
An Abelian Theorem for Completely Monotone Functions

Peter J. Downey

TR 93-15

ABSTRACT

When an average is taken over any completely monotone function using Poisson weights, the result is asymptotically equal to the completely monotone function evaluated at the Poisson mean. Let N be a Poisson random variable with rate $\mathbf{E}N$. Then if f is any function completely monotone on $[0, \infty)$, we show that

$$\mathbf{E}[f(N)] = f(\mathbf{E}N) + O(\mathbf{E}N \cdot f''(\mathbf{E}N)) \qquad \mathbf{E}N \to \infty .$$

For completely monotone functions that do not decay too rapidly (e.g., regularly varying functions), the error term will be of the order of $f(\mathbf{E}N)/\mathbf{E}N$. An application is given to the expectation of random extremes. Let $Z_{(n)}$ denote the maximum of n independent random variables each with the distribution of Z. Since it can be shown that $\mathbf{E}Z_{(n)}/n$ is completely monotone for any random variable Z with finite expectation, it follows that

$$\mathbf{E}_N[\mathbf{E}_Z Z_{(N)}] = \mathbf{E}Z_{(\mathbf{E}N)} + o(1) \qquad \mathbf{E}N \to \infty ,$$

where $\mathbf{E}_N$ and $\mathbf{E}_Z$ represent expectations with respect to the d.f. of N and Z respectively.

June 15, 1993

Department of Computer Science

The University of Arizona

Tucson, Arizona 85721

An Abelian Theorem for Completely Monotone Functions

1. Introduction

Results which allow us to infer the properties of an average of a function f from some property of f are called *Abelian* theorems in honor of Abel's contributions to the theory of series summability.

One may view Jensen's inequality as Abelian: if f is assumed convex, then $\mathbf{E}[f(X)] \geq f(\mathbf{E}X)$, whenever the expectations are defined. Additional assumptions about f allow us to make a stronger assertion, namely that the inequality can be replaced by asymptotic equality. We show that if f is completely monotone and regularly varying, and if X is a Poisson variate, then $\mathbf{E}[f(X)] = f(\mathbf{E}X) + O(f(\mathbf{E}X)/\mathbf{E}X)$ as $\mathbf{E}X \to \infty$. If we assume only that f is completely monotone, then the result is that $\mathbf{E}[f(X)] = f(\mathbf{E}X) + O(\mathbf{E}X \cdot f''(\mathbf{E}X))$ as $\mathbf{E}X \to \infty$.

Extending a result of Teugels [Teu77], Bingham and Hawkes [Bin83] showed that if f is regularly varying of index $\alpha \leq 0$, and X is a Poisson variate, then $\mathbf{E}[f(X)]$ is a regularly varying function of $\mathbf{E}X$ of index α and $\mathbf{E}[f(X)] \sim f(\mathbf{E}X)$ as $\mathbf{E}X \to \infty$. Our result is analogous under the stronger hypothesis of both regular variation and complete monotonicity, and provides an estimate of the error term.

An application of our result to the theory of extremes is given in Section 3.

1.1. Complete Monotonicity

Let $f^{(n)}(x)$ or $\mathbf{D}^n f(x)$ represent n-fold differentiation with respect to real argument x.

DEFINITION [Wid71]. A real function $f(x)$ is *completely monotone* in argument x on the interval (a, ∞), written $f \in$ c.m.(a, ∞), if it is infinitely differentiable on (a, ∞) and for all $k \geq 0$

$$(-1)^k f^{(k)}(x) \geq 0 ,$$

for all $x \in (a, \infty)$. If in addition $f(a+) < \infty$, then $f \in$ c.m.$[a, \infty)$.

A major representation theorem by Bernstein [Ber28] states that completely monotone functions are exactly the Laplace transforms of monotone and bounded functions. A complete proof may be found in [Wid71].

PROPOSITION (*Bernstein's Theorem*).

$$f \in \text{c.m.}[a, \infty) \quad \Leftrightarrow \quad f(x) = \int_0^\infty e^{-xs}\, d\mu(s) , \qquad x \in [a, \infty) , \tag{1.1}$$

where μ is monotone nondecreasing and bounded on $[0, \infty)$ with $\mu(0) = 0$ and $\mu(\infty) = f(a+) < \infty$.

1.2. Regular Variation

Regularly varying functions are those that scale homogeneously for large argument.

DEFINITION. A positive measurable function $f : (0, \infty) \to (0, \infty)$ is *regularly varying* at infinity if for all $\lambda > 1$, the limit

$$\lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} \tag{1.2}$$

exists and is in $(0, \infty)$. f is *rapidly varying* if the limit exists and is 0 or ∞.

The fundamental result about regular variation [Bin87] is that *if* the (finite or infinite) limit (1.2) exists for all $\lambda > 1$, then there is an extended real number α, $-\infty \leq \alpha \leq \infty$ such that

$$\forall \lambda > 1 \qquad \lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha . \tag{1.3}$$

This α is called the *exponent* or *index of variation*. If (1.3) holds, we write $f \in R_\alpha$. Thus with the understanding $\lambda^{-\infty} = 0$ and $\lambda^{\infty} = \infty$, $R_{-\infty}$ and R_∞ are the rapidly varying functions.

Functions like $x^a + \sin x$ where $a > 0$ are in R_a. Functions like $\exp(-x^k)$, $k > 0$ belong to $R_{-\infty}$ and their reciprocals belong to R_∞.

Functions that fail to have a well-defined limit (finite or infinite) in (1.3) are neither regularly varying nor rapidly varying. Examples are $\sin x$ and $e^{-\lfloor x \rfloor}$.

By historical convention, the class R_0 is called the *slowly varying* functions. It includes, for example functions like $(\ln x)^\alpha$ for any $\alpha \geq 0$. From the results above, it is evident that $f \in R_\alpha$ if and only if there is some slowly varying function l such that $f(x) = x^\alpha \cdot l(x)$.

It is easy to see that if $f \in R_\alpha$ and $g \sim f$ then $g \in R_\alpha$. Thus asymptotic equality $\sim$ is the natural equivalence relation on the class of regularly varying functions.

Regularly varying functions need not be smooth. However, if such functions are suitably smooth, their derivatives also vary regularly. A number of useful results are cited here.

PROPOSITION (*Monotone Density Theorem*) [Bin87, p. 39]. Suppose $f(x) = x^\alpha l(x)$ where $l \in R_0$. If f is absolutely continuous with density f', and f' is eventually monotone, then

(a) If $\alpha \neq 0$ then $f' \in R_{\alpha-1}$ and $f'(x) \sim \alpha x^{\alpha-1} l(x)$ as $x \to \infty$.

(b) If $\alpha = 0$ then $f'(x) = o(l(x)/x)$ as $x \to \infty$.

Remark: When $f \in R_0$, we cannot conclude that $f' \in R_{-1}$, as shown by the example $f(x) = \exp[-x^{-a}]$. The monotonicity assumption on f' is essential, as shown by the example $f(x) = \exp[-\sin(x^2)/x]$, which is a slowly varying function whose derivative is neither regularly nor rapidly varying.

The following result is due to Lamperti [Lam58, Theorem 2], although the first part of the result goes back to Von Mises [Mis36]. Proofs may also be found in [Haa70, p. 23, 109] and [Sen73, p. 1057].

PROPOSITION (*Lamperti's Theorem*). Let f be positive and absolutely continuous with density f'.

(a) If

$$\lim_{x \to \infty} \frac{x f'(x)}{f(x)} = \alpha \tag{1.4}$$

then $f \in R_\alpha$; i.e.,

$$f(x) = x^\alpha l(x) \tag{1.5}$$

where $l \in R_0$.

(b) Conversely, if (1.5) holds and f' is eventually monotone, then (1.4) holds, and

(i) if $\alpha \neq 0$ then $(\operatorname{sgn} \alpha) f'(x) \in R_{\alpha-1}$;

(ii) If $\alpha = 0$ then $f'(x) = o(l(x)/x)$.

A smooth class of regularly varying functions is given by the following definition. Let $\alpha^{\underline{n}}$ denote the falling factorial $\alpha(\alpha-1) \cdots (\alpha-n+1)$.

DEFINITION. A positive function f that is infinitely differentiable in some neighborhood of infinity is said to be *smoothly varying* with *index* α, written $f \in SR_\alpha$, if

$$\frac{x^n f^{(n)}(x)}{f(x)} \to \alpha^{\underline{n}} \qquad x \to \infty \quad (n = 1, 2, \cdots). \tag{1.6}$$

Smoothly varying functions are discussed in [Bin87, §1.8]. We have the following connection to complete monotonicity.

THEOREM 1.1. If $f \in$ c.m.$[a, \infty)$ and $f \in R_\alpha$, for some finite α (necessarily $\alpha \leq 0$), then $f \in SR_\alpha$.

Proof: Consider first the case $\alpha < 0$. Suppose $f(x) = x^\alpha l(x)$ for slowly varying l. Since f is c.m., f' is monotone and so by Lamperti's Theorem,

$$x f^{(1)}(x)/f(x) \to \alpha. \tag{1.7}$$

establishing (1.6) for $n=1$. Furthermore $-f'(x) \in R_{\alpha-1}$. Since $-f'(x)$ is again a c.m. function, the argument is repeated to arrive at

$$x(-f^{(2)}(x))/(-f^{(1)}(x)) \rightarrow \alpha-1 ,$$

and together with (1.7), this yields (1.6) for $n=2$. Induction following this line yields the result for all n.

In the case $\alpha=0$, we need to show that for all $n \geq 1$

$$\frac{f^{(n)}(x)}{f^{(n-1)}(x)/x} \rightarrow 0 , \quad f^{(n)}(x) \rightarrow 0 , \quad \text{and} \quad f^{(n-1)}(x)/x \rightarrow 0 ,$$

which is equivalent in this case to (1.6). The basis for $n=1$ is just Lamperti's Theorem. The induction step follows by L'Hôpital's rule and the fact that for any c.m. function f, $f^{(n)}(x) \rightarrow 0$ for $n \geq 1$ [Fel71, XIII.4, Theorem 2]. □

The above results show that higher derivatives of a completely monotone and regular function decay ever more quickly for large x.

COROLLARY 1.2. If f is a c.m. and is regularly varying with finite index $\alpha \leq 0$, then $f^{(n)}(x) \sim \alpha^{\underline{n}} f(x) x^{-n}$. When $\alpha = 0$, this is interpreted as $f^{(n)}(x) = o(f(x)x^{-n})$.

Not all completely monotone functions are regularly varying. If f is given by the Bernstein representation (1.1), then it follows from Karamata's Tauberian Theorem [Bin87, Theorem 1.7.1; Fel71, XIII.5, Theorem 2] that $f \in R_\alpha$ for some $\alpha \leq 0$ if and only if $\mu(1/s) \in R_{-\alpha}$ (i.e., $\mu(s)$ is regularly varying as $s \rightarrow 0$.) Since increasing bounded α may easily be constructed that fail to vary regularly or even rapidly at zero, the c.m. transform f yielded by (1.1) will fail to regularly or rapidly vary for large x.

1.3. Random Variables

Unless otherwise indicated, all random variables are non-negative. If random variables X and Y are identically distributed, we write $X =_d Y$. Similarly, we use the shorthand $X =_d F$ to mean that X has distribution function F. The complementary distribution function $1 - F(x)$ is denoted $\overline{F}(x)$.

Expectations are denoted $\mathbf{E}$. For a non-negative random variable $X =_d F$

$$\mathbf{E}X^p = \int_0^\infty x^p \, dF(x) = p \int_0^\infty x^{p-1} \overline{F}(x) dx .$$

When it is necessary to emphasize the d.f. with respect to which an expectation is taken, a subscript is employed. For example, $\mathbf{E}_Z(X + Z)$ is equivalent to the random variable $\mathbf{E}Z + X$.

The Poisson random variable N with expected value ρ is the discrete random variable with the probability mass function $\mathbf{P}[N = n] = e^{-\rho}\rho^n/n!$.

2. An Abelian Theorem

Let N have a Poisson distribution. We will show that when f is completely monotone, then $\mathbf{E}[f(N)] \sim f(\mathbf{E}N)$ for large mean and will give an estimate for the error term.

THEOREM 2.1. Let $f \in$ c.m.$[0, \infty)$. Define the average

$$I(\rho) := \sum_{n=0}^\infty e^{-\rho} \frac{\rho^n}{n!} f(n) , \tag{2.1}$$

so that $I(\rho) = \mathbf{E}_N[f(N)]$ where N is a Poisson random variable with mean ρ. Then

(a) $I(\rho)$ is completely monotone, strictly decreasing in ρ (unless f is identically constant) and

$$\forall \rho \geq 0 \quad I(\rho) \geq f(\rho) . \tag{2.2}$$

(b) As $\rho \rightarrow \infty$,

$$I(\rho) = f(\rho) + O(\rho f''(\rho)) . \tag{2.3}$$

Proof:

(a). Since f is completely monotone it is convex. Jensen's Inequality yields

$$I(\rho) = \mathbf{E}_N[f(N)] \geq f(\mathbf{E}N) = f(\rho) ,$$

establishing the lower bound of (2.2).

Directly differentiating the series (2.1) with respect to ρ results in

$$(-1)^k I^{(k)}(\rho) = \sum_{n=0}^{\infty} e^{-\rho} \frac{\rho^n}{n!} (-1)^k \Delta^k f(n) .$$

Since f is completely monotone, each term is non-negative, showing that $I(\rho)$ is completely monotone. Unless $f(x) \equiv c$ for some constant c, there is at least one n such that $\Delta^1 f(n) < 0$, so that $I'(\rho) < 0$, and I is strictly decreasing.

(b). Since f is completely monotone on $[0, \infty)$, Bernstein's Theorem guarantees that there is a nondecreasing and bounded $\mu(s)$ on $[0, \infty)$ with $\mu(0) = 0$ and $\mu(\infty) = f(0+) < \infty$ such that

$$f(x) = \int_0^{\infty} e^{-sx}\, d\mu(s) . \tag{2.4}$$

Putting this integral into the sum (2.1) and interchanging provides an integral representation for I:

$$I(\rho) = \sum_{n=0}^{\infty} e^{-\rho} \frac{\rho^n}{n!} \int_0^{\infty} e^{-sn}\, d\mu(s) = \int_0^{\infty} e^{-\rho} \sum_{n=0}^{\infty} \frac{(\rho e^{-s})^n}{n!}\, d\mu(s) = \int_0^{\infty} e^{-\rho(1-e^{-s})}\, d\mu(s) . \tag{2.5}$$

We now proceed to asymptotically estimate the latter integral for large ρ. This is an integral of Laplace type, with the major contribution of the integrand near $s = 0$. However, standard methods for expansion of Laplace integrals [Olv74] do not directly apply here, since μ does not necessarily have a series expansion near the origin.

Call a function $g(\rho)$ *subdominant* if, for every $k \geq 0$, $g(\rho) = o(1/\rho^k)$ as $\rho \to \infty$. For convenience we denote the class of subdominant functions by $\varpi(\rho)$.

As a preliminary, we prove that certain tail integrals are subdominant:

LEMMA 2.1.

$$\int_{1/\sqrt{\rho}}^{\infty} e^{-\rho(1-e^{-s})}\, d\mu(s) = \varpi(\rho) \qquad \rho \to \infty . \tag{2.6}$$

Proof of Lemma 2.1: Suppose $s \geq 1/\sqrt{\rho}$. For this region we have $e^{-s} \leq e^{-1/\sqrt{\rho}}$, and thus $1 - e^{-s} \geq \rho^{-1/2} - \frac{1}{2}\rho^{-1}$, where we have made use of the inequality

$$\forall x \geq 0 \qquad e^{-x} \leq 1 - x + \frac{x^2}{2} . \tag{2.7}$$

This yields $\rho(1 - e^{-s}) \geq \sqrt{\rho}$ and so

$$e^{-\rho(1-e^{-s})} \leq e^{-\sqrt{\rho}} .$$

Using this bound in the integral results in

$$\int_{1/\sqrt{\rho}}^{\infty} e^{-\rho(1-e^{-s})}\, d\mu(s) \leq e^{-\sqrt{\rho}} \int_0^{\infty} d\mu(s) = \mu(\infty) e^{-\sqrt{\rho}} = \varpi(\rho) . \square$$

LEMMA 2.2. For all $k \geq 0$:

$$\int_{1/\sqrt{\rho}}^{\infty} e^{-\rho s} s^k\, d\mu(s) = \varpi(\rho) \qquad \rho \to \infty . \tag{2.8}$$

Proof of Lemma 2.2: When integrated by parts, (2.8) is equivalent to

$$\int_{1/\sqrt{\rho}}^{\infty} e^{-\rho s} s^k \, d\mu(s) = \rho \int_{1/\sqrt{\rho}}^{\infty} \mu(s) e^{-\rho s} s^k ds \; - \; k \int_{1/\sqrt{\rho}}^{\infty} \mu(s) e^{-\rho s} s^{k-1} ds \; + \omega(\rho)$$

Thus we are done if we can establish that for each k both integrals in the right side above are subdominant, i.e., that

$$\int_{1/\sqrt{\rho}}^{\infty} \mu(s) e^{-\rho s} s^k ds = \omega(\rho) \; .$$

But the previous integral is dominated by a constant $\mu(\infty)$ times

$$\int_{1/\sqrt{\rho}}^{\infty} e^{-\rho s} s^k ds \; = \; \rho^{-(k+1)} \int_{\sqrt{\rho}}^{\infty} e^{-z} z^k dz \; .$$

This last integral is just an instance of the incomplete gamma function $\Gamma(k+1; \sqrt{\rho})$, and it is well-known [Spa87, p. 440] that $\Gamma(k+1; x) \sim x^k \cdot e^{-x}$ as $x \to \infty$. Thus the tail integrals a all $\omega(\rho)$. □

We return now to the main theorem. Using (2.6) in (2.5) results in

$$I(\rho) = \int_{0}^{1/\sqrt{\rho}} e^{-\rho(1-e^{-s})} \, d\mu(s) + \omega(\rho) \; . \tag{2.9}$$

Using inequality (2.7) provides the bound

$$I(\rho) \le \int_{0}^{1/\sqrt{\rho}} e^{-\rho s} e^{\frac{1}{2}\rho s^2} \, d\mu(s) + \omega(\rho) \; . \tag{2.10}$$

Now when $x < 1$, $e^x \le 1 + ex$. Since when $s \le 1/\sqrt{\rho}$ we have $\frac{1}{2}\rho s^2 \le \frac{1}{2}$, this bound can be applied to yield

$$I(\rho) \le \int_{0}^{1/\sqrt{\rho}} e^{-\rho s} (1 + \tfrac{1}{2} e \rho s^2) \, d\mu(s) + \omega(\rho) = \int_{0}^{1/\sqrt{\rho}} e^{-\rho s} \, d\mu(s) + \frac{e\rho}{2} \int_{0}^{1/\sqrt{\rho}} e^{-\rho s} s^2 \, d\mu(s) + \omega(\rho) \tag{2.11}$$

If we add to the bound in (2.11) the tail integrals that are known by Lemma 2.2 to be subdominant, we have the result

$$I(\rho) \le \int_{0}^{\infty} e^{-\rho s} \, d\mu(s) + \frac{e\rho}{2} \int_{0}^{\infty} e^{-\rho s} s^2 \, d\mu(s) + \omega(\rho) = f(\rho) + \frac{e\rho}{2} f^{(2)}(\rho) + \omega(\rho) \; . \tag{2.12}$$

Using (2.2), we conclude that $I(\rho)$ is trapped between the lower bound $f(\rho)$ and the upper bound (2.12). The theorem follows. □

EXAMPLE 1:. Let $a \ge 0$ and define $f(x) = 1/x^a$. Then $f^{(2)}(x) = a(a+1)x^{-(a+2)}$. The theorem yields the estimate

$$\sum_{n=0}^{\infty} e^{-\rho} \frac{\rho^n}{n!} \frac{1}{n^a} = \frac{1}{\rho^a} + O(\frac{1}{\rho^{a+1}}) \qquad \rho \to \infty \; .$$

EXAMPLE 2:. Take $f(x) = 1/\ln(x+e)$. Then $f^{(2)}(x) = (x+e)^{-2} f^2(x)[1+2f(x)]$. The theorem yields the estimate

$$\sum_{n=0}^{\infty} e^{-\rho} \frac{\rho^n}{n!} \frac{1}{\ln(n+e)} = \frac{1}{\ln(\rho+e)} + O(\frac{1}{\rho \cdot (\ln \rho)^2}) = \frac{1}{\ln \rho} + O(\frac{1}{\rho \cdot (\ln \rho)^2}) \qquad \rho \to \infty \; .$$

Remarks:

(1) When $f(\rho)$ is itself a subdominant function, the estimate (2.3) is correct but useless. In this case the error terms are of the same order as, or of higher order than, the leading term. In addition, all terms are in the subdominant class $\omega(\rho)$. For example, when $f(n) = \exp[-cn]$, direct summation of the

series yields $I(\rho) = \exp[-\rho(1-e^{-c})]$, which is an exponential multiple of $f(\rho)$. In this case the error term $\rho f''(\rho)$ is $c^2\rho\cdot\exp[-c\rho]$, so the error term actually dominates $f(\rho)$. All terms are $\omega(\rho)$.

(2) Higher order terms in the expansion of $I(\rho)$ are available by using more terms in the expansion of $1-e^{-s}$ in the proof. By methods similar to those of the proof, one obtains

$$I(\rho) = f(\rho) + \rho\frac{f^{(2)}(\rho)}{2!} - \rho\frac{f^{(3)}(\rho)}{3!} + O(\rho^2 f^{(4)}(\rho)) \qquad \rho \to \infty\ .$$

When f is additionally assumed to vary regularly, we can conclude that the error term is asymptotically smaller than f:

THEOREM 2.2. If in the hypothesis of Theorem 2.1 we additionally assume that $f \in R_\alpha$ for some $\alpha \le 0$, then $I(\rho)$ is both c.m. and R_α and (2.3) can be strengthened to

$$I(\rho) = f(\rho) + O(f(\rho)/\rho) \qquad \rho \to \infty. \tag{2.13}$$

Proof: From Corollary 1.2, $f^{(2)}(\rho) = O(\rho^{-2}f(\rho))$ if f is regularly varying. Putting this into (2.3) yields (2.13). Since $I(\rho) \sim f(\rho)$ and f is regularly varying, so is $I(\rho)$. □

3. An Extreme Application

Let $X_1, \cdots, X_n$ be independent and identically distributed random variables each with d.f. F. The *extreme*

$$X_{(n)} = \max(X_1, \cdots, X_n)$$

has the d.f. $F(x)^n$, and $\mathbf{E}X_{(n)}$ is finite if $\mathbf{E}X$ is finite. We will assume here that F has a finite mean.

In several applications, the following problem of "random maxima" arises. If N has a Poisson distribution, what is the the expected value of the maximum of N values $X_1, \cdots, X_N$ chosen independently from a population with d.f. F? In this section we provide an estimate for this expected maximum $\mathbf{E}_N[\mathbf{E}_X X_{(N)}]$ when the Poisson mean $\mathbf{E}N$ is large. We do this by recourse to Theorem 2.1, after first explaining the relevance of complete monotonicity to the subject of expected extremes (Lemma 3.2).

The above problem has arisen in studying the resequencing delay for the $M/G/\infty$ queueing system [Har82, Bac84] and in the performance of parallel processing [Dow93]. It also has the following simple interpretation. Suppose customers arrive from an external source according to a Poisson process of rate λ. These customers accumulate for a time interval t during which the server is absent. At time t service begins, and all all customers are served simultaneously. The server is freed only when all customers have completed service. If customer service intervals are independent and described by a variate $X =_d F$, then the expected time during which the server is busy is $\mathbf{E}_N[\mathbf{E}_X X_{(N)}]$ for N a Poisson distributed variate having mean λt. Our result below shows that for large t, the busy time is $\mathbf{E}X_{(\lambda t)} + o(1)$ under the assumption that $\mathbf{E}X$ is finite.

Since

$$\mathbf{E}X_{(n)} = \int_0^\infty (1 - F(x)^n)\,dx = n\int_0^\infty F(x)^{n-1}\,dF(x)\ ,$$

we may readily extend this sequence to a function $g(\nu)$ of real argument $\nu \ge 1$ by defining:

$$g(\nu) = \int_0^\infty (1 - F(x)^\nu)\,dx = \nu\int_0^\infty xF(x)^{\nu-1}\,dF(x)\ . \tag{3.1}$$

An important question in the study of extremes is the rate at which $\mathbf{E}X_{(n)}$ grows as a function of n [Dow90, Dow91]. A partial result on the rate of growth of $g(\nu) = \mathbf{E}X_{(\nu)}$ is provided by

LEMMA 3.1. If $\mathbf{E}X < \infty$ then for $k = 0, 1, 2, \cdots$

$$(-1)^k\mathbf{D}^{(k+1)}g(\nu) = o(\nu^{-k}) \qquad \nu \to \infty \tag{3.2}$$

and

$$(-1)^k \mathbf{D}^{(k)} \left(\frac{g(\nu)}{\nu} \right) = o(\nu^{-k}) \qquad \nu \to \infty \tag{3.3}$$

Proof: Differentiating under the first integral in (3.1) results in

$$(-1)^k \mathbf{D}^{(k+1)} g(\nu) = \int_0^\infty F^\nu (-\ln F)^{k+1}\, dx, \tag{3.4}$$

for $\nu \ge 1$. The equality is valid provided we establish that the integral on the right—call it $I_{k+1}(0, \infty]$—is uniformly convergent for $\nu \ge 1$.

Break the range of integration of $I_{k+1}(0, \infty)$ into two parts so that $I_{k+1}(0, \infty) = I_{k+1}(0, R] + I_{k+1}(R, \infty)$. We bound each of these integrals in turn.
Since for all $u \ge 0$

$$e^{-u} u^k \le e^{-k} k^k \ , \tag{3.5}$$

we obtain $F^\nu (-\ln F)^{k+1} \le C_1 \nu^{-(k+1)}$ and

$$I_{k+1}(0, R] \le C_1 \nu^{-(k+1)} \int_0^R dx = C_1 R \nu^{-(k+1)} \ .$$

Also since for all u in $(0, 1]$ we have $u \cdot (-\ln u) \le 1 - u$, it follows that since $\nu \ge 1$

$$I_{k+1}(R, \infty) = \int_R^\infty F^{\nu-1} (-\ln F)^k F (-\ln F)\, dx \le \int_R^\infty F^{\nu-1} (-\ln F)^k \overline{F}\, dx \le C_2 \nu^{-k} \int_R^\infty \overline{F}\, dx \ .$$

Since $\mathbf{E}X = \int_0^\infty \overline{F}\, dx < \infty$, it follows that the last integral $\to 0$ as $R \to \infty$, which establishes that all of the integrals $I_{k+1}(0, \infty)$ are uniformly convergent for $\nu \ge 1$, and hence that the derivatives (3.4) are valid. Furthermore, setting $R := \ln \nu$ in the above two bounds results in

$$I_{k+1}(0, \infty) \le C_1 \nu^{-(k+1)} \ln \nu + C_2 \nu^{-k} o(1) = o(\nu^{-k}) \ .$$

This establishes (3.2).

To obtain the bounds in (3.3), differentiate $\nu \cdot \frac{g(\nu)}{\nu}$ a total of $k+1$ times to get that for $k \ge 0$

$$\mathbf{D}^{k+1} \left(\frac{g(\nu)}{\nu} \right) = \frac{1}{\nu} \mathbf{D}^{k+1} g(\nu) - \frac{k+1}{\nu} \mathbf{D}^k \left(\frac{g(\nu)}{\nu} \right) . \tag{3.6}$$

In [Dow90] it is shown that if $\mathbf{E}X < \infty$, then $g(\nu) = o(\nu)$; this establishes (3.3) for $k = 0$. For the induction step, assume (3.3) is true at $k \ge 0$, and apply this hypothesis along with (3.2) in the identity (3.6) to get

$$\mathbf{D}^{k+1} \left(\frac{g(\nu)}{\nu} \right) = \frac{1}{\nu} \cdot o(\nu^{-k}) \ - \ \frac{k+1}{\nu} \cdot o(\nu^{-k}) \ = \ o(\nu^{-(k+1)}) \ .$$

□.

EXAMPLE. For the Weibull distribution $F(x) = 1 - e^{-\sqrt{x}}$, direct calculation establishes that $g(\nu) \sim (\ln \nu)^2$ and that $(-1)^{k+1} g^{(k)}(\nu) \sim 2k! \ln \nu / \nu^k$.

The next result shows that $\mathbf{E}X_{(\nu)}/\nu$ is completely monotone for $\nu \ge 1$.

LEMMA 3.2. $g(\nu)/\nu \in$ c.m.$[1, \infty)$.

Proof: By the second integral in (3.1)

$$\frac{g(\nu)}{\nu} = \int_0^\infty x F^{\nu-1}\, dF \ .$$

Differentiating, we have for $k = 1, 2, \cdots$

$$(-1)^k \mathbf{D}^k \left(\frac{g(\nu)}{\nu} \right) = \int_0^\infty x(-\ln F)^k F^{\nu-1} \, dF \ , \tag{3.7}$$

provided the integral is uniformly convergent for $\nu \geq 1$. Uniform convergence is easily shown via the bound (3.5) and the fact that $\int_R^\infty x \, dF \to 0$ as $R \to \infty$. Finally, $F^{\nu-1}$ is c.m.$[1, \infty)$ and hence $(-\ln F)^k F^{\nu-1} \geq 0$; from this and (3.7) it follows that $g(\nu)/\nu$ is also in c.m.$[1, \infty)$. □

Using the Abelian theorem of the last section, we can now estimate the following expectation of a random Poisson-weighted maximum.

THEOREM 3.1. Let N be a Poisson random variable, and denote its mean $\mathbf{E}N$ by ρ. Let random variables X_i be i.i.d. with d.f. F and assume $\mathbf{E}X < \infty$. Then

$$\mathbf{E}_N[\mathbf{E}X_{(N)}] = \mathbf{E}X_{(\rho)} + o(1) \qquad \rho \to \infty \ . \tag{3.8}$$

Proof: Write $g(\nu)$ for the expected extreme as in (3.1), and define

$$f(\nu) = \frac{g(\nu+1)}{\nu+1} \ .$$

By Lemma 3.2, we have that $f(\nu) \in$ c.m.$[0,\infty)$.

Define $I(\rho) := \mathbf{E}_N[\mathbf{E}X_{(N)}]$. Then by definition (since $\mathbf{E}X_{(0)} = 0$)

$$I(\rho) = \sum_{n=1}^\infty e^{-\rho} \frac{\rho^n}{n!} \mathbf{E}X_{(n)} = \sum_{n=1}^\infty e^{-\rho} \frac{\rho^n}{(n-1)!} \frac{g(n)}{n} = \sum_{j=0}^\infty e^{-\rho} \frac{\rho^{j+1}}{j!} \frac{g(j+1)}{j+1} = \rho \sum_{j=0}^\infty e^{-\rho} \frac{\rho^j}{j!} f(j) \ .$$

The last sum is of the form (2.1) where f is c.m.$[0, \infty)$, and so invoking Theorem 2.1 yields

$$I(\rho) = \rho f(\rho) + O(\rho^2 f^{(2)}(\rho)) \qquad \rho \to \infty \ .$$

Now by Lemma 3.1, $\mathbf{D}^2[g(\nu)/\nu] = o(\nu^{-2})$, and so $\mathbf{D}^2 f(\nu) = o(\nu^{-2})$ as well. Therefore

$$I(\rho) = \rho f(\rho) + o(1) \qquad \rho \to \infty \ . \tag{3.9}$$

By Taylor's theorem, for some θ in $[0, 1]$ we have

$$f(\rho) = f(\rho-1) + f'(\rho-\theta).$$

By Lemma 3.1 again, $\mathbf{D}f(\nu) = o(\nu^{-1})$ and so $f'(\rho-\theta) = o(\rho^{-1})$ as $\rho \to \infty$. Thus

$$f(\rho) = f(\rho-1) + o(\rho^{-1}) = \frac{g(\rho)}{\rho} + o(\rho^{-1}) \qquad \rho \to \infty \ .$$

Putting this into (3.9) yields

$$I(\rho) = g(\rho) + o(1) \qquad \rho \to \infty \ ,$$

which is the desired conclusion. □

4. References

[Bac84] Baccelli, F., E. Gelenbe and B. Plateau. An end-to-end approach to the resequencing problem. *J. ACM 31*, 3 (Jul 1984), 474-485.

[Ber28] Bernstein, S. Sur les fonctions absolument monotones. *Acta Mathematica 51*, (1928), 1-66.

[Bin83] Bingham, N.H. and J. Hawkes. On limit theorems for occupation times. In J.F.C. Kingman and G.E.H. Reuter, *Probability, Statistics and Analysis*, London Math. Soc. Lect. Note Ser. 79, Cambridge University Press (1983), 46-62.

[Bin87] Bingham N.H., C.M. Goldie and J.L. Teugels. *Regular Variation*, volume 27 of *The Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, 1987.

[Dow90] Downey, P.J. Distribution-free bounds on the expectation of the maximum with scheduling applications. *Operations Research Letters 9* (1990), 189-201.

[Dow91] Downey, P.J. and R.S. Maier. Stochastic orderings and the mean growth of extremes. TR 91-13, Department of Computer Science, University of Arizona, April 1991.

[Dow93] Downey, P.J. Bounds and approximations for overheads in the time to join parallel forks, *ORSA Journal on Computing*, to appear.

[Fel71] Feller, W. *An Introduction to Probability Theory and its Applications, Vol. II*. New York: Wiley (1971).

[Haa70] de Haan, L. *On Regular Variation and Its Application to the Weak Convergence of Sample Extremes*. Mathematical Centre Tract No. 32, Mathematisch Centrum, Amsterdam (1970).

[Har82] Harrus, G. and B. Plateau. Queueing analysis of a reordering issue. *IEEE Trans. on Software Engineering SE-8*, 2(Mar 1982), 113-123.

[Lam58] Lamperti, J. An occupation time theorem for a class of stochastic processes. *Trans. AMS 88*, 2(July 1958), 380-387.

[Mis36] von Mises, R. La distribution de la plus grande de *n* valeurs. *Selected Papers II*, AMS, Providence, pp. 271-294.

[Olv74] Olver, F.W.J. *Asymptotics and Special Functions*. New York: Academic Press, 1974.

[Sen73] Seneta, E. A Tauberian theorem of E. Landau and W. Feller, *Ann. Probability 1* (1973), 1057-1058.

[Spa87] Spanier, J. and K.B. Oldham. *An Atlas of Functions*. Washington: Hemisphere Publishing Corporation, 1987.

[Teu77] Teugels, J.L. On the rate of convergence of the maximum of a compound Poisson process. *Bull. Soc. Math. Belg., Ser B 29*, II(1977), 205-216

[Wid71] Widder, D.V. *An Introduction to Transform Theory*. New York: Academic (1971).