Section 3.1: Inductively Defined Sets

To define a set S "inductively", we need to give 3 things:

Basis:

Specify one or more elements that are in S.

Induction Rule:

Give one or more rules telling how to construct a new element from an existing element in S.

Closure:

Specify that no other elements are in S.

(The closure is generally assumed implicitly.)

The basis elements and the induction rules are called ***constructors****.*

Example: Give an inductive definition of S = {3,7,11,15,19,23,…}

Basis:

$3 \in S$

Induction:

If $x \in S$ then $x+4 \in S$

The constructors are "3" and the "add 4" operation.

Note: Without the closure part, lots of sets would satisfy this defn.

For example, $\mathbb{Z}$ works since $3 \in \mathbb{Z}$ and $x+4 \in \mathbb{Z}$.

<u>Example:</u> Find an inductive definition of

$S=\{3,4,5,8,9,12,16,17,20,24,33,\ldots\}$

Solution: Notice that S can be written as a union of simpler sets:

$S = \{3,5,9,17,33,\ldots\} \cup \{4,8,12,16,20,\ldots\}$

Basis:

Induction:

<u>Example:</u> Here is an inductive definition. What does this set look like?

Basis: $2 \in S$

Induction: $x \in S$ implies $x+3 \in S$ and $x-3 \in S$

Solution:

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$3,4 \in S$

Induction:

If $x \in S$ then

If x is odd

then $2x-1 \in S$

else $x+4 \in S$.

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Solution:

$S = \{2,5,8,11,\ldots\} \cup \{-1,-4,-7,-10,\ldots\} = \{\ldots,-10,-7,-4,-1,2,5,8,11\ldots\}$

Example: Find an inductive definition for

$S = \{\Lambda, ac, aacc, aaaccc, \ldots\} = \{ a^n c^n \mid n \in \mathbb{N} \}$

Basis:

Induction:

Example: Find an inductive definition for

$S = \{ a^{n+1} b c^n \mid n \in \mathbb{N} \}$

Basis:

Induction:

Example: What set is defined by this inductive definition?

Basis: $a, b \in S$

Induction: $x \in S$ then $f(x) \in S$.

Solution:

Example: Find an inductive definition for

$S = \{\Lambda, ac, aacc, aaaccc, \ldots\} = \{ a^n c^n \mid n \in \mathbb{N} \}$

Basis: $\Lambda \in S$

Induction: If $x \in S$ then $axc \in S$.

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Example: What set is defined by this inductive definition?

Basis: $a, b \in S$

Induction: $x \in S$ then $f(x) \in S$.

Solution:

$S = \{a, f(a), f(f(a)), \ldots\} \cup \{b, f(b), f(f(b)), \ldots\}$

$= \{ f^n(a) \mid n \in \mathbb{N} \} \cup \{ f^n(b) \mid n \in \mathbb{N} \}$

$= \{ f^n(x) \mid x \in \{a, b\} \text{ and } n \in \mathbb{N} \}$

Example: Describe the set S defined inductively by:

Basis: <0> ∈ S

Induction: x ∈ S implies cons(1,x) ∈ S.

Solution: S = { <0>, <1,0>, <1,1,0>, <1,1,1,0>, … }

Giuseppe Piano's Definition of The Set of Natural Numbers, $\mathbb{N}$:

Define the "successor" function, **succ**.

Basis: $0 \in \mathbb{N}$

Induction: If $x \in \mathbb{N}$ then succ(x) $\in \mathbb{N}$.

Closure: There are no other elements in $\mathbb{N}$.

$\mathbb{N}$ = { 0, succ(0), succ(succ(0)), succ(succ(succ(0))), … }

List Functions:

head (<a,b,c,d>) = a
tail (<a,b,c,d>) = <b,c,d>
cons (a,<b,c,d>) = <a,b,c,d>

Notation:

cons(x,y) can be written with the infix operator ::

x **::** y

Examples:

a::<b,c,d> = <a,b,c,d>
a :: (b :: (c :: <>)) = <a,b,c>
<a,b> :: <c,d> = <<a,b>,c,d>

Most operators are **left-associative**:

x ÷ y ÷ z = (x ÷ y) ÷ z

Assume that :: is **right-associative**:

x :: y :: z = x :: (y :: z)

So:

a :: b :: c :: <> = <a,b,c>

Example: Find an inductive definition for

S = { <>, <a,b>, <a,b,a,b>, … }

Solution:

Basis:

Inductive Step:

Example: Find an inductive defition for

S = { <>, <<>>, <<<>>>, … }

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Binary Trees

The set of binary trees B is defined as follows:
(Assume A is an alphabet: the labels for the nodes.)

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Induction: If L, R ∈ B and x ∈ A then <L,x,R> ∈ B.

Example: Here is a set S, which is a subset of B. What is in S?

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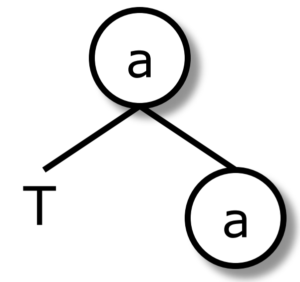

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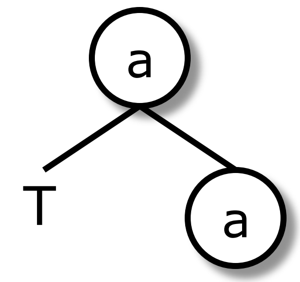

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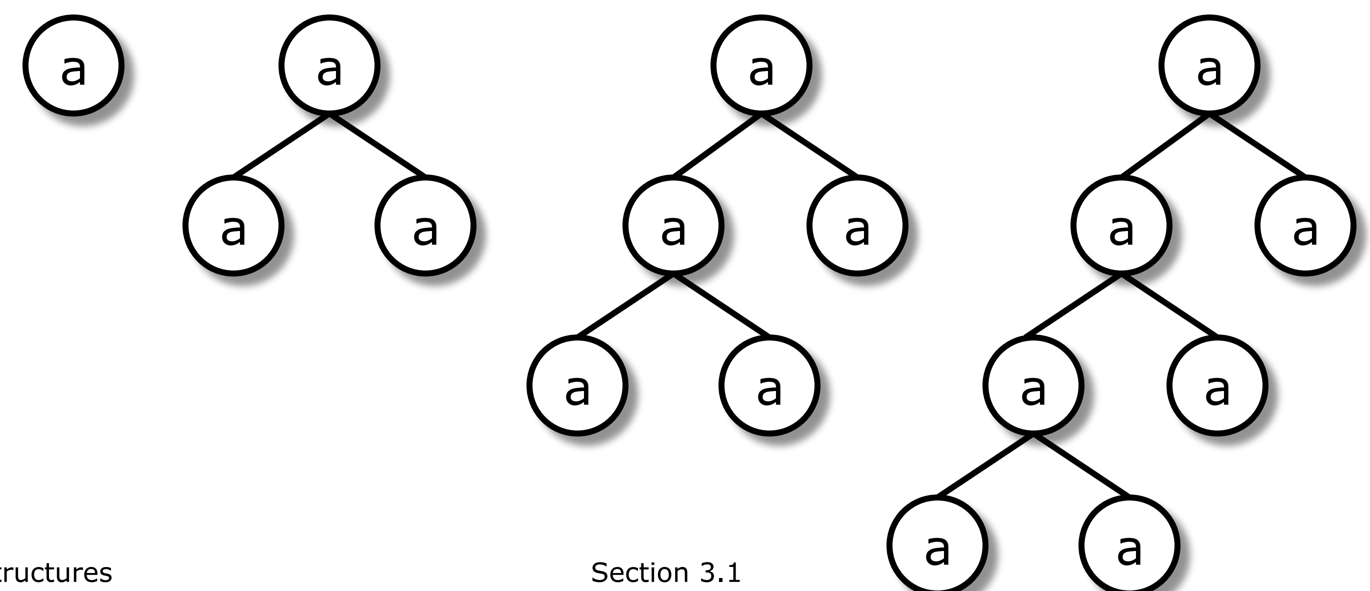

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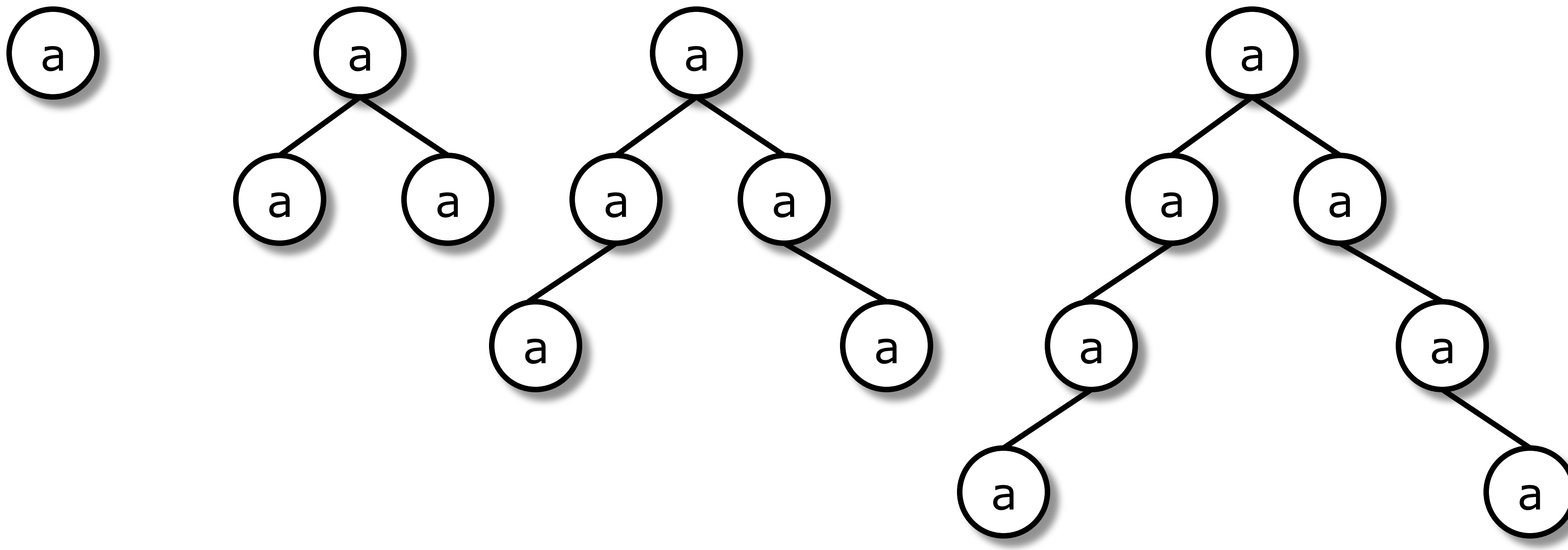

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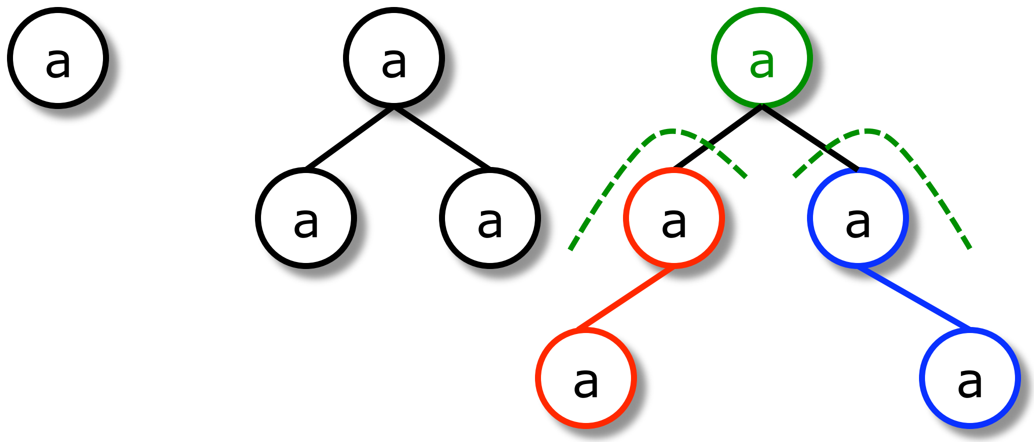

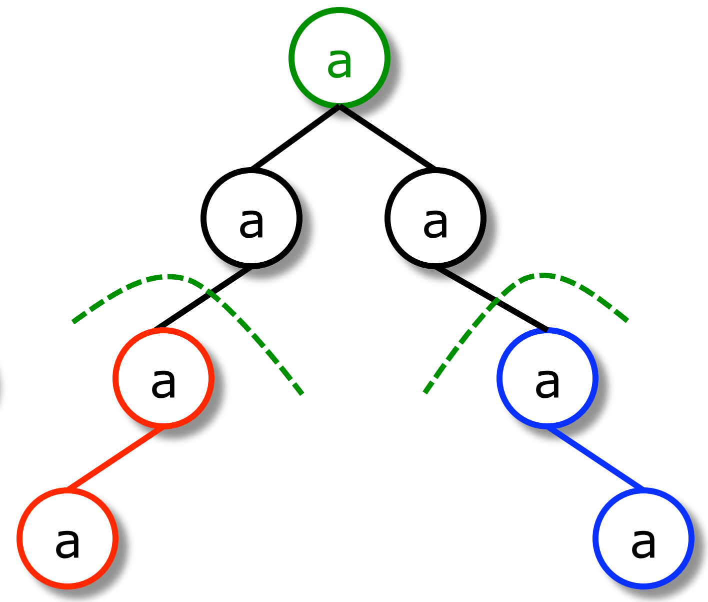

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Induction: <L,a,R> ∈ S implies

<<L,a,<>>,a,<<>,a,R>> ∈ S

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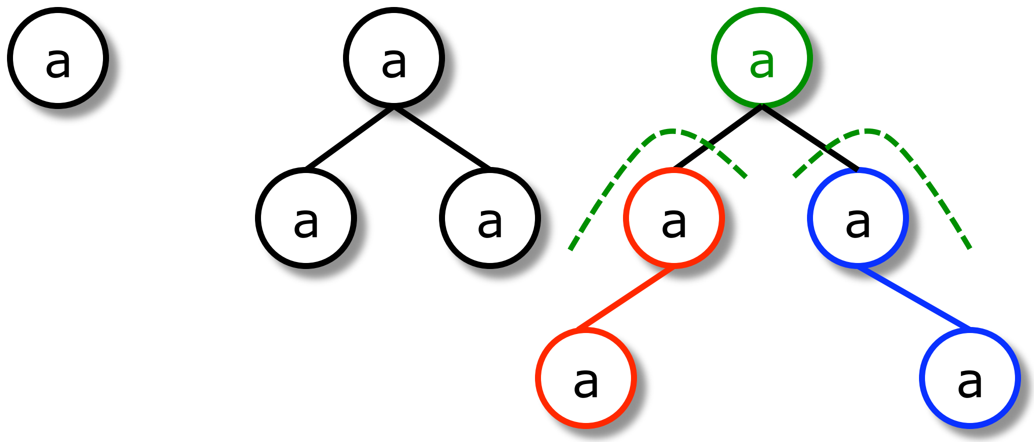

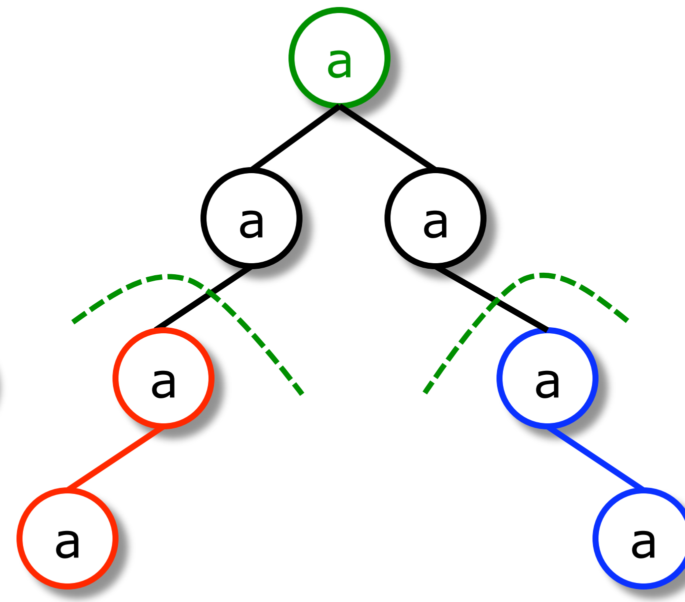

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Induction: <L,a,R> ∈ S implies
    <<L,a,<>>,a,<<>,a,R>> ∈ S

Define function **left**,**right**: Trees→Trees as
    left (<L,a,R>) = L
and
    right (<L,a,R>) = R

Revised Inductive Step: T ∈ S implies
    <<left(T),a,<>>,a,<<>,a,right(T)>> ∈ S

Example: Find an inductive definition for the set $S=\{a\}^* \times \mathbb{N}$.

Solution:

Basis:

Induction:

Example: Find an inductive definition for the set $S=\{a\}^* \times \mathbb{N}$.

Solution:

Basis: $(\Lambda,0) \in S$

Induction: $(s,n) \in S$ implies $(as,n) \in S$ and $(s,n+1) \in S$.

Example: Find an inductive definition for the set

$S = \{(x,-y) \mid x,y \in \mathbb{N} \text{ and } x \geq y\}$

Let's try to understand S by writing out some tuples:

Here is a graphical representation of S:

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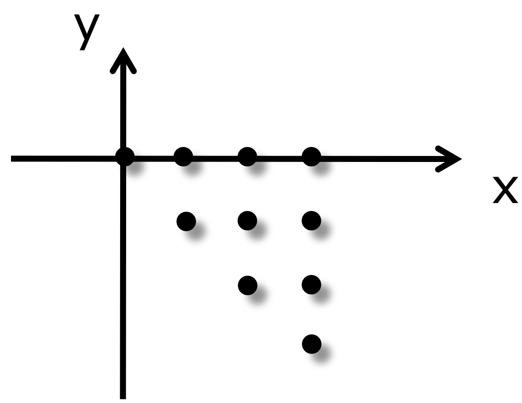

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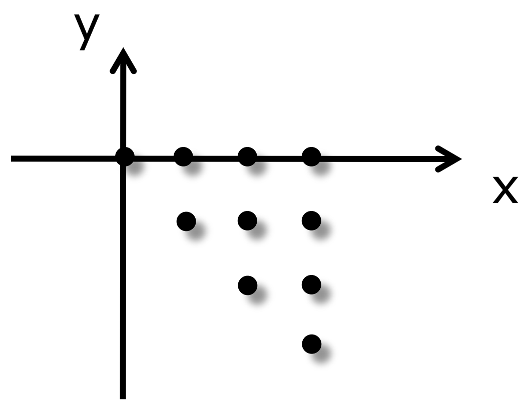

Basis: (0,0) ∈ S

Induction: (x,y) ∈ S implies (x+1,y) ∈ S and (x+1,y-1) ∈ S

Notice that the inductive step will construct some points two ways.

(1,-1) → (2,-1)

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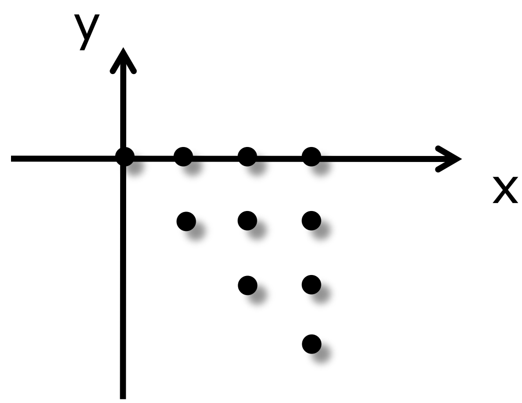

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In-Class Quiz: Try to find a solution that does not construct repeated points.

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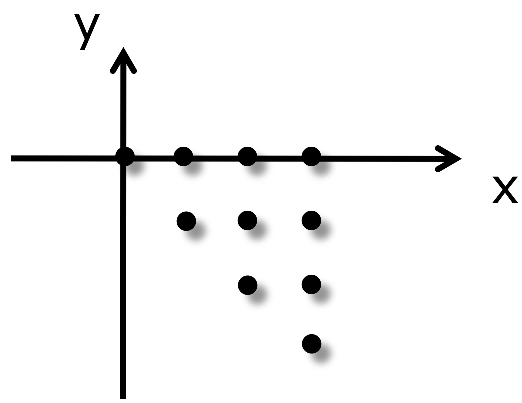

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Approach: Construct the diagonal. Then construct horizontal lines.

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