–a, a–e–c–d–b–f–a, and a–f–c–e–b–d–a shown in Fig. 3. In both cases, the union forms a subdivided $K_{3,3}$ and must have a crossing by Theorem 1 in any drawing. Each edge in the union belongs to two paths (or cycles). Such a crossing must then be between two pairs of paths (or cycles). Since there are only three paths (or cycles) and fixed edges are being used, one must self-intersect. □



(a) 3 paths on 9 vertices (b) 3 cycles on 6 vertices

Fig. 3. Triples of graphs without a SEFE.

This shows that the set of triples for SEFE is fairly restricted. As a result, we can focus our attention to forbidden pairs of graphs without a SEFE.

2.2 Minimal Forbidden Pairs

Suppose a pair of graphs $G_1(V, E_1)$ and $G_2(V, E_2)$ does not have a SEFE as in Fig. 4(a). If deleting any edge from either graph allows a SEFE, then G_1 and G_2 are *edge minimal* as is Fig. 4(b). If a degree-2 vertex v (adjacent to u and w) in the union is not a degree-1 vertex in either G_1 or G_2, then we can *unsubdivide* the vertex by deleting v and replacing edges (u, v) and (v, w) with the edge (u, w) in G_1 and/or G_2. A pair of graphs for which this can no longer be done is *vertex minimal* as is Fig. 4(c). A *minimal forbidden pair* does not have a SEFE and is edge and vertex minimal.

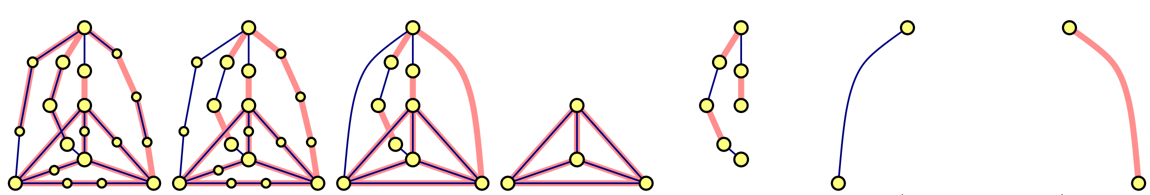
We define the *union* $G_1 \cup G_2$ and the *intersection* $G_1 \cap G_2$ to be the graphs with edge sets $E_1 \cup E_2$ and $E_1 \cap E_2$, respectively. Suppose then that the union of the pair is homeomorphic to a graph G without any degree-2 vertices. Let $u \rightsquigarrow v$ denote the path corresponding to the subdivided edge (u, v) in G. Path $u \rightsquigarrow v$ *is incident to* $x \rightsquigarrow y$ if and only if (u, v) and (x, y) are incident in G.

An *alternating edge* is a $u \rightsquigarrow v$ path in which the edges strictly alternate between being in either G_1 and G_2, but not both. An *exclusive edge* is a $u \rightsquigarrow v$ path composed of the single edge (u, v) only in G_1 or G_2, while an *inclusive edge* is composed of the single edge (u, v) in the intersection.

Claim 3 *Any pair of graphs $G_1(V, E_1)$ and $G_2(V, E_2)$ can be reduced to a pair in which every $u \rightsquigarrow v$ path is either an inclusive edge, an exclusive edge or an alternating edge.*

Proof. While a minimal forbidden pair already has this property, we show how this can be done for an arbitrary pair. We examine each $u \rightsquigarrow v$ path p in the union. If p is in $G_1 \cap G_2$ we replace it with a single inclusive edge (u, v) in both G_1 and G_2. If p is in G_i but is missing edges in G_j, we replace it with the single exclusive edge (u, v) in G_i for $i \neq j$. If p is missing edges from both graphs, we make p into an alternating edge by deleting edges from p in either G_1 or G_2 until each edge along p is no longer in $G_1 \cap G_2$. Then we unsubdivide p until it is strictly alternating. Since we can always avoid crossings along edges of $u \rightsquigarrow v$ paths contained in $G_1 \cap G_2$ reduced in this way, neither of these operations change whether the pair has a SEFE. Any pair of graphs for which this has been done for all $u \rightsquigarrow v$ paths is called a *reduced pair*. $\square$

Suppose G_1 and G_2 are a reduced pair. The *alternating edge subgraph*, denoted $G_1 \uplus G_2$, is the subgraph of $G_1 \cup G_2$ consisting only of alternating edges. The *exclusive edge subgraph* of G_1, denoted $G_1 \setminus G_2$, is the subgraph of $G_1 \cup G_2$, of exclusive edges from G_1, where $G_2 \setminus G_1$ is defined analogously. Hence, edges of $G_1 \cup G_2$ are partitioned into $G_1 \uplus G_2$, $G_1 \setminus G_2$, $G_2 \setminus G_1$, and $G_1 \cap G_2$; see Fig. 4(c)–(g). Next we see why we only need to consider crossings between nonincident edges.



$G_1 \cup G_2$ $G_1 \cap G_2$ $G_1 \uplus G_2$ $G_1 \setminus G_2$ $G_2 \setminus G_1$

(a) (b) (c) (d) (e) (f) (e)

Fig. 4. Removing extra edges from (a) gives (b). Unsubdividing vertices gives (c) with the four subgraphs (d)–(g).

Observation 4 *Crossings in a nonplanar drawing between a pair of incident edges can be removed without affecting the number of crossings of nonincident edges.*

This can be done by swapping Jordan curves from the incident vertex to the first intersection point p. Separating the curves at p by a distance ε eliminates the crossing at p without affecting the rest of drawing. Repeating this process removes all crossings of incident edges. Hence, we only need to consider crossings of nonincident edges in simultaneous drawings with fixed edges. Applying this observation to the minimal examples K_5 and $K_{3,3}$ of Theorem 1 gives the following corollary.

Corollary 5 *(a) Every drawing of K_5 or $K_{3,3}$ has at least one crossing between nonincident edges. (b) Any K_5 and $K_{3,3}$ can be drawn with only one crossing between any pair of nonincident edges.*

We use this corollary to produce a sufficient condition for SEFE.

Lemma 6 *Suppose the union of a reduced pair G_1 and G_2 is homeomorphic to K_5 or $K_{3,3}$. Let $u \rightsquigarrow v$ and $x \rightsquigarrow y$ be nonincident paths in $G_1 \cup G_2$ but not in $G_1 \cap G_2$. G_1 and G_2 share a SEFE if either path belongs to $G_1 \uplus G_2$ or one belongs to $G_1 \setminus G_2$ and the other belongs to $G_2 \setminus G_1$.*

Proof. By Corollary 5(b), a K_5 or $K_{3,3}$ can always be drawn so that only (u, v) and (x, y) cross. Hence, there is a SEFE in which an alternating edge in $G_1 \uplus G_2$ only crosses an edge in either $G_1 \setminus G_2$ or $G_2 \setminus G_1$ or an edge in $G_1 \setminus G_2$ only crosses an edge in $G_2 \setminus G_1$. □

Applying Lemma 6, we next determine when a pair of graphs forming K_5 or $K_{3,3}$ has a SEFE.

Corollary 7 *Suppose the union of a reduced pair G_1 and G_2 is homeomorphic to K_5 or $K_{3,3}$. G_1 and G_2 do not share a SEFE if and only if (i) every nonincident edge of an alternating edge in $G_1 \uplus G_2$ is in $G_1 \cap G_2$ and (ii) every nonincident edge of an exclusive edge in $G_1 \setminus G_2$ is also in G_1.*

Proof. For necessity, suppose G_1 and G_2 do not have a SEFE. If there is a nonincident edge $x \rightsquigarrow y$ of an alternating edge $u \rightsquigarrow v$ that is not in $G_1 \cap G_2$, by Lemma 6, G_1 and G_2 would have a SEFE since $u \rightsquigarrow v$ is in $G_1 \uplus G_2$ and neither path is in $G_1 \cap G_2$. If there is a nonincident edge $x \rightsquigarrow y$ of an edge $(u, v) \in G_1 \setminus G_2$ that is not in G_1, then again by Lemma 6, G_1 and G_2 would have a SEFE since $x \rightsquigarrow y$ is either in $G_1 \uplus G_2$ or $G_2 \setminus G_1$.

For sufficiency, suppose conditions (i) and (ii) hold. Since the union forms a subdivided K_5 or $K_{3,3}$, by Corollary 5(a) at least one pair of nonincident paths $u \rightsquigarrow v$ and $x \rightsquigarrow y$ cross. If either is in $G_1 \cap G_2$, then there must be a crossing in G_1 or G_2. If either is in $G_1 \uplus G_2$, then by (i) the other would be in $G_1 \cap G_2$, again giving a crossing in G_1 or G_2. If both are in $G_i \setminus G_j$ for $i \neq j$, then there is a crossing in G_i. Finally, (ii) prevents one edge being in $G_1 \setminus G_2$ and the other edge being in $G_2 \setminus G_1$. Hence, G_1 and G_2 do not have a SEFE. □

Theorem 8 *There are 16 minimal forbidden pairs whose union is homeomorphic to K_5 or $K_{3,3}$.*

Proof. Let $G_{i,j}$ denote the 16 pairs of graphs for $i \in \{1, \ldots, 16\}$ and $j \in \{1, 2\}$ shown in Figs. 5 and 6. The union of the first ten pairs is homeomorphic to K_5, while the union of the remaining six is homeomorphic to $K_{3,3}$. One can verify that all the nonincident edges of any alternating edge are in the intersection and every edge nonincident to an exclusive edge of $G_{i,1}$ is also in $G_{i,1}$. This satisfies Corollary 7 implying that none of these pairs has a SEFE.

Removing an edge from any pair either means (i) their union no longer forms a K_5 or a $K_{3,3}$ or (ii) their intersection does not contain all nonincident edges of $G_{i,1} \uplus G_{i,2}$ or of $G_{i,1} \setminus G_{i,2}$ (other than those already in $G_{i,1}$). This then implies by Corollary 7 they would then have a SEFE, which shows that all 16 pairs are minimal forbidden pairs.

To show that our set of 16 pairs are complete, assume w.l.o.g. (G_1, G_2) are a reduced minimal forbidden pair whose union forms a K_5 or $K_{3,3}$ where $G_1 \setminus G_2$ has no more edges than $G_2 \setminus G_1$. We consider all the possibilities for edges in $G_1 \setminus G_2$ or $G_1 \uplus G_2$ in turn.

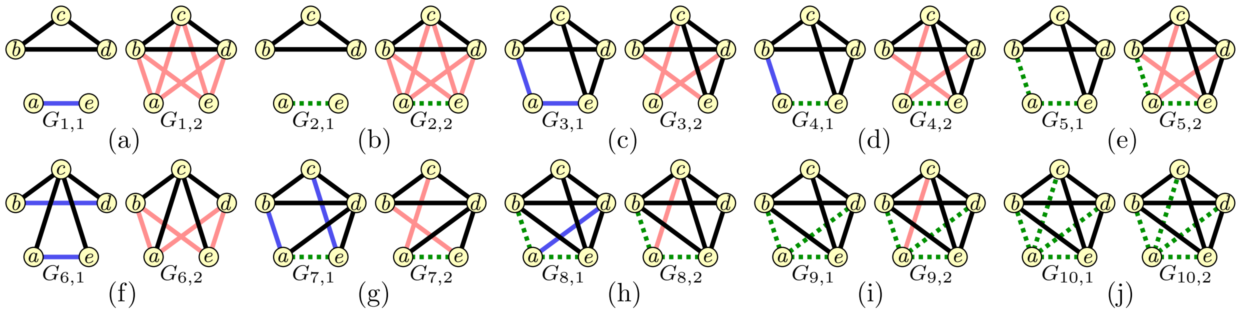


Fig. 5. Ten K_5 minimal forbidden pairs. The dark edges are in $G_1 \cap G_2$ and the dashed edges are in $G_1 \uplus G_2$.

Pairs $(G_{1,1}, G_{1,2})$, $(G_{2,1}, G_{2,2})$, $(G_{11,1}, G_{11,2})$, and $(G_{12,1}, G_{12,2})$ are the only possibilities in which there is exactly one exclusive edge in G_1 or one alternating edge in $G_1 \uplus G_2$. Two nonincident alternating edges would violate Corollary 7, so the other case for a pair of nonincident edges are two exclusive edges in G_1 given by pairs $(G_{6,1}, G_{6,2})$ and $(G_{13,1}, G_{13,2})$. Three nonincident edges are only possible in a $K_{3,3}$, but including all nonincident would mean that G_1 was the whole $K_{3,3}$.

For a pair whose union is homeomorphic to K_5, pairs $(G_{3,1}, G_{3,2})$, $(G_{4,1}, G_{4,2})$, and $(G_{5,1}, G_{5,2})$ give the three possibilities for two incident edges in $(G_1 \setminus G_2) \cup (G_1 \uplus G_2)$, namely, two exclusive, one exclusive and one alternating, and two alternating. Two incident exclusive edges with a third edge, exclusive or alternating, is not possible since $G_{3,1}$ with two exclusive edges already has seven edges. Adding another exclusive or alternating edge and its nonincident edge, would leave only one edge for $G_2 \setminus G_1$, contradicting our assumption of $G_1 \setminus G_2$ being no larger. Two incident exclusive edges with a third alternating edge is given by the pair $(G_{7,1}, G_{7,2})$. Finally, three and four incident alternating edges are given by pairs $(G_{9,1}, G_{9,2})$ and $(G_{10,1}, G_{10,2})$.

For a pair whose union is homeomorphic to $K_{3,3}$, if $(G_1 \setminus G_2) \cup (G_1 \uplus G_2)$ contains two incident edges, then all remaining edges except for the third incident edge $u \rightsquigarrow v$ (in the union) are nonincident and must be in G_1. Since edges nonincident to $u \rightsquigarrow v$ are also in G_1, $G_2 \setminus G_1$ only contains $u \rightsquigarrow v$, so $G_1 \setminus G_2$ has at most one edge by assumption. Thus, pairs $(G_{14,1}, G_{14,2})$ with one exclusive edges and one alternating edge and $(G_{15,1}, G_{15,2})$ with two alternating edges are the only possibilities for two incident edges. Adding a third edge, means that it must be an alternating edge. However, $G_{15,2}$ already has one exclusive edge with two incident alternating edges leaving three incident alternating edges given by pair $(G_{16,1}, G_{16,2})$ as the final possibility. □

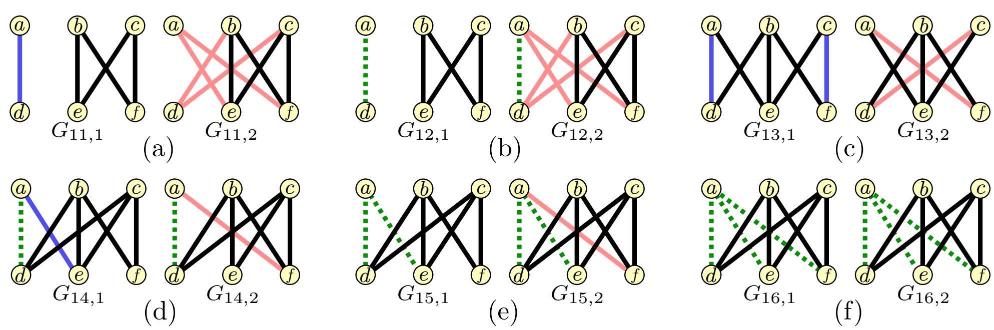


Fig. 6. Six $K_{3,3}$ minimal forbidden pairs. The dark edges are in $G_1 \cap G_2$ and the dashed edges are in $G_1 \uplus G_2$.

3 Characterizing Simultaneous Embeddings with Planar Graphs

In this section we determine which graphs always have a SEFE with any planar graph and show how to produce a simultaneous drawing. Let $\mathcal{P}$ be the set of planar graphs and $\mathcal{P}_{\mathsf{SEFE}} \subset \mathcal{P}$ be the subset of forests, circular caterpillars and subgraphs of K_4.

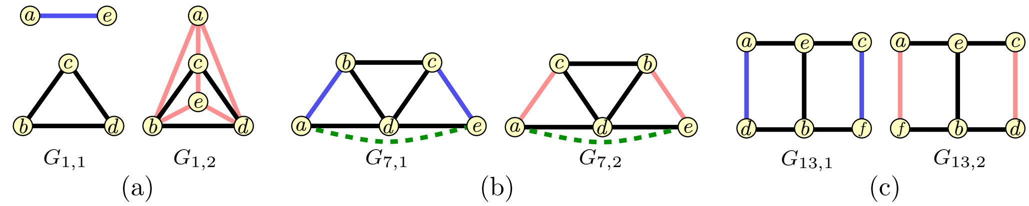


Fig. 7. Planar drawings of each of the graphs in the pairs $(G_{1,1}, G_{1,2})$, $(G_{7,1}, G_{7,2})$, and $(G_{13,1}, G_{13,2})$.

Lemma 9 *The set $\mathcal{P}_{\mathsf{SEFE}}$ are the only graphs that can always have a SEFE with any planar graph.*

Proof. Let $G_1 \in \mathcal{P}_{\mathsf{SEFE}}$ and $G_2 \in \mathcal{P}$. Both graphs of each of 16 pairs of Theorem 8 have a subgraph homeomorphic to $G_{1,1}$, which is a K_3 and a disjoint edge; see Fig. 7(a). First, we show that G_1 does not contain a subgraph homeomorphic to $G_{1,1}$. This can allow G_1 to have a SEFE with G_2 since they cannot match any of the 16 pairs. Clearly, a forest cannot since $G_{1,1}$ has a cycle. A circular caterpillar has a single cycle and all other edges are incident to the cycle unlike $G_{1,1}$ that has an edge nonincident to its K_3. Finally, K_4 only has four vertices and $G_{1,1}$ has five.

Next, we show that any graph in $\mathcal{P} \setminus \mathcal{P}_{\mathsf{SEFE}}$ must contain a subgraph homeomorphic to $G_{1,1}$. If $G \in \mathcal{P}$ does not have a cycle, it is a forest. Otherwise, assume G has a cycle C and let C be maximal in terms of the number of edges. If any other edge e in G is nonincident to C, G has a subgraph homeomorphic to $G_{1,1}$, namely C and e. Hence, all edges are incident to C. If any edge e forms a chord of C where C has length greater than four, then one of the cycles C' containing e would have a nonincident edge e' on C such that C' and e' would be homeomorphic to $G_{1,1}$. If C has length four or less, then G is a subgraph of K_4. For C to be maximal, no pair of edges can be incident to C and share a common endpoint. Hence, all the edges not in C are degree-1 vertices forming a circular caterpillar. □

Next we show that the graphs $\mathcal{P}_{\mathsf{SEFE}}$ from Lemma 9 that can have a SEFE with any planar graph, do indeed have one with the next theorem.

Theorem 10 *Pair $G_1 \in \mathcal{P}_{\mathsf{SEFE}}$, $G_2 \in \mathcal{P}$ on n vertices have a SEFE computable in $O(n^2 \lg n)$ time.*

Proof. Frati [10] gave an algorithm using the dual of the graph for a forest and a planar graph without bounding the time complexity. Our algorithm is based on finding Euclidean shortest paths in the plane with a set of line segments incident only at endpoints as potential obstacles.

Let $G_1 \in \mathcal{P}_{\mathsf{SEFE}}$ and $G_2 \in \mathcal{P}$. First, we draw G_2 in $O(n)$ time. We can find an embedding of G_2 in $O(n)$ time [3] and then draw it on an $(n-2) \times (n-2)$ grid in $O(n)$ time [4]. Some of the edges of G_1 are then drawn simultaneously with G_2. We then ignore the edges in $G_2 \setminus G_1$ and draw the remainder of G_1. In the case of a forest or circular caterpillar in which the cycle is not yet drawn, there is a single face giving some shortest Euclidean path between two vertex locations. In the case of the circular caterpillar with the cycle already drawn, the remaining points either lie interior or exterior to the cycle. All edges are incident with the cycle, so a Euclidean path always exists from the cycle vertices to the degree-1 vertices. Finally, for a K_4 there is a cycle of length 4 in which all vertices lie on the cycle. We first draw this cycle if not already drawn. The other two edges, which form chords, can always be drawn since one can be drawn inside and the other outside the cycle.

Using the line segments of the edges of G_1 already drawn as obstacles, we proceed to add edges using an optimal Euclidean shortest path algorithm [13] that runs in $O(n \lg n)$ for each step. For each step i, a new bend $b_{i,k}$ is either caused by an endpoint p_k of an original line segment or a bend $b_{j,k}$ from a previous step $2 \leq j < i$. However, for each such bend $b_{i,k}$ only at most two points in the set $\{p_k, b_{2,k}, \ldots b_{i-1,k}\}$ (the inner and outer ones) can contribute since bends added more recently hide bends caused by the original point p_k in previous steps. Hence, each time we add edges, at most $O(n)$ new bends are being introduced giving an overall running time of $O(n^2 \lg n)$. □

4 Characterizing Simultaneous Embeddings with Outerplanar Graphs

We next determine which outerplanar graphs always have a SEFE with any other outerplanar graph and produce simultaneous drawings when possible. A K_3*-cycle* is a biconnected outerplanar graph such that the endpoints of every chord are at a distance 2 from each other along the outerface. A *cubic* graph has maximum degree 3. Let $\mathcal{O}$ be the set of outerplanar graphs and $\mathcal{O}_{\mathsf{SEFE}} \subset \mathcal{O}$ be the set of outerplanar graphs that consist of a single K_3-cycle or have a cubic K_3-cycle for each biconnected subgraph.

Lemma 11 *The set $\mathcal{O}_{\mathsf{SEFE}}$ are the only outerplanar graphs that can always have a SEFE with any outerplanar graph.*

Proof. Let $G_1 \in \mathcal{O}_{\mathsf{SEFE}}$ and $G_2 \in \mathcal{O}$. Of the 16 pairs from Theorem 8, only both graphs of pairs $(G_{7,1}, G_{7,2})$ and $(G_{13,1}, G_{13,2})$ are outerplanar. Both $G_{13,1}$ and $G_{13,2}$ have an outerface of length six with one chord forming two C_4's, which we call a *bi-*C_4; see Fig. 7(b). The non-alternating edges of $G_{7,1}$ and $G_{7,2}$ each form a K_3-cycle with an outerface of length five and two chords, both incident creating a degree-4 vertex, which we call a *tri-*K_3; see Fig. 7(c). Hence, by Theorem 8 any pair of outerplanar graphs with a SEFE cannot each have subgraphs both isomorphic to $G_{13,1}$, a bi-C_4, or $G_{7,1}$, a tri-K_3 (with an extra edge with at most one endpoint in the tri-K_3). Clearly, a K_3-cycle cannot contain either of these since there is at most one C_4 and no extra edges. If each biconnected component is only a cubic K_3-cycle, it has at most one C_4 and no degree-4 vertices so G_2 cannot contain either a bi-C_4 or a tri-K_3.

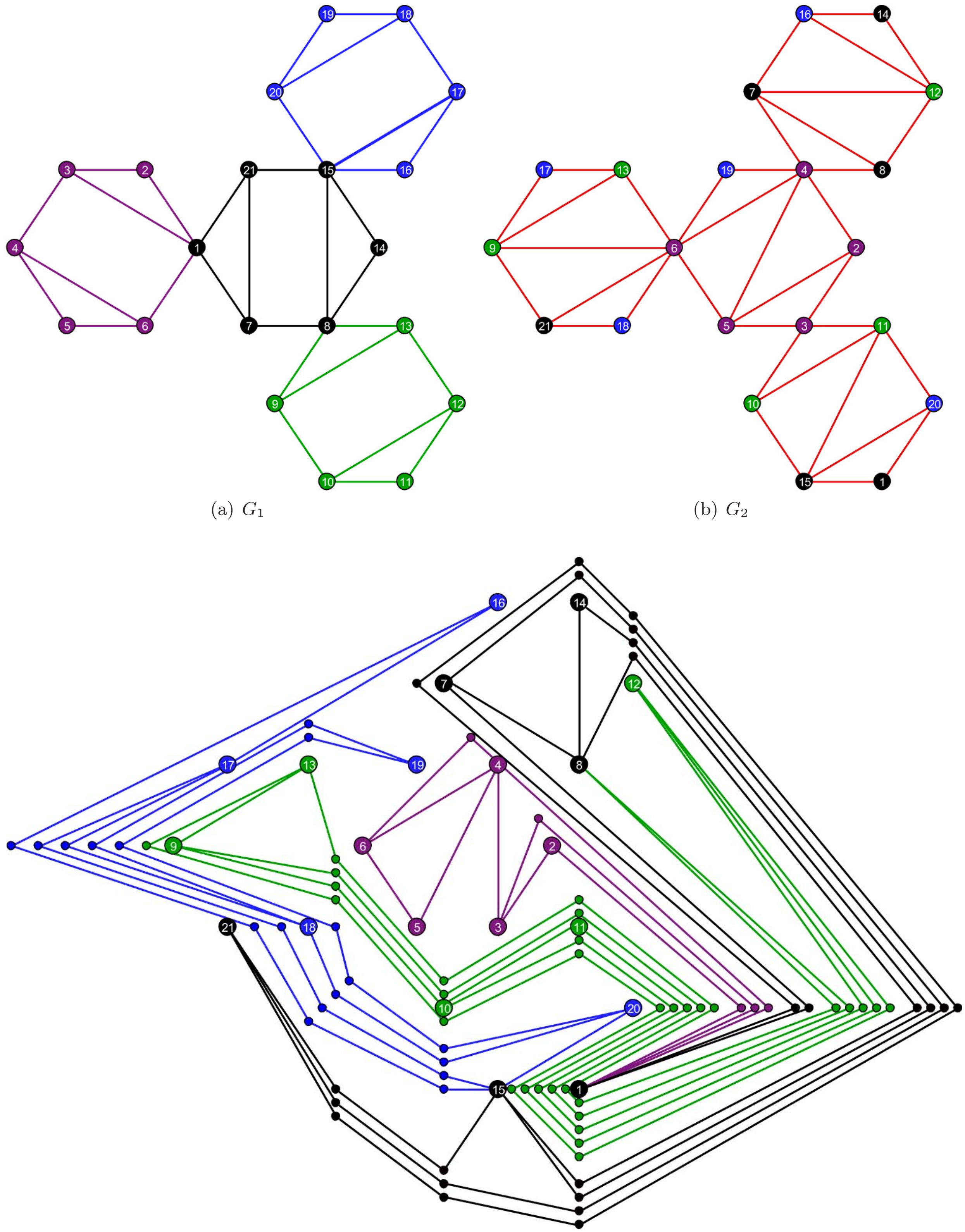
Next we show that all graphs in $\mathcal{O} \setminus \mathcal{O}_{\mathsf{SEFE}}$ have a subgraph homeomorphic to $G_{7,1}$ or $G_{13,1}$. If G is biconnected, all the vertices must lie along a single cycle C along the outerface. If each chord (u, v) has endpoints u and v at a distance 2 along the outerface, then G is a K_3-cycle. Otherwise, the vertices u and v must have at least two immediate vertices along the outerface in both directions, so that C and edge (u, v) are homeomorphic to $G_{13,1}$.

Otherwise, G is not biconnected. If G has no biconnected components, then G is a tree in $\mathcal{O}_{\mathsf{SEFE}}$. So G must have at least one biconnected component H with cycle C for its outerface. If C has no chords, then it is trivially a cubic K_3-cycle. If C has one chord such that its endpoints (u, v) are separated by more than one vertex along C, then C and (u, v) are isomorphic to $G_{13,1}$. Otherwise, they are separated by one vertex w and are at a distance 2 along C, which means H is a K_3-cycle. If H has two incident chords (u, v) and (u, w), then C and the two chords and another edge not in H are homeomorphic (which must exist since G is not biconnected) to $G_{7,1}$. If all chords are nonincident, then H has maximum degree 3 and H is a cubic K_3-cycle. Otherwise, it has one chord (u, v) separated by at least two vertices in both directions so that C and (u, v) are $G_{13,1}$. □

We show all the outerplanar graphs $\mathcal{O}_{\mathsf{SEFE}}$ of Lemma 11 have a SEFE with any outerplanar graph with the next theorem.

Theorem 12 *Pair $G_1 \in \mathcal{O}_{\mathsf{SEFE}}$, $G_2 \in \mathcal{O}$ on n vertices have a SEFE computable in $O(n^2 \lg n)$ time.*

Proof. We augment the drawing algorithm from Theorem 10 for the case of outerplanar graphs. However, we need to be careful when closing cycles along the outerface so as not to include any vertices of any other biconnected component. By Lemma 11, each biconnected component is a K_3-cycle. First we draw G_2 in $O(n)$ time as before and ignore edges in $G_2 \setminus G_1$. We transverse the biconnected components in a depth first order. For each biconnected component B that has not been drawn, we proceed along the outerface of B by drawing each edge (u, v) with a Euclidean path (with at least a distance ε from any point causing a bend) that proceeds in a clockwise direction except for the last edge closing the cycle of the outerface. For that last edge (u, v) we must close the cycle without enclosing any other points of any other biconnected component. We follow the



(a) G_1 (b) G_2

(c) Planar drawing of G_1 with fixed edgess using vertex locations of G_2.

Fig. 8. SEFE of two outerplanar graphs. The edges of the four biconnected component of G_1 are colored distinctly.

boundary of B from u to v in the clockwise direction. Any interior chord (x, z) not already drawn must have a common neighbor y, such that the chord can follow the paths from (x, y) and (y, z) within a distance ε on the interior of the cycle. The overall running time remains $O(n^2 \lg n)$.

See Fig. 8 for a non-trivial example of this algorithm. Please note that the bends are placed at a fixed distance apart (for readability) which distorts the drawing from the algorithm. The purple cycle 1–2–3–4–5–6–1 of the first biconnected component was drawn first, and then the black cycle 1–7–8–14–15–21–1 of the interior biconnected component was drawn second. Then the third biconnected component with the green cycle 8–9–10–11–12–13–8 was drawn next leaving the blue cycle 15–16–17–18–19–20–15 for last. □

5 Deciding Simultaneous Embeddings for Outerplanar Graphs

Theorem 13 *Deciding if a pair of outerplanar graphs have a* SEFE *can be done in $O(n^2)$ time.*

Proof. We spend $O(n)$ time reducing the pair by spending $O(n)$ time removing extra edges along degree-2 paths in the union and unsubdividing accordingly. Let $G_1, G_2 \in \mathcal{O}$. By Lemma 11 and Theorem 12, we know that the pairs $(G_{13,1}, G_{13,2})$ and $(G_{7,1}, G_{7,2})$ determine which pairs of outerplanar graphs do not have a SEFE. All we need to do is examine all possible pairs of subgraphs whose union is homeomorphic to K_5 or $K_{3,3}$ to see if any pair meets the appropriate conditions, which determines whether G_1 and G_2 have a SEFE.

For the pair G_{13}, the intersection is a tree on six vertices with a pair of adjacent degree-3 vertices; see Fig. 7(b). So we examine each of the $O(n)$ chords (u, v) in each biconnected component in G_1 with outerface cycle C in which u and v are separated by at least two vertices in each direction along C. Let $\{x, y\}$ and $\{w, z\}$ be the vertices adjacent to u and v, where (x, u, y, w, v, z) are their ordering along C. Then if all six vertices are in the same biconnected component of G_2 (which we can test in constant time with a preprocessing step that takes $O(n)$ time to find all biconnected components of each graph in which we number all vertices in clockwise order for each biconnected component), and (y, u, x, w, v, z) is the order along the cycle in G_2 (that can also be determined in constant time), then we have the pair being homeomorphic to $(G_{13,1}, G_{13,2})$. Hence, we can find all possible pairs that can form $(G_{13,1}, G_{13,2})$ in $O(n)$ time.

For the pair $(G_{7,1}, G_{7,2})$, the intersection is a K_3, $\{(u, v), (v, w), (u, w)\}$, with two edges (x, v) and (v, y) incident to v; see Fig. 7(c). For each vertex v with degree-4 or greater along each biconnected component in G_1 with outerface cycle C, we pick (u, v) and (v, w) to be the chords in which v and w have the greatest separation along the C in the intersection, and take x and y to be the vertices adjacent to v, where (x, v, y, w, u) are their order along C. We need to perform two checks with G_2. First, we check that all five are in the same biconnected component in G_2 with cycle order (y, v, x, w, u) (again done in constant time with an $O(n)$ time preprocessing step). Second, we check that x and y are connected to each other in the union minus the vertices $\{u, v, w, x, y\}$. This is also done in $O(n)$ time. Thus, we find all possible pairs forming $(G_{7,1}, G_{7,2})$ in $O(n^2)$ time. □

6 Conclusion

We gave a necessary condition for two graphs to have a SEFE. Using this condition we characterized which graphs always have a SEFE with any planar graphs as well as which outerplanar graphs always have a SEFE with any outerplanar graph. This allowed us to give a polynomial time algorithm for deciding if a pair of outerplanar graphs has a SEFE. Finding an analogous polynomial time algorithm for deciding if a pair of planar graphs has a SEFE remains open.

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