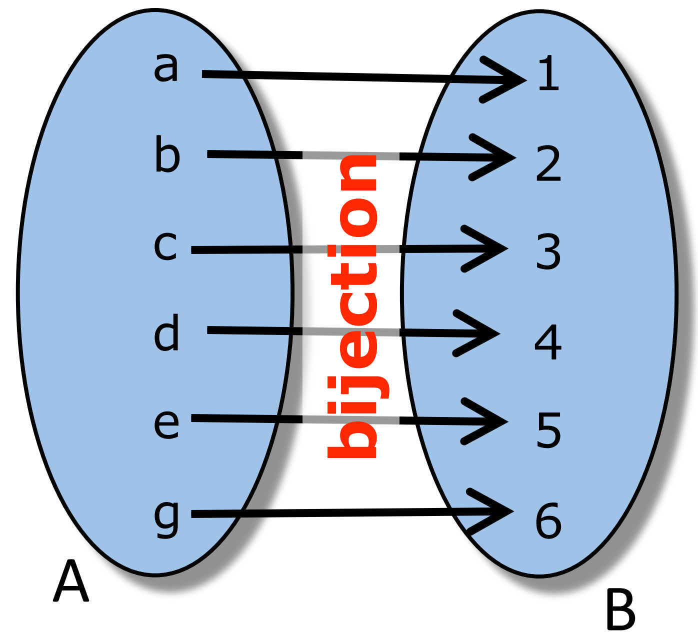
Section 2.4: Countability

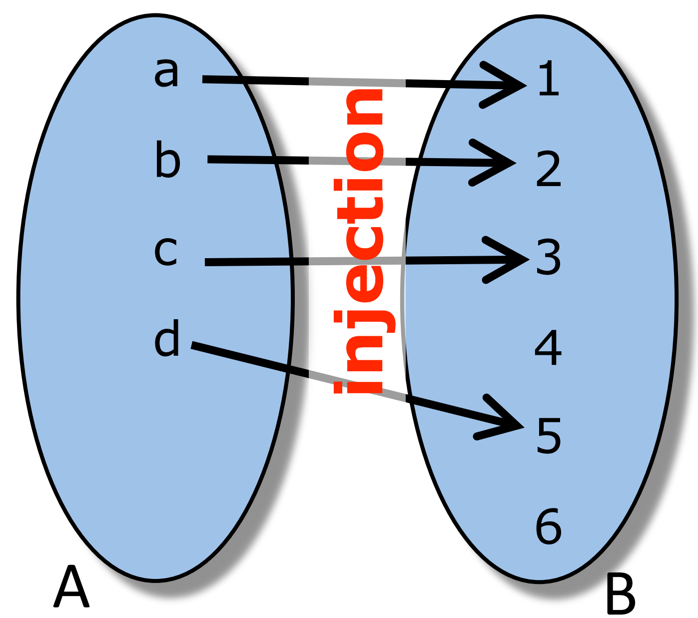
We can compare the cardinality of two sets.

$|A| = |B|$ means there is a bijection between A and B.

$|A| \le |B|$ means there is an injection from A to B.

$|A| < |B|$ means $|A| \le |B|$ and $|A| \ne |B|$





Example:

Let $A = \{ 3x+4 \mid x \in \mathbb{N} \text{ and } 0 \le 3x+4 \le 1000 \}$

What is $|A|$?

Solution:

$0 \le 3x+4 \le 1000$

iff $-4/3 \le x \le 996/3$

iff $-4/3 \le x \le 332$

So we can rewrite the definition of A as

$A = \{3x+4 \mid x \in \mathbb{N} \text{ and } 0 \le x \le 332\}$

We can define a function

$f: \{0,1,\ldots,332\} \rightarrow A$

$f(x) = 3x+4$

Notice f is a bijection. So $|A| = 333$.

Countable

Finite sets:

The set S is **countable** if $|S| = |\mathbb{N}_n|$ for some n. {0,1,2,…,n}

Countably Infinite Sets:

The set S is **countable** if $|S| = |\mathbb{N}|$

Uncountably Infinite Sets:

If a set is not countable, it is **uncountable**.

$\mathbb{Z}$ is countable: $|\mathbb{Z}| = |\mathbb{N}|$

To prove this we need to find a bijection between $\mathbb{Z}$ and $\mathbb{N}$.

Let $f:\mathbb{N} \rightarrow \mathbb{Z}$ be defined by $f(2k)=k$ and $f(2k+1)=-(k+1)$.

x	0	1	2	3	4	5	6	7	
f(x)	0	-1	1	-2	2	-3	3	-4	…

Some Countability Results

$S \subseteq \mathbb{N}$ implies S is countable.

S is countable iff $|S| \leq |\mathbb{N}|$.

Subsets and images of countable sets are countable.

If each of the sets S_0, S_1, … S_n, … is countable,
then so is their union: $S_0 \cup S_1 \cup \ldots S_n \cup \ldots$

$\mathbb{N} \times \mathbb{N}$ is countable.

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$\mathbb{N} \times \mathbb{N}$ is countable.

Proof: Step 1: List all pairs of natural numbers like this:

0,0			
0,1	1,0		
0,2	1,1	2,0	
0,3	1,2	2,1	3,0
…			

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$\mathbb{N} \times \mathbb{N}$ is countable.

Proof: Step 1: List all pairs of natural numbers like this:

Step 2: Assign a number from $\mathbb{N}$ to each pair.

0: 0,0

1: 0,1   2: 1,0

3: 0,2   4: 1,1   5: 2,0

6: 0,3   7: 1,2   8: 2,1   9: 3,0

…

This is Cantor's Bijection: $n = ((x+y)^2 + 3x + y)/2$

Example:

Is $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ countable?

Let $S_n = \{ (x,y,z) \in \mathbb{N}^3 \mid x+y+z = n \}$

Each of the sets $S_0, S_1, \ldots S_n, \ldots$ is countable (actually, each is finite).

Their union $S_0 \cup S_1 \cup \ldots S_n \cup \ldots$ is the set $\mathbb{N}^3$.

Since their union is countable, $\mathbb{N}^3$ is countable.

Facts:

Countably Infinite Sets

The set of rational numbers $\mathbb{Q}$ is countably infinite.

The set A^* of all finite strings over a finite alphabet
is countably infinite.

Uncountably Infinite Sets

The set of real numbers is not countable!
$|\mathbb{N}| < |\mathbb{R}|$

Let [0,1] be the closed interval of real numbers $0 \leq x \leq 1$.
The closed interval [0,1] is uncountably infinite!

The powerset of a countably infinite set is uncountable!
$|\mathbb{N}| < |\text{power}(\mathbb{N})|$

Diagonalization Example: Alphabet = {a,b,c}

Consider the set of all (possibly infinite) strings over A.

Assume it is countable.

$\{S_0, S_1, S_2, \ldots\}$ = {abcabc…, bbbbb…, cacacac…, …}

Then we can list all the elements in some order.

Diagonalization Example: Alphabet = {a,b,c}

Consider the set of all (possibly infinite) strings over A.

Assume it is countable.

$\{S_0, S_1, S_2, \ldots\}$ = {abcabc…, bbbbb…, cacacac…, …}

List each string as a row in a matrix, in order.

	0	1	2	3	…	n	…
S_0	a	b	c	a	…	b	…
S_1	b	b	b	b	…	b	…
S_2	c	a	c	a	…	c	…
S_3	c	b	a	c	…	b	…
S_4	c	c	c	c	…	c	…
S_5	a	a	a	a	…	a	…
…	…						

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S_1	b	b	b	b	…	b	…
S_2	c	a	c	a	…	c	…
S_3	c	b	a	c	…	b	…
S_4	c	c	c	c	…	c	…
S_5	a	a	a	a	…	a	…
…	…						

Consider the diagonal:

abcc…a…

Change every character.

a→b

b→a

c→a

to create a new string.

Now we have the string

baaa…b…

This string differs from every string in the table.

This string cannot be in the table.

Contradiction!

Diagonalization

(Cantor's Technique to show that $\mathbb{R}$ is uncountable)

Let A be an finite alphabet of symbols with at least 2 elements {**a**,**b**,…}
Look at all strings over A, including infinitely long strings.
Assume that this set of strings is countable.
Then let $S_0, S_1, \ldots, S_n, \ldots$ be a listing of all these strings.
List all strings in rows in an infinite 2-D matrix.
Each string S_n fills one row of the matrix.
$S_n = (a_{n0}, a_{n1}, \ldots a_{nn}, \ldots)$
Now construct the following string (this is the diagonal):
$S = (a_{00}, a_{11}, \ldots a_{nn}, \ldots)$
Now modify S to give S'.
$S' = (f(a_{00}), f(a_{11}), \ldots f(a_{nn}), \ldots)$
where f(x) is a function that changes the symbol to a different symbol.
Example: if x=**a** then f(x) is **b**; if x≠**a** then f(x) is **a**.
S' cannot be anywhere in the matrix, since it will differ from every string by at least one symbol.
But we have listed all elements in the matrix.
Contradiction! The set must be **uncountably infinite**!

Application to Real Numbers

Just consider the interval [0.0, 1.0].
All numbers have a decimal representation.
  0.025
  0.31415926536… = Pi/10
  0.99999999999… = 1.0
Irrational numbers have an infinitely long decimal expansion.

Apply Cantor's Diagonalization Method:
  Alphabet={0,1,2,3,4,5,6,7,8,9}
  Assume the set is countable.
  List these numbers in a table in any order.
  Look at the digits on the diagonal.
  Construct a new number that differs from each number.
  This new number can't be in the table.
  Proof by contradiction: Therefore the set must be uncountable!

Application to Powersets

Consider a countably infinite set, S.

$S = \{ a_0, a_1, \dots \}$

power(S) = the set of all subsets of S.

Is power(S) countable?

Represent a subset of S as a bit string.

$\{a_0, a_1, a_2, \quad , a_4, \quad , a_6, \quad , a_8 \}$

1 1 1 0 1 0 1 0 1 0 0 0 0 …

1 if the element is in the subset; 0 if it is not.

Each subset corresponds to an infinitely long bit string.

Is |power(S)| = |set of bitstrings| countable?

Apply Cantor's Diagonalization Method:

Alphabet={0,1}

Assume the set is countable.

List these numbers in a table in any order.

Look at the bits on the diagonal.

Construct a new bitstring that differs from all others.

This new sitstring can't be in the table.

Proof by contradiction: Therefore power(S) must be uncountable!

Cantor's Result:

$|A| < |\text{power}(A)|$ for any set A

$$\underset{\textbf{Countable}}{|\mathbb{N}|} < \underset{\textbf{Uncountable}}{|\text{power}(\mathbb{N})|} < \underbrace{|\text{power}(\text{power}(\mathbb{N}))| < \ldots}_{\textbf{Beyond Uncountable!!!}}$$

Example: Show that $|[0,1]| = |\text{power}(\mathbb{N})|$

We can represent each number in [0,1] as a binary string.

$0.8125 = 0.11010000\ldots$

We can also look at that string as a representation of a subset of $\mathbb{N}$:

$11010000\ldots = \{0,1,3\}$

There is a bijection between [0,1] and power($\mathbb{N}$).

So $|[0,1]| = |\text{power}(\mathbb{N})|$

The Continuum Hypothesis

Is there a set whose cardinality is between that of $\mathbb{N}$ and the reals $\mathbb{R}$?

That is: does there exist a set S such that

$$|\mathbb{N}| < |S| < |\mathbb{R}|$$

No one knows!

The hypothesis is “no”.