Minimum Level Nonplanar Patterns for Trees

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Abstract. Minimum level nonplanar (MLNP) patterns play the role for level planar graphs that the forbidden Kuratowksi subdivisions K_5 and $K_{3,3}$ play for planar graphs. We add two MLNP patterns for trees to the previous set of tree patterns given by Healy *et al.* [4]. Neither of these patterns match any of the previous patterns. We show that this new set of patterns completely characterizes level planar trees.

1 Introduction

Level graphs model hierarchical relationships. A level drawing has all vertices in the same level with the same y-coordinates and has all edges strictly y-monotone. Level planar graphs have level drawings without edge crossings. Hierarchies are special cases in which every vertex is reachable via a y-monotone path from a source in the top level. Many natural hierarchies occur in the sciences including biological taxonomies, linguistic universal grammars, object-oriented design, multi-tiered social structures, and mathematical hierarchies. In general, any *directed acyclic graph* (DAG) yields a hierarchy by using a topological sort as a ranking mechanism. Planar graphs are characterized by forbidden subdivisions of K_5 and $K_{3,3}$ by Kuratowksi's Theorem [8]. The counterpart of this characterization for level planar graphs proposed by Healy, Kuusik, and Liepert [4] are the minimum level nonplanar (MLNP) patterns. These are minimal obstructing subgraphs with a set of level assignments that force one or more crossings.

While Jünger *et al.* provide linear time recognition and embedding algorithms [6, 7] for level planar graphs, swapping the vertices between levels while maintaining planarity can be difficult. Heath and Rosenberg showed that deciding if a planar graph has a proper k-leveling is NP-hard [5]. Finding a matching subgraph of a MLNP pattern can provide a set of candidate vertices to reassign to different levels in order to achieve planarity. Such a method could improve existing hierarchical approaches to drawing DAGs, such as Sugiyama's algorithm [9] that greedily assigns vertices to levels. Determining the minimum number of edges to remove so that a graph becomes level planar is known as the *level planarization problem.* Eades and Whitesides showed that this is NP-hard even for the case of a 2-leveling in which the placement of the vertices of one of the levels is given [2].

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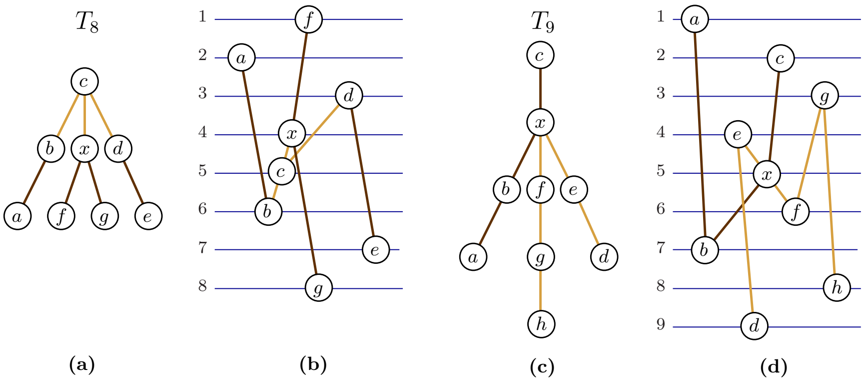

Fig. 1. Labelings preventing the forbidden ULP trees T_8 and T_9 from being level planar.

Di Battista and Nardelli [1] provided three level nonplanar patterns for hierarchies (HLNP patterns) showing they formed a necessary and sufficient condition for level nonplanarity; cf. Fig. 3. These patterns consist of three (not necessary) disjoint paths linking a pair of levels that are joined by three pairwise bridges. If none of the linking paths cross, this condition forces a crossing between one or more bridges. Showing any level nonplanar hierarchy must match one of these patterns was done by considering the cases in which their PQ-tree algorithm fails to provide an embedding if the hierarchy is level nonplanar by generating an edge crossing. They use the paths from the two edges that cross to a common ancestor in order to always construct one of the three HLNP patterns completing their characterization of level planar hierarchies.

Since these patterns are adequately general, this approach can be extended to determine when level graphs are nonplanar. Healy *et al.* adapted these HLNP patterns to MLNP patterns for level graphs. However, the completeness of their characterization was based on the claim that all MLNP patterns must contain a HLNP pattern, which does not hold for a counterexample we provide.

Estrella *et al.* [3] characterized the set of unlabeled level planar (ULP) trees on n vertices that are level planar over all possible $n!$ labelings of the vertices from 1 to n in terms of a pair of forbidden subtrees T_8 and T_9; cf. Fig. 1. The given labelings show that these trees are level nonplanar. Each vertex is assigned

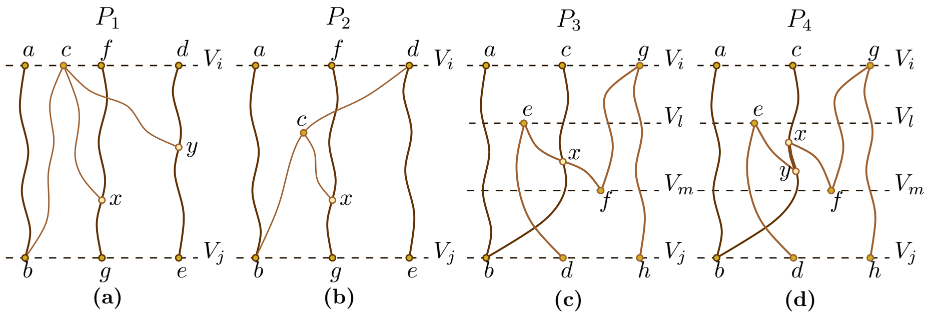

Fig. 2. Four minimum level nonplanar (MLNP) patterns for level nonplanar trees.

to its own level so that its y-coordinate is based on its level. The level nonplanar assignment for T_9 can be shown not to match any of the three HLNP patterns forming the basis of our counterexample. For every set of three paths linking any pair of levels in T_9, two of the three linking paths always has a bridge that shares a vertex with the other path. This violates the condition that forces a crossing between the third linking path and the bridge. As a result, this level nonplanar tree does not match any of the MLNP patterns given by Healy *et al.*

Healy *et al.* provide two of the MLNP patterns, P_1 and P_2, for trees that are also HLNP patterns; cf. Fig. 2(a) and (b). Both have three disjoint paths linking the top and bottom levels with the three pairwise bridges that form a subdivided $K_{1,3}$. We provide two more MLNP patterns, P_3 and P_4 for level nonplanar trees; cf. Fig. 2(c) and (d) using our counterexample. Both of these patterns consist of two paths that have a common vertex x or subpath $x \rightsquigarrow y$ that lies between two intermediate levels. A crossing is forced between the two paths since x or $x \rightsquigarrow y$ must lie between two different sections of path that they are on in order to avoid a self-crossing of that path.

2 Preliminaries

A *k-level graph* $G(V, E, \phi)$ on n vertices has *leveling* $\phi : V \to [1..k]$ where every $(u, v) \in E$ either has $\phi(u) < \phi(v)$ if G is directed or $\phi(u) \neq \phi(v)$ if G is undirected. This leveling partitions V into $V_1 \cup V_2 \cup \cdots \cup V_k$ where the *level* $V_j = \phi^{-1}(j)$ and $V_i \cap V_j = \varnothing$ if $i \neq j$. A *proper level graph* only has *short edges* in which $\phi(v) = \phi(u) + 1$ for every $(u, v) \in E$. Edges spanning multiple levels are *long*. A *hierarchy* is a proper level graph in which every vertex $v \in V_j$ for $j > 1$ has at least one incident edge $(u, v) \in E$ to a vertex $u \in V_i$ for some $i < j$.

A *path* p is a non-repeating ordered sequence of vertices $(v_1, v_2, \ldots, v_t)$ for $t \geq 1$. Let $\textsc{min}(p) = \min\{\phi(v) : v \in p\}$, $\textsc{max}(p) = \max\{\phi(v) : v \in p\}$, and $\mathcal{P}(i, j) = \{p : p \text{ is a path where } i \leq \textsc{min}(p) < \textsc{max}(p) \leq j\}$ are the paths between levels V_i and V_j. A *linking path*, or *link*, $L \in \mathcal{L}(i, j)$ is a path $x \rightsquigarrow y$ in which $i = \textsc{min}(L) = \phi(x)$ and $\textsc{max}(L) = \phi(y) = j$, and $\mathcal{L}(i, j) \subseteq \mathcal{P}(i, j)$ are all paths linking the *extreme levels* V_i and V_j. A *bridge* b is a path $x \rightsquigarrow y$ in $\mathcal{P}(i, j)$ connecting links $L_1, L_2 \in \mathcal{L}(i, j)$ in which $b \cap L_1 = x$ and $b \cap L_2 = y$.

Any *improper level graph* can be made proper by subdividing all long edges into short edges. A *level drawing* of G has all of its *level-j* vertices in the j^{th} level V_j placed along the *track* $\ell_j = \{(x, k - j) \mid x \in \mathbb{R}\}$, and each edge $(u, v) \in E$ is drawn as a continuous strictly y-monotone sequence of line segments downwards. A level drawing drawn without edge crossings shows that G is *level planar*. Any level graph can be made into hierarchy by adding a new source with paths to all vertices unreachable via a y-monotone path to a source. A *pattern* is a set of level nonplanar graphs sharing structural similarities. Removing any edge from the underlying graph matching a *minimum level nonplanar* (MLNP) *pattern* gives a level planar graph. A *hierarchy level nonplanar* (HLNP) *pattern* is a level nonplanar pattern in which every matching graph is a hierarchy. The next theorem gives the set of the three distinct HLNP patterns.

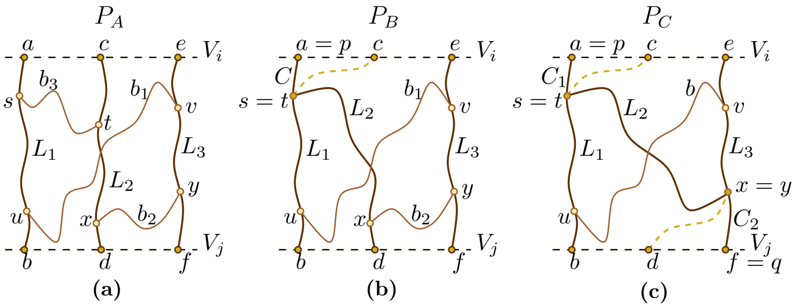

Fig. 3. The three patterns characterizing hierarchies. Patterns P_B and P_C are special cases of P_A. The dashed curves in (b) and (c) are extraneous paths highlighting the relationship P_B and P_C have with P_A if one or more bridges have no edges.

Theorem 1 *[Di Battista and Nardelli [1]] A hierarchy $G(V, E, \phi)$ on k levels is level planar if and only if there does not exist three paths $L_1, L_2, L_3 \in \mathcal{L}(i, j)$ linking levels V_i and V_j for $1 \leq i < j \leq k$ where one of the following hold:*

(P_A) *Links L_1, L_2, and L_3 are completely disjoint, $L_1 \cap L_2 = L_1 \cap L_3 = L_2 \cap L_3 = \varnothing$, and pairwise connected by bridges b_1 from L_1 to L_3, b_2 from L_2 to L_3, and b_3 from L_2 to L_3 such that $b_1, b_2, b_3 \in \mathcal{P}(i, j)$ where $b_1 \cap L_2 = b_2 \cap L_1 = b_3 \cap L_1 = \varnothing$; cf. Fig. 3(a).*

(P_B) *Links L_1 and L_2 share a path $C = L_1 \cap L_2 \in \mathcal{P}(i, j)$ starting from endpoint $p \in V_i \cup V_j$ that is disjoint from L_3, where $L_1 \cap L_3 = L_2 \cap L_3 = \varnothing$ are connected by bridges b_1 from L_1 to L_3 and b_2 from L_1 to L_3 such that $b_1, b_2 \in \mathcal{P}(i, j)$ and such that $b_1 \cap L_2 = b_2 \cap L_1 = \varnothing$; cf. Fig. 3(b).*

(P_C) *Links L_1 and L_2 share a path $C_1 = L_1 \cap L_2 \in \mathcal{P}(i, j)$ starting from endpoint $p \in V_i$ and links L_2 and L_3 share a path $C_2 = L_2 \cap L_3 \in \mathcal{P}(i, j)$ starting fromendpoint $q \in V_j$ such that $C_1 \cap C_2 = \varnothing$. Bridge $b \in \mathcal{P}(i, j)$ connects L_1 and L_3 where $b \cap L_2 = b \cap C_1 = b \cap C_2 = \varnothing$; cf. Fig. 3(c).*

A HLNP pattern P of Theorem 1 is not necessarily minimal in that it does not minimize the number of levels required to force level nonplanarity. However, P becomes minimal if both of the extreme levels V_i and V_j each contain a vertex from one of the bridges or the point at which two links merge. If this were the case, the removal of any further edge from the subgraph of a level graph matching P would then violate one of the structural requirements of the Theorem 1. This is because each extreme level plays an essential role in that the next closest level to the opposite extreme level cannot be substituted for it in the description in the pattern. Generalizing this notion gives the following observation regarding minimality.

Observation 2 *A LNP pattern P between extreme levels V_i and V_j for some $1 \leq i < j \leq k$ is minimal only if the adjacent levels V_{i+1} or V_{j-1} cannot be substituted for V_i or V_j, respectively, in the description of the pattern.*

Observation 2 implies the extreme levels of a MLNP pattern are defined in terms of the roles of their vertices, which we will see in the following descriptions of the four MLNP patterns for trees in the next section.

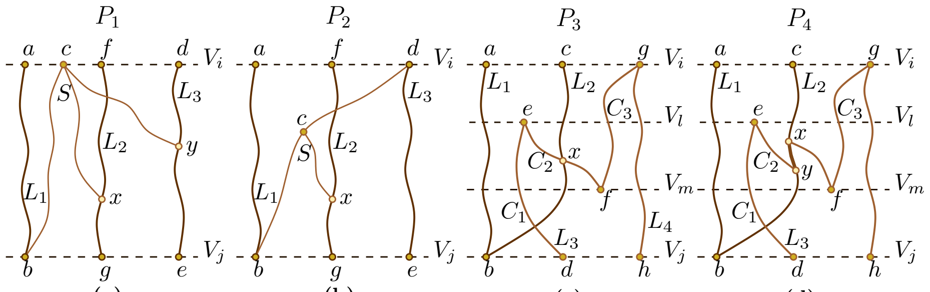

Fig. 4. P_1 of (a) and P_2 of (b) are MLNP patterns $T1$ and $T2$ given by Healy *et al.* [4], respectively. P_3 matches T_9 in [3]. P_4 splits the degree 4 vertex x of P_3 into path $x \rightsquigarrow y$.

3 MLNP Patterns for Trees

We begin by providing an extended set of MLNP patterns for trees.

Theorem 3 *A level tree $T(V, E, \phi)$ on k levels is minimum level nonplanar if*

(1) there are three disjoint paths $L_1, L_2, L_3 \in \mathcal{L}(i, j)$ linking levels V_i and V_j for $1 \le i < j \le k$ where P_A of Theorem 1 applies and the union of the three bridges $b_1 \cup b_2 \cup b_3$ forms a subdivided $K_{1,3}$ subtree S with vertex c of degree 3 where either

(P_1) $c \in V_i$ (or V_j) and there is a leaf of S in V_j (or V_i) as in Fig. 4(a) or

(P_2) one leaf of S is in V_i and another leaf of S is in V_j as in Fig. 4(b),

or

(2) there are four paths $L_1, L_2, L_3, L_4 \in \mathcal{L}(i, j)$ linking levels V_i and V_j for $1 \le i < j \le k$ where $L_1 \cap L_4 = \varnothing$, $L_1 \cap L_2 \in V_j$ (or V_i) and $L_3 \cap L_4 \in V_i$ (or V_j) where $L_1 \cup L_2$ and $L_3 \cup L_4$ form paths with both endpoints in V_i and V_j (or V_j and V_i), respectively, and there exist levels V_l and V_m for some $i < l < m < j$ in which either L_2 or L_3 consists of threee subpaths C_1, C_2, and C_3 such that $C_1 \in \mathcal{L}(i, m)$ links V_i to V_m $\big(d \rightsquigarrow e$ as in Fig. 4(c)$\big)$, $C_2 \in \mathcal{L}(l, m)$ links V_l to V_m $\big(e \rightsquigarrow f$ as in Fig. 4(c)$\big)$, and $C_3 \in \mathcal{L}(l, j)$ links V_l to V_j $\big(f \rightsquigarrow g$ as in Fig. 4(c)$\big)$ where either

(P_3) $L_2 \cap L_3 = x$ where $l \le \phi(x) \le m$ as in Fig. 4(c), or

(P_4) $L_2 \cap L_3$ is path $x \rightsquigarrow y$ where $l \le \{\phi(x), \phi(y)\} \le m$ and $L_2 = c \rightsquigarrow x \rightsquigarrow y \rightsquigarrow b$ where $c \in V_i$ (or V_j) and $b \in V_j$ (or V_i) as in Fig. 4(d).

Proof. The description of patterns P_1 and P_2 are more succinctly stated and more closely match notation used in Theorem 1 from [1] than the Healy *et al.* characterization of MLNP T1 and T2 tree patterns given in Section 3.1 of [4]; see the appendix for the original descriptions of T1 and T2. Patterns P_1 and P_2 are MLNP given they match the patterns of T1 and T2 of Healy *et al* since they meet the four conditions given for T1 and T2 in [4]. Hence, we can conclude that P_1 and P_2 are MLNP. The argument in [3] used by Estrella *et al.* to show T_9 is level nonplanar easily generalizes for P_3 and P_4. To see that P_3 is minimal (the argument for P_4 is similar), we try the seven clearly distinct ways of removing

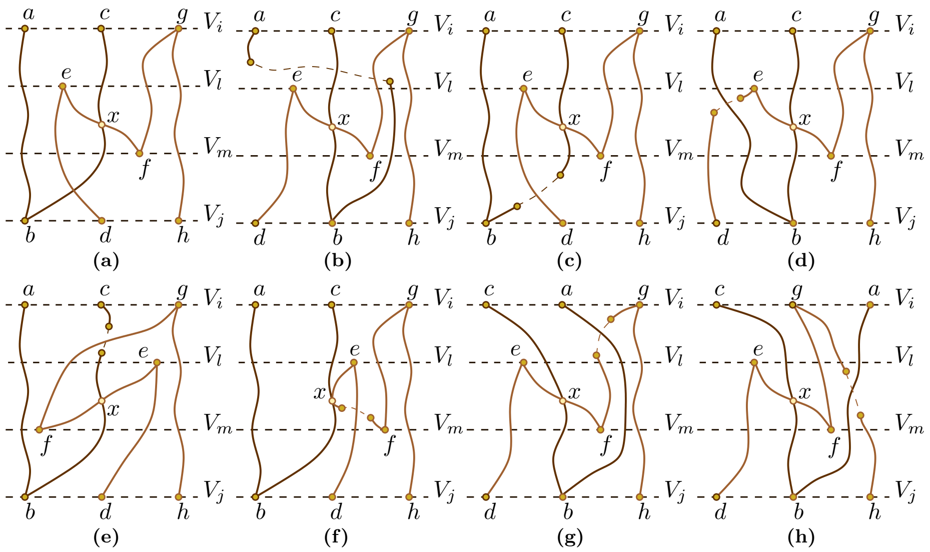

Fig. 5. The seven cases of deleting an edge from pattern P_3 in (a). The dashed curves represent the removed edges.

an edge; cf. Fig. 5. In each case crossings can be avoided by rearranging vertices on the tracks. Given that both MLNP trees patterns T1 and T2 in [4] have one vertex of degree 3, neither can match P_3 with a vertex of degree 4 or P_4 has two vertices of degree 3. Hence, all four MLNP patterns are distinct. □

The proof of Theorem 15 of Healy *et al.* [4] argues that every MLNP pattern must match some HLNP pattern. We show why this argument fails for P_3.

Lemma 4 *P_3 augmented to form a hierarchy has a subtree matching P_2.*

Proof. Fig. 6 shows the highlighted subtrees that match P_2 when P_3 is augmented either above or below to form a hierarchy. In each case, the additional path being added from the source is an essential part of the pattern P_2. Since

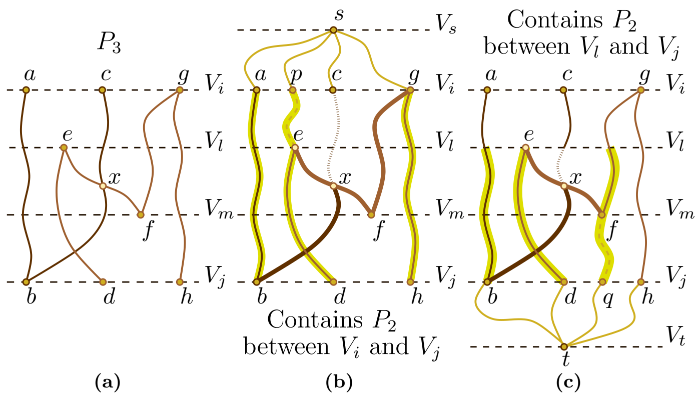

Fig. 6. P_3 in (a) is augmented from above in (b) and from below in (c) to form hierarchies with subtrees matching P_2 in both (b) and (c).

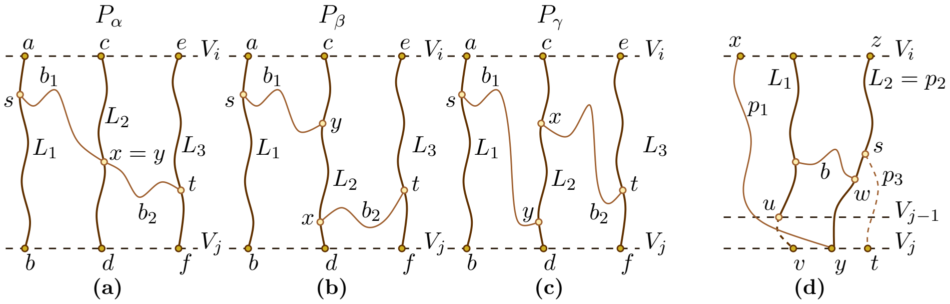

Fig. 7. The three minimal patterns (a)–(c) that must be part of any MLNP pattern for trees and a pattern (d) with at most two disjoint linking paths.

P_2 does not match P_3 by Theorem 3, P_2 is clearly being introduced by the augmentation in which it was not previously being present. □

The next lemma gives the minimal conditions for a MLNP tree pattern.

Lemma 5 *A level nonplanar tree $T(V, E, \phi)$ on k levels contains three disjoint paths $L_1, L_2, L_3 \in \mathcal{L}(i, j)$ linking levels V_i and V_j for $1 \leq i < j \leq k$ with bridges b_1 from L_1 to L_2 and b_2 from L_2 to L_3 with $x = b_1 \cap L_2$ and $y = b_2 \cap L_2$ so that either (P_α) $x = y$, (P_β) $L_2 = c \rightsquigarrow y \rightsquigarrow x \rightsquigarrow d$, or (P_γ) $L_2 = c \rightsquigarrow x \rightsquigarrow y \rightsquigarrow d$ hold where $c \in V_i$ and $d \in V_j$ as in Fig. 7(a), (b), (c).*

Proof. We observe that these conditions match P_A of Theorem 1 except for one bridge. By Lemma 5 of [4], P_A is the only HLNP pattern that can match a tree. Assume that P is an MLNP pattern between levels V_i and V_j in which $|i - j|$ is minimum and there are at most two disjoint paths $L_1, L_2 \in \mathcal{L}(i, j)$. There could be at most one bridge b joining L_1 and L_2 without forming a cycle. Let w be the endpoint of b in L_2.

Let P' be $P - (u, v)$ where (u, v) is the short edge connecting L_1 to V_j in which $v \in V_j$. In order for P to be MLNP, there must exist two linking paths $p_1, p_2 \in \mathcal{L}(i, j)$ in P' with endpoints $x, z \in V_i$ and common endpoint $y \in V_j$ such that for any level planar embedding of P', u is contained in the region bounded by p_1, p_2 and the track ℓ_i; cf. Fig. 7(d).

Assume w.l.o.g. that L_2 is p_2. In order for p_1 not to be embeddable on the other side of p_2 (allowing edge (u, v) to be drawn in P without crossing), there must be a path p_3 from s in L_2 to $t \in V_j$ in which s lies between z and w blocking this direction. Then there are at least three disjoint paths in P in $\mathcal{L}(i, j)$: p_1, L_1 and the path $z \rightsquigarrow s \rightsquigarrow t$, contradicting our assumption of there only being two.

Let $L_1, L_2, L_3 \in \mathcal{L}(i, j)$ be three disjoint paths. At least one of the three paths, say it is L_2, must be joined by bridges b_1 and b_2 to the other two paths L_1 or L_3, respectively, or P would be disconnected contradicting the minimality of P. If $b_1 \cap b_2$ form a nonempty path, then $b_1 \cup b_2$ would form a subtree homeomorphic to $K_{1,3}$, yielding pattern P_1 or P_2 of Theorem 3. Thus, b_1 and b_2 can share at most one vertex as in P_α of Fig. 7(a). Otherwise there must have been endpoints $x = b_1 \cup L_2$ and $y = b_2 \cup L_2$ along the path $c \rightsquigarrow d$ forming L_2 where either y proceeds x as in P_β of Fig. 7(b) or x proceeds y as in P_γ of Fig. 7(c). We observe that P_α matches P_3 and P_γ matches P_4. □

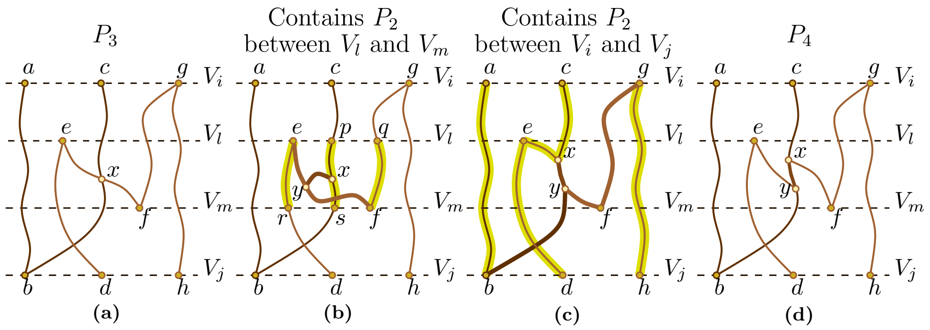

Fig. 8. The three ways of splitting the degree-4 vertex of P_3 into two vertices of degree 3, the last of which yields P_4. The other two match P_2.

We next show that P_4 is easily derived from P_3 by considering the ways in which the degree-4 vertex of P_3 can be split.

Lemma 6 *P_4 is the only distinct MLNP pattern for trees that can be formed from P_3 (by splitting the degree-4 vertex) not containing a subtree matching P_2.*

Proof. Fig. 8 shows the three ways the degree-4 vertex of P_3 can be split into two degree-3 vertices. Two contain subtrees that match P_2. □

Finally we complete our characterization for level nonplanar trees.

Theorem 7 *A level tree T is level nonplanar if and only if T has a subtree matching one of the minimum level nonplanar patterns P_1, P_2, P_3, or P_4.*

Proof. Once a MLNP pattern P is augmented to form a hierarchy, one of the HLNP patterns must apply. Since this augmentation does not introduce a cycle between levels V_i and V_j, either pattern P_1 or P_2 must match a subtree of the augmented pattern by Lemma 5 of [4].

Assume there is a MLNP tree pattern P containing P_α or P_γ (P_β is equivalent to P_γ under vertical reflection) of Lemma 5 that does not match P_1 or P_2. For P_α there are two cases: (i) $x \in V_i$ or $x \in V_j$ or (ii) $x \notin V_i$ and $x \notin V_j$. Assume w.l.o.g. that $b_1 \cap L_1 \in V_j$ for both (otherwise P is not minimal since the portion of L_1 to $L_1 \cup b_1$ is extraneous) that $x \in V_i$ for (i), and that $b_2 \cap L_2 \in V_i$ for (ii). Similarly, for P_γ there are three cases: (i) $x \in V_i$ and $y \in V_j$, (ii) $x \in V_i$ and $y \notin V_j$, and (iii) $x \notin V_i$ and $y \in V_j$, Assume w.l.o.g. that $b_1 \cap L_1 \in V_j$ for (ii) and and that $b_2 \cap L_3 \in V_i$ for (iii). We augment P to form a hierarchy to illustrate how either P must match P_1 or P_2 or contain a cycle preventing it from matching a tree.

Suppose that a bridge of P_α or P_γ in P is not strictly y-monotone. Then P could either have a bend at e in level V_l in one bridge or a bend at f in level V_m in the other as in Fig. 9(a) for some $i < l < m < j$. Each bend would require augmentation to a path from the source when forming a hierarchy from above or below as was the case with P_3 in Fig. 6.

We augment P with a path $p \rightsquigarrow e$ from V_i to V_l to form P', a hierarchy, that must match P_1 or P_2. We observe that between levels V_i and V_m, we have four linking paths. A third bridge $u \rightsquigarrow v$ must be present in P' that is part of a subtree S homeomorphic to $K_{1,3}$. Fig. 9(b) gives one such example. While P'

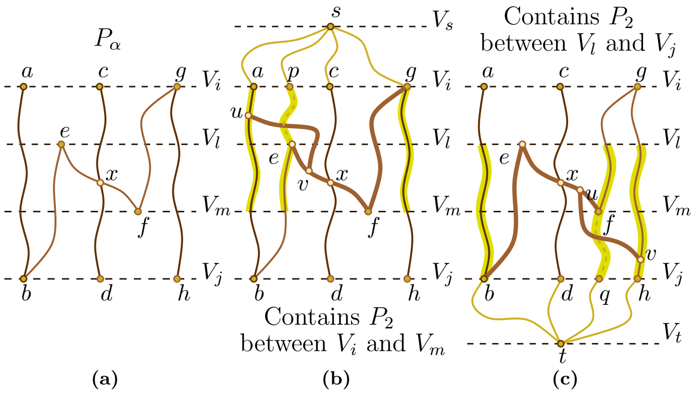

Fig. 9. Examples of pattern P_α in (a) being augmented to form a hierarchy in (b) and (c).

matches P_2 between levels V_i and V_m, we see that between levels V_i and V_j, P must have had the cycle $u \rightsquigarrow v \rightsquigarrow e \rightsquigarrow b \rightsquigarrow u$, contradicting P being a tree pattern. By inspection, any other placement of $u \rightsquigarrow v$ to connect three of the four linking paths to form P_1 or P_2 similarly implies a cycle in P.

Hence, P cannot contain any more edges than those of P_α without matching P_1 or P_2. We observe that P_α consists of two paths sharing a common vertex x. Given the minimality of P in minimizing $|i - j|$, one path has both endpoints in V_i with one vertex in V_j that can be split into linking paths $L_1, L_2 \in \mathcal{L}(i, j)$. Similarly, the other has both endpoints in V_j with one vertex in V_i that can also be split into the linking paths $L_3, L_4 \in \mathcal{L}(i, j)$. In P_3 of Fig. 9(a), L_1 is $a \rightsquigarrow b$, L_2 is $b \rightsquigarrow e \rightsquigarrow x \rightsquigarrow c$, L_3 is $d \rightsquigarrow x \rightsquigarrow f \rightsquigarrow g$, and L_4 is $g \rightsquigarrow h$.

For P to be level nonplanar, a crossing must be forced between these two paths. This is done by having L_2 or L_3 meet the condition of P_3 of three subpaths $C_1 \in \mathcal{L}(i, m)$ linking V_i to V_m, $C_2 \in \mathcal{L}(l, m)$ linking V_l to V_m, and $C_3 \in \mathcal{L}(l, j)$ linking V_l to V_j. This is not the case for P_α in Fig. 9(a) since the $x \rightsquigarrow c$ portion of L_2 does not reach level V_m, and the $x \rightsquigarrow d$ portion of L_3 does not reach level V_l. So for P not to match P_3, at least one subpath of both L_2 and L_3 from x to V_i or V_j must strictly monotonic as was the case in Fig. 9(a). However, in this case P can be drawn without crossings. This leaves P_3 as the only possibility of a MLNP pattern matching P_α that does not match P_1 or P_2. □

4 Conclusion and Future Work

The sufficiency argument of the MLNP patterns used by Healy *et al.* is flawed in its contention that all MLNP patterns contain a HLNP pattern. Given this flaw, there remains the very likely possibility of the characterization of Healy *et al.* omitting some MLNP patterns with cycles.

We provided two new MLNP patterns for trees and showed that the new set of four was sufficient. We presented a new approach for showing sufficiency based upon pattern augmentation to form HLNP patterns. However, our approach heavily relied on the underlying graph of the pattern forming a tree and avoiding cycles. For future work remains the open problem of finding the remaining set, if any, of MLNP patterns for graphs with cycles and proving they are sufficient to complete the characterization for all level planar graphs.

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Appendix

Characterization of patterns T1 and T2 from Healy *et al.* in Section 3.1 of [4]:

"Let i and j be the extreme levels of a pattern and let x denote a root vertex with degree 3 that is located on one of the levels $i, \ldots, j$. From the root vertex emerge 3 subtrees that have the following common properties (cf. Fig. 2 for illustrations of two typical patterns):

- each subtree has at least one vertex on both extreme levels;
- a subtree is either a chain or it has two branches which are chains;
- all the leaf vertices of the subtrees are located on the extreme levels, and if there is a leaf vertex v of a subtree S on an extreme level $l \in \{i, j\}$ then v is the only vertex of S on the extreme level l;
- those subtrees which are chains have one or more non-leaf vertices on the extreme level opposite to the level of their leaf vertices.

The location of the root vertex distinguishes the two characterizations.

(T1) The root vertex x is on an extreme level $l \in \{i, j\}$ (cf. Fig. 2(a)):
- at least one of the subtrees is a chain starting from x, going to the opposite extreme level of x and finishing on x's level;

(T2) The root vertex x is on one of the intermediate levels $l, i < l < j$ (cf. Fig. 2(b)):
- at least one of the subtrees is a chain that starts from the root vertex, goes to the extreme level i and finishes on the extreme level j;
  at least one of the subtrees is a chain that starts from the root vertex, goes to the extreme level j and finishes on the extreme level i."

Note that Fig. 2(a) and (b) of [4] correspond to our Figs. 4(a) and (b).

Next we state Theorem 2 and Lemmas 3, 4, and 5 of [4] with slight rewording to match our own terminology and previous theorems.

Theorem 8 (Healy *et al.* Theorem 2) *A subgraph matching either of the two tree characterizations $T1$ or $T2$ is MLNP.*

Lemma 9 (Healy *et al.* Lemma 3) *If HLNP pattern P_A of Theorem 1(a) matches a tree then each one of the paths L_1, L_2, L_3 contains only one vertex being the end vertex of a bridge.*

Lemma 10 (Healy *et al.* Lemma 4) *If HLNP pattern P_A of Theorem 1(a) matches a tree then its bridges must form a subgraph homeomorphic to $K_{1,3}$.*

Lemma 11 (Healy *et al.* Lemma 5) *The only HLNP pattern that can be matched to a tree is P_A of Theorem 1.*