Simultaneous Graph Embedding with Bends and Circular Arcs*

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Abstract. We consider the problem of simultaneous embedding of planar graphs. We demonstrate how to simultaneously embed a path and an n-level planar graph and how to use radial embeddings for curvilinear simultaneous embeddings of a path and an outerplanar graph. We also show how to use star-shaped levels to find 2-bends per path edge simultaneous embeddings of a path and an outerplanar graph. All embedding algorithms run in $O(n)$ time.

1 Introduction

Simultaneous embedding techniques are useful in the visualization of graphs that evolve through time, e.g., in the context of visualization of the evolution of software [6]. Simultaneous embedding of planar graphs is also motivated by its relationship with problems of graph thickness, geometric thickness, and contour tree simplification. Embedding trees and other classes of planar graphs on predetermined point-sets, small integer grids, and levels is motivated by graph layout algorithms and applications in visualizing hierarchical information. Level and radial embeddings can also be used for simultaneous embedding of graphs which are in turn useful in dynamic graph visualization. Simultaneous embedding of planar graphs is also motivated by its relationship with problems of graph and geometric thickness.

A *geometric simultaneous embedding* of two vertex-labeled planar graphs on n vertices in the xy-plane is possible if there exists a labeled point set of size n such that each of the graphs can be realized on that point set (using the vertex-point mapping defined by the labels) with straight-line edges without crossings. For example, any two paths can be simultaneously embedded, while there exist pairs of outerplanar graphs that do not have a simultaneous embedding [5]. Geometric simultaneous embeddings are quite restrictive: pairs of trees and triples of paths may not have such embeddings. Less restrictive versions allow for larger classes of graphs to be embedded without crossings, using few bends per edge [13].

Suppose an n-vertex path P is labeled 1 to n from one endpoint to the other. In this paper, we show how to simultaneously embed P with an n-vertex planar graph G (also labeled from 1 to n) that remains planar when the y-coordinate

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of each vertex of G equals its label. We can restrict each vertex of G to lie on the distinct horizontal line, or *level*, $\ell_j = \{(x, j) \mid x \in \mathbb{R}\}$ given by its label $j \in \{1, 2, \ldots, n\}$. Such graphs are called *level planar graphs* with respect to the labeling of G. The ability to simultaneously embed P and G in this way depends on the particular labeling of G. If G is not level planar for the given labeling, then we give alternative simultaneous embedding techniques provided that G is outerplanar: we retain the straight-line edges for G, but relax the edges of P to be either composed of one circular arc each or to have 2 bends per path edge.

1.1 Motivating Example

An example of a simultaneous embedding of a tree and an outerplanar graph for the 2002 Football World Cup is shown in Fig. 1. While the union of these two (underlying undirected) graphs contains a subdivided K_5, demonstrating it is non-planar, each individual graph can be drawn without crossings in which the edges of the second graph have bends in (a), but not in (b). This example illustrates that sometimes it is preferable to restrict the embedding to a predetermined point set. In this case, the point set of (a) lies on a set levels corresponding to the stage that a country reached before being eliminated. While (a) has two bends-per-edge in its second graph, and (b) does not, (a) is more readable since it preserves this relative vertical ranking of teams for the tree.

1.2 Related Work

The existence of plane (straight-line, crossings-free) drawings for a single planar graphs is well known [15, 25, 26]. Moreover, plane drawings for n-vertex planar graphs can be found in $O(n)$ time using $O(n^2)$ area with vertices placed at integer grid points as shown by de Fraysseix, Pach and Pollack [7] and Schnyder [24]. Brass *et al.* [5] describe linear time algorithms for geometric simultaneous embeddings of pairs of paths, cycles, and caterpillars on an $O(n) \times O(n)$ grid. Geyer *et al.* [17] show that tree-tree pairs do not always have a geometric simultaneous embedding, and the status of tree-path pairs is open. If bends on the edges are allowed, Erten and Kobourov [13] show that tree-path pairs can be embedded simultaneously using at most one bend per tree edge and no bends of path edges on an $O(n) \times O(n^2)$ grid. Moreover, pairs of planar graphs can be embedded simultaneously using at most 3 bends per edge using an $O(n^2) \times O(n^2)$ grid.

A related problem is that of level planarity, where the goal is to display graphs according to a given hierarchical ordering of the vertices. The class of graphs where this is possible are called level planar graphs, which have been widely investigated [8, 19]. In particular, Jünger and Leipert [20] present a linear time planarity embedding algorithm for level planar graphs using PQ-trees, where the resulting embedding is a set of linear orderings of vertices on each level. Once a graph is determined level planar, Eades *et al.* [12] can produce a plane drawing in $O(|V|)$ time, though it may require exponential area. In [14], a characterization of trees that are level planar for any possible labeling of the vertices is given. These trees are called *unlabeled level planar* (ULP).

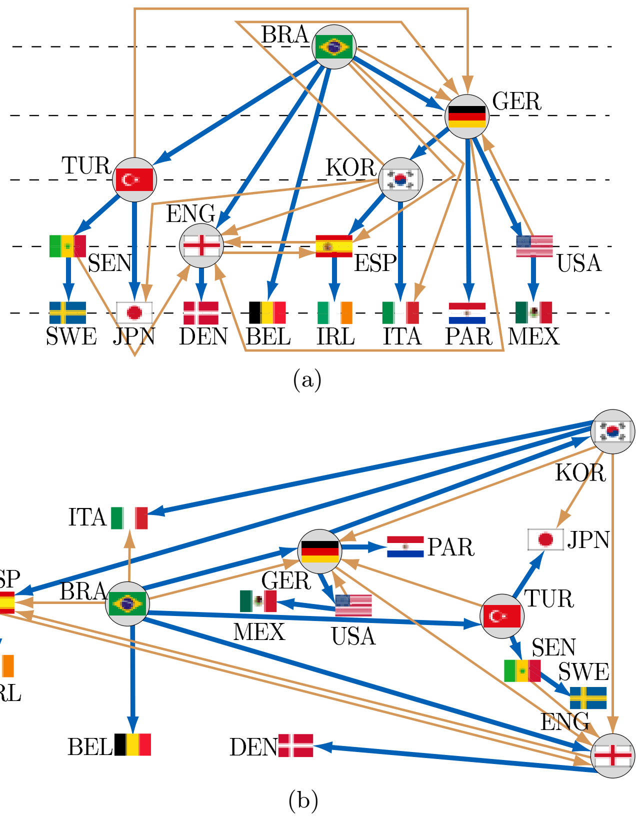


Fig. 1. In (a) the dark thick directed edges with no bends show the outcome of the 2002 Football World Cup in which edges are directed from the victorious to the defeated teams of the top 16 countries who reached the knockout stages. The light thin directed edges with bends depict the foreign nationalities for the eight quarter finalist teams in which an edge goes from originating country that exports one or more players to play in the league of another country during 2006. A geometric simultaneous embedding of (a) is given in (b). While all bends are eliminated in both graphs of (b), the relative vertical positioning of the teams indicating which stage they reached before being eliminated is lost. The five countries with light gray circles correspond to the five vertices of the subdivided K_5 contained in the union of the two (underlying undirected) graphs.

Track graphs are a generalization of level graphs that can have edges between vertices on the same level. Bachmaier *et al.* [1] provide a reduction from *track graphs* on k levels to level graphs on $3k$ levels preserving planarity. Analogous to level graphs and track graphs, *radial level graphs* and *circle graphs* use concentric circles instead of horizontal lines for levels. Bachmaier *et al.* [2] extend the linear time testing and embedding algorithms to radial level graphs, and provide a similar reduction from circle tracks on k levels to radial level graphs on $3k$ levels, provided that the edges are composed of radially monotone polylines.

Related to simultaneous embedding is the problem of *graph thickness*—the minimum number of planar subgraphs into which the edges of the graph can be partitioned; see survey by Mutzel *et al.* [22]. Dillencourt *et al.* [9] study *geometric thickness* of graphs, where the edges are required to be straight-line segments, which directly relates: If two graphs have a geometric simultaneous embedding, then their union has geometric thickness two just as the union of any two planar graphs has graph thickness two. Duncan *et al.* [11] use geometric simultaneous embedding techniques to show that degree-four graphs have geometric thickness two. *Book thickness* adds the further restriction that the vertices must be in convex position [3].

Simultaneous drawing of multiple graphs is also related to the problem of *fixed point-set embedding* of planar graphs. Gritzman *et al.* [18] and Bose [4] show that if the mapping between the vertices V and the points P is not fixed, then trees and outerplanar graphs can be drawn without crossings using straight-line edges, which is not the case for planar graphs in general. Bose [4] shows that if the mapping between the vertices V and the points P is not fixed, then outerplanar graphs can be drawn without crossings using straight-line edges, which is not the case for planar graphs in general. If bends are allowed, Kaufmann and Wiese [21] show that two bends per edge suffice for a planar graph. However, if the mapping between V and P is predetermined, Pach and Wenger [23] show that $O(n)$ bends per edge are necessary to guarantee planarity, where $n = |V|$. Felsner *et al.* [16] characterize the trees that can be drawn on the $n \times 2$ grid and describe a universal point-set for outerplanar graphs in 3D. In the context of 3D layout, Dujmovic *et al.* [10] study the (k,t)-*track layouts* of graphs, where the graph is vertex t-colored and edge k-colored. They examine the relationship between such layouts and geometric thickness.

1.3 Our Contributions

We present results about simultaneous embeddings of pairs of graphs where we allow the use of curvilinear edges and piecewise-linear edges[1] It is straightforward to simultaneously embed a pair of n-vertex planar graphs without crossings, where one has straight-line edges and the other has $O(n)$ bends per edge, using the result by Pach and Wenger [23]. With level and radial embeddings of graphs, we improve on this result to obtain a planar simultaneous embedding of an n-vertex planar graph G with no edge bends and an n-vertex path P, which may have circular arcs or bends depending on the nature of G. Our results illustrate the following trade-offs between the class of graphs to which G can belong and the type of edges used for P:

1. *Simultaneous embedding of a level planar graph G w.r.t. the labeling of P.* If G is level planar, then we show how both G and P can be simultaneously embedded with straight-line edges in $O(n)$ time.

[1] The generic term *simultaneous embedding* refers to geometric simultaneous embedding unless otherwise qualified. An example of such a qualifier would be "simultaneous embedding with bends" in which the restriction of straight-line edges is relaxed to edges with bends for one or both graphs. Regardless of the qualifier, the restriction of realizing each graph without crossings on the same point set is *never* relaxed.

Pair (P, G)	Edges in P	Edges in G	Condition	Reference
(Path, Path)	0 bends	0 bends	none	[5]
(Path, Tree)	0 bends	1 bend	none	[13]
(Path, Planar)	0 bends	0 bends	G is ULP	Theorem 1
(Path, Planar)	0 bends	0 bends	G level planar w.r.t. P	Theorem 2
(Path, Outerplanar)	circular arcs	0 bends	none	Theorem 3
(Path, Outerplanar)	2 bends	0 bends	none	Theorem 8
(Path, Tree)	0 bends	0 bends	none	Open

Table 1. Summary of results. ULP here stands for unlabeled level planar, as defined above.

2. *Simultaneous embedding of an outerplanar graph G and a path P with circular arcs.*
   If G is outerplanar, we show how to find a plane drawing for G simultaneously with a drawing for P that uses one circular arc per edge in $O(n)$ time.
3. *Simultaneous embedding of an outerplanar graph G and a path P with bends.*
   If G is outerplanar and piecewise-linear edges are desirable, we show how to obtain a plane drawing for G simultaneously with a drawing for P that uses two bends per edge in $O(n)$ time.

Table 1 summarizes our current results regarding simultaneous embedding of pairs of planar graphs and relevant previous results.

2 Geometric Simultaneous Embedding

We try to use standard notation when discussing level graphs while focusing on the aspects of level graphs that give us simultaneous embeddings using straight-line segments. In doing so, we omit from our definitions certain properties of level graphs that are not directly relevant to our problem domain. For instance, level graphs can, and usually do, have fewer than n levels, and generally, are in terms of *short* directed edges, which are restricted to be only between vertices on adjacent levels. However, we are not interested in such level graphs since for our application we require n levels, direction is immaterial, and the only graph on n levels with short edges is a y-monotone path.

Taking all this into consideration, the following definitions are akin to those given in [19]. Let $G(V, E)$ be an n-vertex undirected graph with a labeling $\mathcal{L} : V \xrightarrow[\text{onto}]{1:1} \{1, 2, \ldots, n\}$, which induces a *level graph* $G(V, E, \phi)$ with a bijective *level assignment* $\phi = \mathcal{L}$ onto n levels. A vertex v in V is a j-level vertex if $\phi(v) = j$. A *level drawing* is *level planar* if it admits a plane drawing such that each j-level vertex v can be embedded onto the horizontal line $\ell_j = \big\{(x, j) \,|\, x \in \mathbb{R}\big\}$. A level graph is level planar if it has a level drawing.

Any planar graph G admits some labeling for which it is level planar, which can be obtained via the y-ordering of any plane drawing of G perturbed so that no two vertices have the same y-coordinate. For instance, the graph in Fig. 2(a), (b) is level planar for the given labeling. However, only some planar graphs have the property of being level planar regardless of the labeling used. Such graphs are called *unlabeled level planar* (ULP) [14] and they can be characterized in terms of

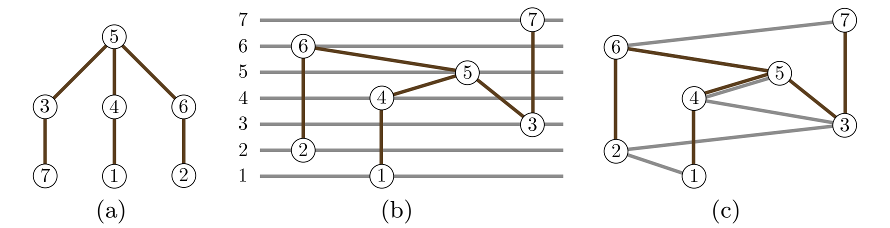


Fig. 2. The tree on the left (a) is embedded on parallel levels (b) and then simultaneously embedded with path $P = \{1, 2, 3, 4, 5, 6, 7\}$ in (c).

a pair of forbidden subtrees; see Fig. 3. This results in the only ULP trees being (i) *caterpillars*, trees such that the removal all degree-1 vertices results in a path, (ii) *radius-2 stars*, a $K_{1,k}$ in which each edge is subdivided at most once, and (iii) *degree-3 spiders*, an arbitrarily subdivided $K_{1,3}$. Given this characterization, the linear-time recognition and embedding algorithms for ULP trees yield a straightforward way to simultaneously embed an n-vertex path and an n-vertex ULP tree as illustrated in the following theorem.

Theorem 1 *A geometric simultaneous embedding of an n-vertex graph G and an n-vertex path P can be computed in $O(n)$ time, provided G is ULP.*

Proof. Label the vertices of P sequentially 1 to n, and label the vertices of G so that $\mathcal{L} = \phi$ for any n-level bijective assignment ϕ of the vertices in G and P. If G happens to be a ULP tree, it is either a caterpillar, radius-2 star, or degree-3 spider, each of which has an $O(n)$ time algorithm [14] to produce a compact straight-line level planar drawing of G. If G is not a tree, the $O(n)$ time algorithm of Eades *et al.* [12] can provide a planar straight-line drawing of G with level assignment $\phi = \mathcal{L}^2$. Regardless, the y-coordinate of each j-vertex of G matches its label j in a level drawing of G. Then we draw the path P in a y-monotone fashion zig-zagging upward from one level to the next in $O(n)$ time; see Fig. 2(c). This completes our geometric $O(n)$ time simultaneous embedding of P and G since no path edges of P can cross given its y-monotone nature. □

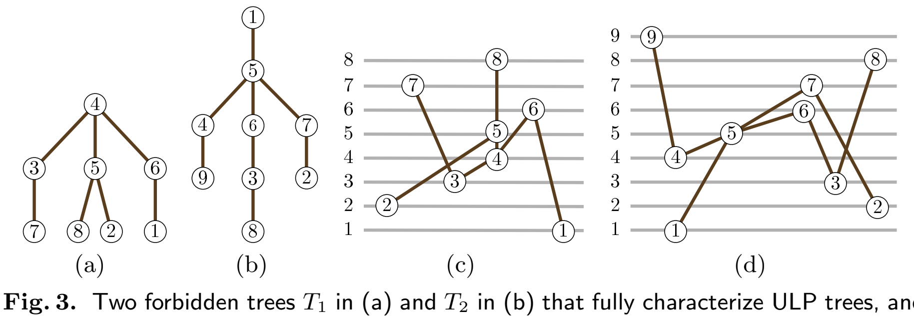


Fig. 3. Two forbidden trees T_1 in (a) and T_2 in (b) that fully characterize ULP trees, and their embeddings with crossings in (c) and (d), respectively, for the given labelings.

[2] It should be noted that the distribution of the vertices of G in the drawing produced by Eades *et al.* can vary depending on the order in which they are picked in the algorithm, which is why it is preferable to use the custom embeddings provided in [14] if G happens to be a tree.

The requirement for G to be ULP is overly restrictive. One can use the same approach to simultaneously embed a planar graph G and a path P, provided that G is level planar with respect to the labeling induced by P. This gives the following theorem.

Theorem 2 *A geometric simultaneous embedding of an n-vertex graph G and an n-vertex path P can be computed in $O(n)$ time, provided G is level planar with respect to the labeling given by P.*

The disadvantage of this approach to simultaneously embed a path and a planar graph is that most planar graphs, including most trees, are not level planar for some labeling. This is not surprising since it is a strong restriction to have a predetermined order of the y-coordinates of the vertices. What is surprising, however, is that introducing curvature to the levels, by using concentric circles in lieu of horizontal lines, is enough to allow us to embed all trees and outerplanar graphs with circular arcs for path edges. We show this in the next section.

3 Simultaneous Embedding with Curves

This section combines straight-line embeddings of outerplanar graphs with paths consisting of circular arcs to produce curvilinear simultaneous embeddings. First, we describe how to obtain a plane drawing of an n-vertex outerplanar graph G on a set of concentric circles such that each vertex lies on a distinct circle, determined by the labeling of G. We then use this straight-line crossings-free drawing of G to simultaneously embed G with an n-vertex path P, such that each path edge consists of a circular arc that lies between adjacent concentric circles. This will give the primary theorem of this section.

Theorem 3 *An n-vertex outerplanar graph G can be simultaneously embedded with an n-vertex path P in $O(n)$ time such that G forms a plane drawing and P is drawn without crossings using one circular arc per edge in which P forms a radially monotone polyline from some center point o.*

However, this method only works if we can fix the radii of the circles to be relatively close to each other. We conclude this section by showing that it is not always possible to even draw a tree with straight-line edges without crossings on a set of concentric circles in which the radii of the circles are uniformly distributed.

3.1 Embedding an Outerplanar Graph on Concentric Circles

In this section, we describe how to embed a radial plane drawing of an labeled n-vertex outerplanar graph G on a set of distinct n concentric circles using the labeling of G. This is similar to a radial level planar embedding on n radial levels, i.e., circles, except that we are using straight lines instead of radially monotone polylines for edges. However, as straight-line edges are not necessarily radially monotone, the radial level planarity test and embeddings algorithms of Bachmaier *et al.* [2] cannot be directly applied here. Rather, we use the following result from Bose [4].

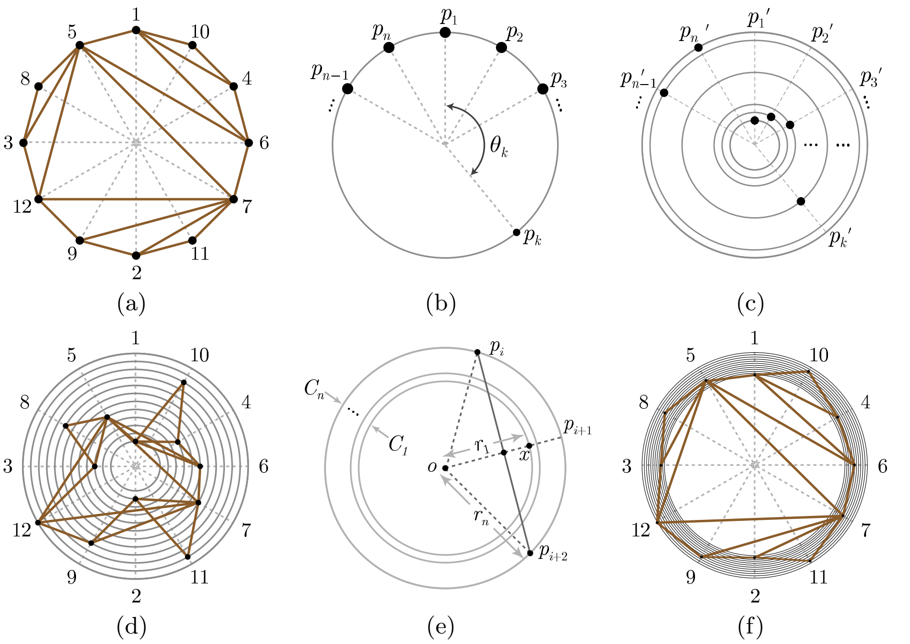


Fig. 4. The vertices of a 12-vertex outerplanar graph G are embedded on a circle in (a). This embedding follows the general point set $\mathcal{P}$ given in (b). The points are then perturbed radially inward so that each vertex with label j lies on C_j for $j = 1, 2, \ldots, 12$, where C_{12} is the outermost circle, yielding point set $\mathcal{P}'$ in (c). Drawing G on $\mathcal{P}'$ gives (d). Crossings are avoided by restricting the ratio between the radii of the innermost circle C_1 to the outermost circle C_n in (e) giving (f).

Theorem 4 *(**Theorem 6.2** of [4]) If the input point set $\mathcal{P}$ is in convex position then $O(n)$ time and space is sufficient to straight-line embed G into $\mathcal{P}$.*

Using this theorem we can obtain our linear time radial straight-line embedding of G onto n concentric circles given by the next lemma.

Lemma 5 *Given a set $\mathcal{C}$ of n concentric circles $\{C_1, C_2, \ldots, C_n\}$ centered at the origin o with monotonically increasing radii $r_1, r_2, \ldots, r_n$, it is a sufficient condition that $\frac{r_1}{r_n} > \cos\frac{2\pi}{n}$ in order to obtain in $O(n)$ time a radial plane drawing of an n-vertex outerplanar graph G with vertices labeled 1 to n such that each vertex with label j is embedded on circle C_j.*

Proof Sketch: Our strategy is to first embed G using straight-line edges onto a circle C with radius r centered at the origin o without crossings in $O(n)$ time via Theorem 4. We want the vertices of G evenly distributed along the circle, i.e., on a point set $\mathcal{P}$ such that there is a point p_i in $\mathcal{P}$ at a distance r from the origin o, i.e., $|\overline{op_i}| = r$, at every radian angle $\theta_k = (k-1)\frac{2\pi}{n}$, where θ_k is the angle $\angle p_1 o p_k$ for $k = 1, 2, \ldots, n$. Clearly, $\mathcal{P}$ forms a convex set; see Fig. 4(b).

Then we perturb the vertices in $O(n)$ time in a radial direction so that each one lies on its own circle according to the labeling of G, i.e., a vertex v labeled j is placed on C_j where $C_n = C$. Finally, we determine that if our perturbation is sufficiently small, i.e., $\frac{r_1}{r_n} > \cos\frac{2\pi}{n}$, then the radial drawing remains free of crossings. We do this by perturbing the point set $\mathcal{P}$ to match the new locations of the vertices of G. We call this perturbed point set $\mathcal{P}'$; see Fig. 4(c).

We note that while Theorem 4 works for any convex point set $\mathcal{P}$, which vertex is embedded at which point of $\mathcal{P}$ is determined by the algorithm. We can show that when rerunning the algorithm on point set $\mathcal{P}'$ instead of $\mathcal{P}$, it makes the same choices. We omit including a description of this algorithm in this abstract, except to note that it works by recursively subdividing $\mathcal{P}$ into smaller point sets by recursively picking a pair of points and taking the line passing through them as a dividing line into which to split the remaining points. In order to do that it suffices to show that when we perturb the point set $\mathcal{P}$ to $\mathcal{P}'$ the following two conditions hold:

– all the points of $\mathcal{P}'$ remain in convex position *and*
– the order in which each point p' of $\mathcal{P}'$ sees all the other points $\mathcal{P}' - p'$ using a radial line sweep centered at p' remains the same.

Given these conditions and the greedy nature of the algorithm of Theorem 4, if a vertex v of G is embedded on point $p_i \in \mathcal{P}$, then v will also be embedded onto the corresponding point $p'_i \in \mathcal{P}'$.

Since the points $\mathcal{P}$ are uniformly distributed over all radial angles, retaining the convexity of $\mathcal{P}'$ also achieves the condition of retaining relative positioning provided that the points of $\mathcal{P}'$ are only perturbed in a radial direction. This is because the order in which a clockwise radial line sweep centered at point p_i in $\mathcal{P}$ encounters the other points will be $p_{i+1}, \ldots, p_n, p_1, \ldots, p_{i-1}$, which is the order that the vertices $\mathcal{P}$ are encountered along the convex hull of $\mathcal{P}$ after p_i. Maintaining the convexity of $\mathcal{P}'$ so that its convex hull is $p'_1, p'_2, \ldots, p'_n$ means than a radial line sweep centered at point p'_i in $\mathcal{P}'$ encounters the other points in the same order as p_i did in $\mathcal{P}$, namely $p'_{i+1}, \ldots, p_n, p_1, \ldots, p_{i-1}$.

Next, we show that $\frac{r_1}{r_n} > \cos\frac{2\pi}{n}$ is a sufficient condition in maintaining the convexity and same convex hull of $\mathcal{P}$ when perturbing the points to $\mathcal{P}'$. Perturbing the point p_{i+1} of $\mathcal{P}$ to lie on the innermost circle C_1, while letting its neighboring points p_i and p_{i+2} along the convex hull of $\mathcal{P}$, remain on $C = C_n$, gives a worst case in terms of affecting the convexity of $\mathcal{P}$. Fig. 4(e) illustrates this. Let x denote the midpoint of $\overline{p_i p_{i+2}}$. In order for p_{i+1} to remain on the convex hull of $\mathcal{P}$, it is sufficient that the distance from o to x is less than r_1, the radius of the innermost circle C_1. Since the angle $\angle xop_i$ is $\frac{2\pi}{n}$, if the ratio of $\frac{r_1}{r_n} > \frac{|\overline{ox}|}{r_n} = cos\frac{2\pi}{n}$, then p_{i+1} will lie in the outer half-plane formed by the line passing through p_i and p_{i+2}, i.e., the half-plane not containing the origin o. $\square$

3.2 Embedding a Path of Circular Arcs Between Concentric Circles

From Lemma 5, we have that given n distinct concentric circles $C_1, C_2, \ldots, C_n$ of monotonically increasing radii $r_1, r_2, \ldots, r_n$, we can create a plane drawing of any n-vertex outerplanar graph $G(V, E)$ with labeling $\mathcal{L} : V \xrightarrow[\text{onto}]{1:1} \{1, 2, \ldots, n\}$ such that $v \in C_{\mathcal{L}(v)}$ for all $v \in V$ provided $\frac{r_1}{r_n} > \cos\frac{2\pi}{n}$. Here we assume the

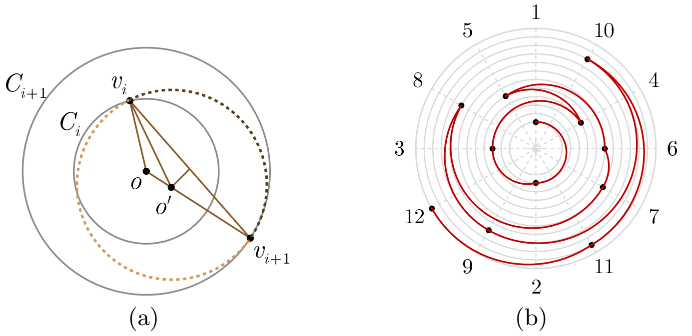


Fig. 5. Routing one circular arc per edge so that fits inside two consecutive concentric circles is in (a). The concentric circles are centered at o where o' is the center of a circle that gives a curve connecting v_i and v_{i+1} that stays within the annulus defined by C_i and C_{i+1}. An example of this is given in (b) for the outerplanar graph from Fig. 4(a).

n-vertex path P is labeled sequentially 1 to n. We show how to route the edges of the path using exactly one circular arc per path edge so that the arcs of P form a radially monotone polyline, which implies that no two circular arcs intersect.

Lemma 6 *A radially monotonically increasing crossings-free drawing of an n-vertex path $P(V, E)$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \big\{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)\big\}$ can be realized on n concentric circles $C_1, C_2, \ldots, C_n$, where $v_i \in C_i$ for $1, 2, \ldots, n$ with one circular arc per edge.*

Proof. It suffices to show that one circular arc always can be used to connect two consecutive vertices on the path, v_i and v_{i+1}, such that the arc lies strictly outside circle C_i and inside circle C_{i+1} except for the end points of the arc at v_i and v_{i+1}; see Fig. 5(a).

Let o be center of the circles. We compute o', the center of the circle that forms the desired circular arc connecting v_i and v_{i+1} as follows. The center o' is the intersection of the perpendicular bisector of $\overline{v_i v_{i+1}}$ and the line segment $\overline{o v_{i+1}}$. The radius of the circle centered at o' is given by the distance from o' to v_i. The shorter circular arc between v_i and v_{i+1} connects the two vertices and is located in the annulus between circles C_i and C_{i+1}. Furthermore, the distance from c to any point along the arc from v_i to v_{i+1} is monotonically increasing. Therefore, the entire path P can be realized as a radially monotone polyline, implying no edge crossings, using one circular arc per edge; see Fig. 5(b). □

Lemma 5 together with Lemma 6 gives us Theorem 3. However, as noted, this only works when we can restrict the radii of the concentric circles to be in a small range. One might wonder whether it is possible to use radially uniform concentric circles instead. The next subsection shows this is not the case.

3.3 Trees on Radially Uniform Concentric Circles

In this section we give an example of a 406-vertex tree with a labeling from 1 to 406 that cannot be straight-line embedded on a set of 406 radially uniform concentric circles such that each vertex lies on its respective circle.

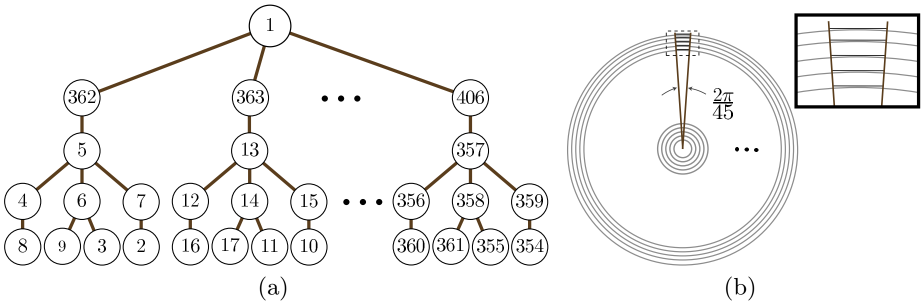


Fig. 6. The 406-vertex tree in (a) cannot be drawn on radially uniform concentric circles since there must exist one subtree that is a copy of T_1 from Fig. 3(a) that fully resides in a sector such that tangents of circles do not intersect any other circle. We can rotate this sector so that it lies directly above o so that any vertices placed on the concentric circular arc in this sector must have strictly increasing y-coordinates as shown in (b).

Lemma 7 *There exists an* 406*-vertex tree* $T(V, E)$ *with labeling* $\mathcal{L} : V \xrightarrow[onto]{1:1} \{1, 2, \ldots 406\}$ *that cannot be straight-line embedded on a set* $\mathcal{C}$ *of* 406 *radially uniform concentric circles* $\{C_1, C_2, \ldots, C_n\}$ *centered at* o *with radii* $r_1, r_2, \ldots, r_{406}$ *such that* $r_i = (i-1)\Delta$ *for* $i = 1, 2, \ldots, 406$ *for any* $\Delta > 0$*, where each vertex with label* j *is embedded on circle* C_j*.*

Proof. Here we use the ULP forbidden tree T_1 with 8 vertices from Fig. 3(a) to construct a 406 vertex tree T with root x which has 45 subtrees of 9 vertices each; see Fig. 6(a). Each of the 45 neighbors of x is a degree-2 vertex connected to a copy of T_1. We start by labeling x with 1 placing it on C_1, which has radius 0, so x must be embedded at the center o.

Then we label each of its 45 neighbors with $362, 363, \ldots, 406$ so that at least one subtree, which is a copy of T_1, call it T_1', must lie within the radian angle $\frac{2\pi}{45}$. W.l.o.g we assume that this sector is centered along a vertical line passing through the center o since we can rotate the drawing of T as needed.

Within this narrow sector, we observe that the tangents to the circles do no intersect any other circle; see Fig. 6(b). This is because the radius of r_{i-1} is strictly less than $r_i \cos \frac{2\pi}{2 \cdot 45} = r_i \cos \frac{\pi}{45}$ for $i = 1, 2, \ldots, 405$. In particular, $r_{405} = 404 < r_{406} \cos \frac{\pi}{45} = 405 \cdot 0.997564 = 404.013$.

Then we label the k^{th} copy of T_1 with the labels from Fig. 3(c) adding the value of $(k-1)8+1$ to the labels for $k = 1, 2, \ldots, 45$. This preserves the y-ordering of the labels such that T_1' (the copy of T_1 lying strictly within the radian angle $\frac{2\pi}{45}$ sector) must have strictly increasing y-coordinates. Hence, if T_1' could be embedded on its circles, then it could be level planarly embedded, which is not the case for the given labeling of T_1'. The inability to level planarly embed T_1' forbids a straight-line embedding of T. □

4 Simultaneous Embedding with Bends

This section combines straight-line embeddings of outerplanar graphs with paths whose edges are drawn with bends. We introduce the notion of *star-shaped levels* which allows us to obtain the main result of this section:

Theorem 8 *An n-vertex outerplanar graph G can be simultaneously embedded with an n-vertex path P in $O(n)$ time such that G forms a plane drawing and P is drawn without crossings with at most 2 bends per path edge.*

First, we describe how to project this 3-dimensional embedding of G onto the 2-dimensional xy-plane. Next we provide an example to illustrate how to simultaneously embed G with a path P of n vertices using as most 2 bends per path edge by carefully navigating star-shaped levels. Once the general idea of how these levels work is understood, we finish this section by giving the detailed geometric construction of the star-shaped levels that insures no crosses.

4.1 Projecting a 3D Outerplanar Graph onto the 2D xy-plane

We start by using the following theorem to get a 3-dimensional embedding of an outerplanar graph onto a 3-side prism.

Theorem 9 *(Felsner, Liotta and Wismath [16]) Every outerplanar graph $G(V, E)$ with n vertices admits a crossings-free straight-line grid drawing in three dimensions in optimal $O(n)$ volume that can be computed in $O(n)$ time and with the vertices of G drawn on the grid points of a prism.*

The ability to do this projection can be used to give our next lemma.

Lemma 10 *There exists a projection of a 3-dimensional outerplanar graph G on a 3-sided regular prism onto the xy-plane that preserves the number of straight-line edge crossings of G.*

Proof Sketch: The embedding of [16] uses the shortest-path distance from some arbitrary root vertex r to every other vertex in the outerplanar graph G. Let $d_{\text{mod }3}(v)$ denote the shortest-path distance from r modulo 3. Fig. 7 gives an example of an outerplanar graph to be used in illustrating the embedding on star-shaped levels. The 3-dimensional regular prism used for this embedding can be visualized as standing vertically on a triangular base in the xy-plane in which the vertical edges are numbered 0, 1, and 2 in clockwise order looking down from the positive z-direction. Then G is "wrapped" around the prism such that each vertex v is embedded along the edge $d_{\text{mod }3}(v)$ of the prism; see Fig. 8.

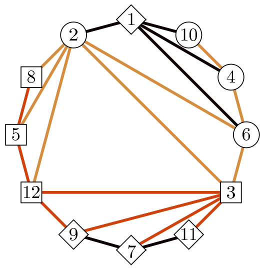
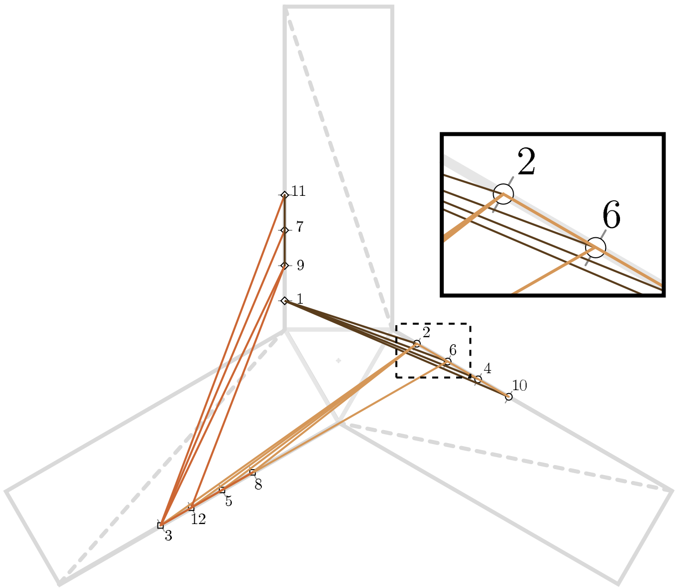
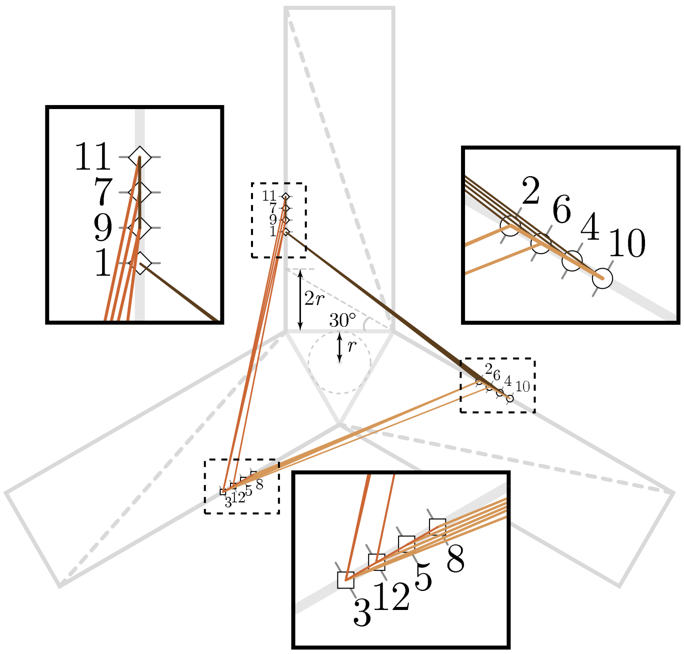


Fig. 7. Outerplanar graph G used as an example for embedding on star-shaped levels. The vertices in this figure and Fig. 8 are in three different shapes (diamonds, circles, and squares) according to whether the vertex is at a shortest-path distance of 0, 1, or 2 modulo 3 from vertex 1, respectively. The edges are colored one of three colors in a similar fashion according to the shortest-path distance of the closest endpoint to vertex 1.



(a)



(b)

Fig. 8. Outerplanar graph G from Fig. 7 projected onto the xy-plane from its 3-dimensional embedding. In (a) there are crossings since the vertices are not above the $2r$ threshold, where r is radius of the circumscribed circle of the base of the prism. In (b) there are no crossings since all the vertices achieve this threshold.

We will use the prism to construct the star-shaped levels. Assume that the base of the prism is an equilateral triangle with side length ℓ and that the height of the prism is 3ℓ. Additionally, let $r = \ell/2\sqrt{3}$ be the radius of the circle that is inscribed within the triangle; see Fig. 8(b). We need to shift all the vertices so that they all lie along a fairly narrow band above the distance $2r$ from the base. This is so that when we "unfold" each vertical side of the prism by laying it flat on the xy-plane we do not introduce a crossing; compare Fig. 8(a) to (b).

Since each vertical edge corresponds to two sides of two distinct rectangles, we pick the side of the rectangular face that is in a 90° counter-clockwise direction from the prism's base onto which to map the vertices along the prism edge. The positions of the vertices are then mapped directly to their corresponding positions along this projected edge in the xy-plane so that the edges on a prism face map directly to straight-line segments in the xy-plane.

While this unfolding operation is a linear transformation, there is the possibility that extra crossings will be introduced between edges incident to the same pair of prism edges in which one or both of the endpoints lie within a distance of $2r$ from this base. Once all the endpoints lie above this threshold, which is the point at which an *extension* of an adjacent prism edge at an angle of 120° would intersect the edge in the xy-plane, then no crossings can occur. This is illustrated by projecting the outerplanar graph G from Fig. 7 onto the xy-plane in which all the vertices are not above this threshold in Fig. 8(a), but achieve this $2r$ threshold in Fig. 8(b) eliminating all crossings. □

4.2 Simultaneous Embedding using Star-Shaped Levels

In this section we show how to use the star-shaped levels generated by the outerplanar graph G from the previous section to simultaneously embed a path P with exactly 2 bends per edge giving our next lemma.

Lemma 11 *There exists a 2-bend per path edge crossings-free drawing of a path P using star-shaped levels for any vertex labeling.*

Proof Sketch: Let $V_i = \{v \in V \mid d_{\text{mod } 3}(v) = i\}$ be the partitioning of V by prism edges, and let $n_i = |V_i|$ for $i \in \{0, 1, 2\}$ be the total number of vertices of G at a shortest-path distance of i modulo 3 from r where $n_{\max} = \max\{n_0, n_1, n_2\}$ is then the total number of star-shaped levels required for this simultaneous embedding. Next we need to perturb the n_i vertices that lie along edge i (after performing the above projection to the xy-plane) to lie along one of $n_{\max}$ closely adjacent nested star-shaped levels. How close these levels need to be is given by the subsequent lemma. Each star-shaped level has 6 sides and is a scaled-down version of the outermost level (w.r.t. the center of the circle circumscribed within the triangular prism base). This includes the 3 *prism edges* projected onto the xy-plane, and the 3 edges that each connect the top of one projected prism edge to the bottom of the next in a clockwise direction, termed *connecting edges*; see the dashed 3-pointed star in Fig. 9.

When perturbing the vertices, we are careful to move a vertex in a direction perpendicular to the prism edge (as well as all of the adjacent edges of the nested levels) on which it resided. The problem with perturbing the vertices too much

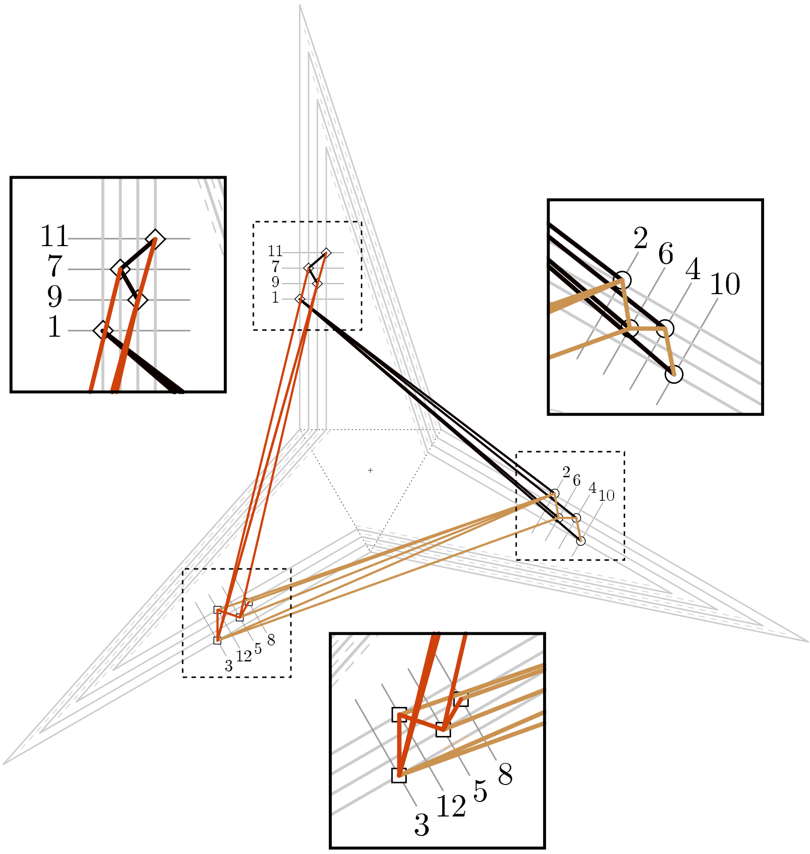


Fig. 9. Outerplanar graph G from Fig. 7 embedded onto the star-shaped levels.

is the introduction of crossings in G. For example, the outerplanar graph G in Fig. 7 is shown in Fig. 9 on a set of star-shaped levels in which the levels are not spaced sufficiently close enough, resulting in several crossings.

Unfortunately, it is nearly impossible to directly illustrate a true-to-scale non-trivial example of this perturbation without introducing crossings in the final simultaneous embedding since the required perturbation is so small that when drawn on the printed page, all the nested star-shaped levels appear to coincide. For instance, the path shown in Fig. 10 will induce crossings when simultaneously embedd with G from Fig. 7.

The vertices are placed clockwise from outermost to innermost star-shaped level by the order given by the path labeling. However, in order to be able to route the path back and forth between vertices that alternate back and forth between adjacent sides of the prism, an extra connecting edge needs to be inserted to lie half-way between every pair of adjacent connecting edges. Since we need one for *each* level, an extra connecting edge also needs to be added to lie just interior to the ${n_{\max}}^{\text{th}}$ innermost connecting edge. In Fig. 9 these are shown as dashed line segments. We denote the levels along which the vertices lie as *regular levels*,

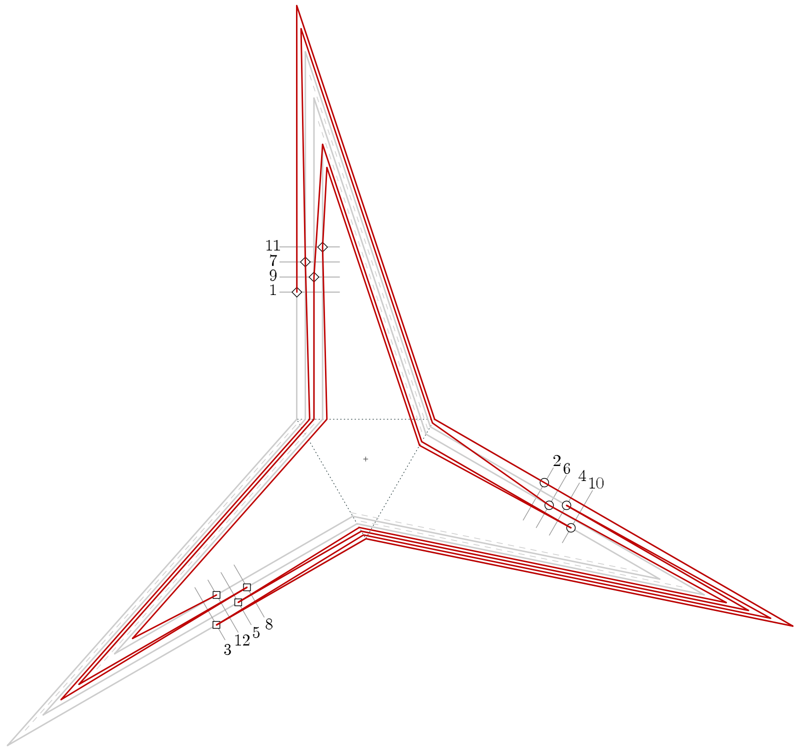


Fig. 10. The corresponding simultaneous embedding of the path for the graph G in Fig. 9.

which are depicted in Fig. 9 with solid gray edges, and the in-between levels as *half-levels* consisting of dashed edges.

The rule then for going from one vertex u to the next vertex v is that if going clockwise, then v's connecting regular level's edge is used, otherwise, its connecting half-level's edge is used. Hence, if going clockwise, 2 bends are introduced at each endpoint of the connecting regular level's edge corresponding to the *destination* vertex. Otherwise, 2 bends are introduced at each endpoint of the connecting half-level's edge of the destination vertex. □

4.3 Geometric Construction of Star-Shaped Levels

In this section we provide a detailed description of the geometry of star-shaped levels. In particular, we determine how small the separation between the star-shaped levels need to be and how close the vertices of G along a prism edge need to be in order to prevent crossings as given by our final lemma.

Lemma 12 *Let G be an n-vertex outerplanar graph. Let δ be the maximum separation between two vertices of G along the same prism edge* before *perturbing vertices onto the other star-shaped levels, and let Δ be the maximum separation between the nested star-shaped levels. Then G will be crossings-free when embedded onto the star-shaped levels provided*

$$\Delta < \frac{\ell}{10n} \text{ and } \delta < \frac{\ell}{(n-1)^2}.$$

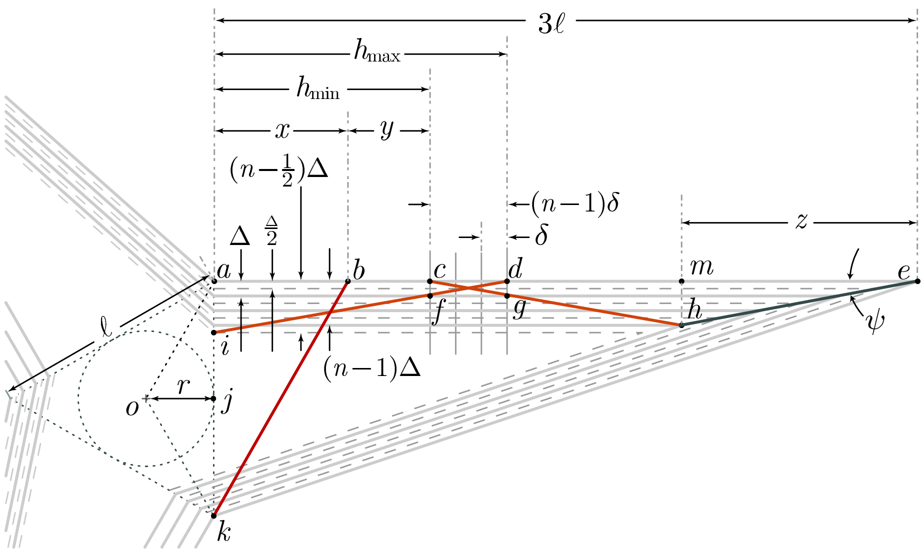


Fig. 11. Geometry of the star-shaped levels.

Proof Sketch: We assume the worst case of $n = n_{\max}$, the number of nested star-shaped levels. Fig. 11 shows the geometry for one of the three-pointed sides of the star-shaped levels. Further, we assume that each of the n parallel regular level edges (corresponding to prism edges) are evenly spaced at a distance Δ from each other. We shift the n parallel half-level edges in a normal direction for a distance of $\frac{\Delta}{2}$ so that each half-level lies between two regular levels except for the innermost half-level. The total width of all $2n$ regular levels and half-levels is then $(n - \frac{1}{2})\Delta$.

The line segment $\overline{bk}$ is an extension from an adjacent side of the star-shaped levels, which intersects the regular level edge $\overline{ae}$. Denote the length $|\overline{ab}| = x$. If r is the radius of the circumscribed circle centered at o, we observe $|\overline{ak}| = \ell = 2|\overline{aj}| = 2\sqrt{3}r$, which implies $x = 2r$ using the properties of 30°-60°-90° triangles $\triangle ajo$ and $\triangle kab$. The distance $x = 2r$ also represents the minimum threshold distance that all the vertices of G must lie from the base of the prism.

Let the term *height* denote the distance of a vertex along the prism edge from the base before it is perturbed. In particular, before being perturbed in an inward perpendicular direction, each of the n vertices along all of the prism edges are evenly spaced at a distance of δ between minimum and maximum heights c to d. Hence, $|\overline{cd}| = (n-1)\delta$. Denote the length $|\overline{bc}| = y$, the distance from the threshold point b to the vertex of minimum height before the perturbation.

The distance $h_{\max}$ of maximum height must be sufficiently high enough, and the distance $h_{\min}$ of minimum height must be sufficiently low enough, such that the point f lies on the interior side of $\overline{di}$. This is so that a vertex at maximum height on the outermost level at point d can route an edge counter-clockwise to the endpoint of the innermost half-level at point i, and not prevent another vertex at minimum height in the second-to-outermost level at point f from routing a path edge. This implies that

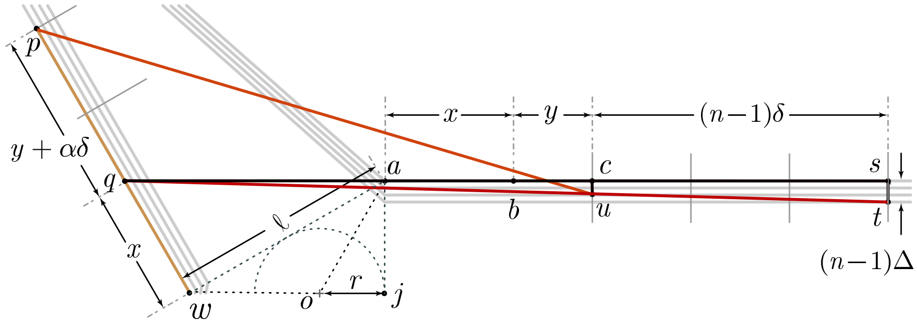


Fig. 12. Geometrical relationship between the Δ of level width and the δ of level height of star-shaped levels.

$$\frac{|\overline{ai}|}{|\overline{ad}|} = \frac{\left(n - \frac{1}{2}\right)\Delta}{x + y + (n-1)\delta} < \frac{\Delta}{(n-1)\delta} = \frac{|\overline{cf}|}{|\overline{cd}|}.$$

If we let $\kappa = (n-1)\delta$, and set $x + y = \ell > 2r$, then we get the inequalities

$$\frac{n - \frac{1}{2}}{\ell + \kappa} < \frac{1}{\kappa} \iff \kappa\left(n - \tfrac{3}{2}\right) < \ell \iff \left(n-1\right)\left(n - \tfrac{3}{2}\right)\delta < \ell$$
$$\iff \delta < \frac{\ell}{\left(n-1\right)\left(n - \frac{3}{2}\right)} \implies \delta < \frac{\ell}{\left(n-1\right)^2}.$$

Conversely, the distance $h_{\min}$ must be sufficiently high enough, and the distance $h_{\max}$ must be sufficiently low enough, such that the point g lies on the interior side of $\overline{ch}$. This is so that a vertex at minimum height on the outermost level at point c can route an edge clockwise to the endpoint of the innermost regular level at point h, and not prevent another vertex at maximum height in the second-to-outermost level at point g from routing a path edge. This implies

$$\frac{|\overline{mh}|}{|\overline{mc}|} = \frac{\left(n-1\right)\Delta}{3\ell - x - y - z} < \frac{\Delta}{(n-1)\delta} = \frac{|\overline{dg}|}{|\overline{dc}|}.$$

Again if we set $x + y = \ell > 2r$, this gives

$$\tfrac{1}{2\ell - z} < \tfrac{1}{\delta} \iff \delta < 2\ell - z,$$

which is an easy condition to meet since the above condition implies $\delta < \frac{\ell}{(n-1)^2} < \ell < 2\ell - z$ provided $n > 1$ and $z < \ell$. The length $\tan\psi = \frac{(n-1)\Delta}{z}$, where $\psi = \frac{1}{2}\arctan\frac{1}{3}$ is one half of the angle of the point of the star. Hence, $z \approx 6.16(n-1)\Delta$, which gives a bounding constraint on the size of Δ, namely $n\Delta < \frac{\ell}{10}$, or $\Delta < \frac{\ell}{10n}$. This value of Δ ensures that $2\ell - z > \ell$ since $z < 7n\Delta < \ell = 10n\Delta$ for $n > 1$.

The final condition is that the angle $\angle sqt$ in Fig. 12 must be sufficiently small enough so that the endpoint u (on the second-to-innermost level) of minimum height of the edge $\overline{pu}$ lies above the line segment $\overline{qt}$, where p is at height $x+y+\alpha\delta$ for some $\alpha \in \{1, 2, \ldots, n-1\}$. This is a worst case of a point of minimum height on the outermost level of one prism edge at point q having an edge to a point of maximum height on the innermost level at point t of the next prism edge. This condition ensures that this edge $\overline{qt}$ does not cross the edge $\overline{pu}$.

This condition gives the inequality $\frac{|\overline{st}|}{|\overline{qs}|} > \frac{|\overline{cu}|}{|\overline{qc}|}$. Note that $|\overline{qa}| = \sqrt{x^2 + \ell^2} = \sqrt{(2r)^2 + (2\sqrt{3}r)^2} = 4r$. This implies that $|\overline{qc}| = |\overline{qa}| + x + y = 4r + \ell$ in which we can again set $x + y = \ell$. Also, $|\overline{qs}| = |\overline{qc}| + (n-1)\delta = 4r + \ell + (n-1)\delta$. Clearly, $|\overline{st}| = (n-1)\Delta$ and $|\overline{cu}| = (n-2)\Delta$. All this gives

$$\begin{aligned}
\frac{(n-1)\Delta}{4r+\ell+(n-1)\delta} > \frac{(n-2)\Delta}{4r+\ell} &\iff \frac{n-1}{n-2} > \frac{4r+\ell+(n-1)\delta}{4r+\ell} \\
&\iff \frac{n-1}{n-2} - 1 > \frac{(n-1)\delta}{4r+\ell} \\
&\iff \delta < \frac{4r+\ell}{(n-1)(n-2)} \\
&\implies \delta < \frac{\ell}{(n-1)^2},
\end{aligned}$$

which is satisfied by the same condition as before. □

5 Conclusions and Open Problems

We presented results in simultaneous embeddings of path and outerplanar graphs with circular arc edges or a small number of bends. Other open problems include:

1. Do all tree-path pairs have geometric simultaneous embedding?
2. What is the complexity of determining whether two planar graphs admit a geometric simultaneous embedding?
3. What is the complexity of determining whether a pair of graphs can be simultaneously embedded?

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