

LOGIC

SECTION  
6.2

First Order Predicate Logic

" $\forall x \exists y [x > y \Rightarrow y < x]$ "

FORMULAS

Strings with a certain syntax

UNIVERSE

RELATIONS

MODEL

Truth of a formula

SOME FORMULAS ARE TRUE

How to Prove?

AXIOMS, RULES OF INFERENCE,

LOGICAL DEDUCTION.

CAN WE AUTOMATE THIS PROCESS?

IS THE SET OF TRUE FORMULAS  
DECIDABLE?

## FIRST-ORDER PREDICATE LOGIC

$$\forall q \exists p \forall x, y [p > q \wedge x, y > 1 \Rightarrow x \cdot y \neq p]$$

"There are infinitely many prime numbers."

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$$\forall a, b, c, n [a, b, c > 0 \wedge n > 2 \Rightarrow a^n + b^n \neq c^n]$$

"Fermat's Last Theorem:  
 $a^n + b^n = c^n$  has no integer solutions  
for  $n > 2$ ."

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$$\forall q \exists p \forall x, y [p > q \wedge (x, y > 1 \Rightarrow (x \cdot y \neq p \wedge x \cdot y \neq p + 2))]$$

"The TWIN PRIME CONJECTURE:  
There are infinitely many  
prime pairs, eg, 29, 31."

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ALPHABET:

$$\Sigma = \{ \forall \exists ( ) \wedge \vee \Rightarrow \neg x R_1 R_2 \dots \}$$

BOOLEAN OPERATIONS

$\wedge$  AND

$\vee$  OR

$\neg$  NOT

$\Rightarrow$  IMPLIES

EBCDIC  
 $\sim \bar{A}$

QUANTIFIERS

$\forall$  FOR ALL

$\exists$  THERE EXISTS

VARIABLES

x y z ...

~~x<sub>0</sub>~~ x<sub>1</sub> x<sub>2</sub> x<sub>3</sub> ...

xx xxx xxxx xxxxx

RELATIONS

R<sub>1</sub> R<sub>2</sub> ... R<sub>N</sub>

$\leq + \neq \times^n$

$x + y = z \equiv R_{add}(x, y, z)$

# FORMULAS

## SYNTAX

- Each Relation symbol must have the right # of args.

$R_{ADO}(x, y, z, w)$  ← NOT OKAY.

- A FORMULA is.. #

- A relation ("atomic formula")

$R(x, 4, 7)$

- OF THE FORM:

$F_1 \wedge F_2$

$F_1 \vee F_2$

$F_1 \Rightarrow F_2$

$\neg F_1$

- $\forall x [F_1]$

$\exists x [F_1]$

$(F_1)$

} Where  $F_1$   
and  $F_2$   
are smaller  
formulas.  
and  $x$  is  
any variable.

$\forall x. (---)$   
 $\exists y. (---)$

$\neg \equiv \sim$

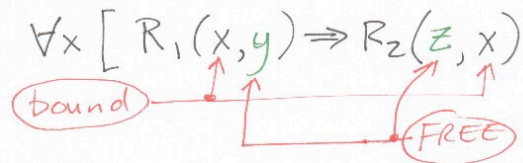
CAN WE CHECK IF A STRING IS  
A LEGAL FORMULA?

(WFF = "Well Formed Formula")

Sure. Decidable. Simple Parsing  
Problem.

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Free Variables.



**DEFN**

A "STATEMENT" is a formula  
with NO free variables.

All variables are quantified.

$\forall x \exists y. R(x, y)$  ← STATEMENT

$\forall x. R(x, y)$  ← NOT A  
"STATEMENT" 5



ALGEBRAIC MANIPULATIONS  
(Don't change meaning)

$$\sim \exists x. P \equiv \forall x. \sim P$$

$$\sim \forall x. P \equiv \exists x. \sim P$$

$$\sim (P \wedge Q) \equiv \sim P \vee \sim Q$$

etc.

PRENEX FORM

Without loss of generality...

Assume all statements in  
PRENEX form.

All quantifiers are at the front.

$$\forall x \exists y \forall z. \left( \begin{array}{c} \vee \wedge \neg \rightarrow \\ \dots x, y, z \dots \end{array} \right)$$

WHAT DOES A STATEMENT "MEAN"?

How do we INTERPRET it?

Is it TRUE or FALSE?

WE NEED...

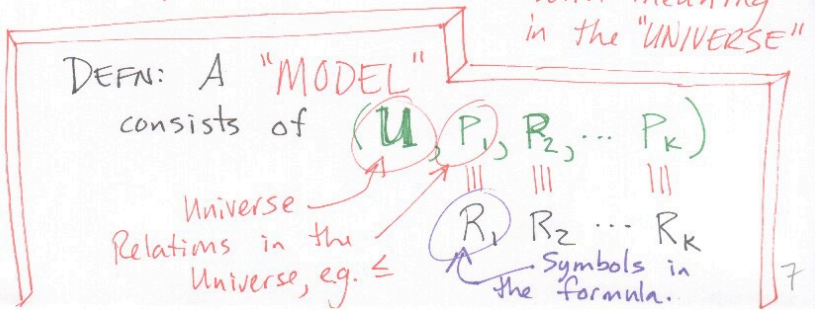
- A UNIVERSE eg:  $\mathbb{N} = \{0, 1, 2, \dots\}$
- INTERPRETATIONS FOR THE RELATION SYMBOLS.

$$R_1(x, y) \equiv x \leq y$$

$$R_2(x, y, z) \equiv x + y = z$$

Symbols in the alphabet

Relations with meaning in the "UNIVERSE"



TRUTH

IS A GIVEN STATEMENT TRUE?  
mmmm

Must specify which Model!

YES (eg: Statement of Infinite # of Primes)

NO

MAYBE/UNKNOWN (eg: Twin Prime Conjecture)

EXAMPLE

$$\forall x, y, z. (R(x, y) \wedge R(y, z) \Rightarrow R(x, z))$$

TRUE

INTERPRETATION #1:  $R \equiv <$   $U \equiv \mathbb{N}$

$$\forall x, y, z (x < y \wedge y < z \Rightarrow x < z)$$

FALSE

INTERPRETATION #2:  $R \equiv \text{succ}$   $U \equiv \mathbb{N}$

$$\forall x, y, z (x+1=y \wedge y+1=z \Rightarrow x+1=z)$$



TAUTOLOGY

Always true  
... IN ANY MODEL.

$$\forall x. (R_1(x) \wedge R_2(x) \Rightarrow R_1(x))$$

## TRADITIONAL APPROACH TO MATH PROOFS

### Axioms

A given set of statements assumed to be true without proof.

### Rules of inference/deduction

A set of rules for transforming one statement into another.

Preserve truth.

Each rule is an algebraic (i.e., Algorithmic, Computable) procedure.

### Proof

A sequence of statements from axioms to theorems, using only the rules of inference.

## Theorems

- Proof is found (creative search)
- Proof is verified (computer?)  
to make sure it is legit.

## Interpretation?

"Intuition" about the model may guide the search for proofs.

Or: The search can be conducted  
WITHOUT ANY UNDERSTANDING.

The symbols in any statement  
are just that:  
MEANINGLESS SYMBOLS.

## Validity of this approach?

How else can one  
define "TRUTH"?

CONSIDER THE TWIN PRIME CONJECTURE.

$x$  and  $x+2$  are both prime.

There are an infinite #  
of these twin primes.

Either it is TRUE or it is NOT TRUE.

Let's fix the Universe and  
interpretation of symbols.  
(the Model).

EXAMPLE

UNIVERSE =  $\mathbb{N} = \{0, 1, 2, \dots\}$

Relations:  $+ - * < = \geq \dots$

[This is "Number Theory"]

Let's ask about the  
Set of true statements  
(It's a language, after all!)

### DEFN

Given a particular model,  $\mathcal{M}$ ,  
[eg: Numbers, +, \*, ...]  
the set of true statements  
is called "The THEORY of  $\mathcal{M}$ ."

$\text{Th}(\mathcal{M})$

[EXAMPLE: ~~#~~ Number Theory

$\text{Th}(\mathbb{N}, +, *)$

is the set of all True statements  
you can make using  
+ \* = etc.

**NOTE:** THIS IS NOT NECESSARILY  
THE SET OF PROVABLE STATEMENTS!  
(eg With AXIOMS, Rules of inference).



### THEOREM

If we limit ourselves to statements we can make only using  $+$  [that is, without  $*$ ]  
Then, the set of true statements is decidable.

$\text{Th}(\mathbb{N}, +)$  is DECIDABLE

Given a statement, there is a procedure to tell whether it is true or not.

$$\forall x, y, z, a, b, c \left( (x+y=z \wedge x+x=a \wedge y+y=b \wedge z+z=c) \Rightarrow a+b=c \right)$$

$$\begin{array}{r} x+y = z \\ x+y = z \\ \hline a+b = c \end{array}$$

$Th(\mathbb{N}, +, *)$

**THEOREM**

Number Theory is Undecidable.

Considering the Universe to be

$$U = \mathbb{N} = \{0, 1, 2, \dots\}$$

and limiting ourselves to simple operations like  $+$  and  $*$ , the set of true statements is undecidable.

**PROOF IDEA**

REDUCE ATM TO THE PROBLEM OF DECIDING  $Th(\mathbb{N}, +, *)$ .

# KURT GÖDEL'S INCOMPLETENESS THEOREM

## FORMAL PROOF

A SEQUENCE OF STATEMENTS.  
STARTING WITH AXIOMS.  
USING PRECISE RULES OF INFERENCE.  
ENDING WITH THEOREM.

PROOF OF  $\phi$ :

$\pi = (S_1, S_2, \dots, S_L = \phi)$

A STATEMENT  
(NOT  $\{\}$ )

## ASSUMPTION: CORRECTNESS

PROOFS CAN BE CHECKED/VERIFIED.

$\{\phi, \pi \mid \pi \text{ is a proof of } \phi\}$   
is decidable

## ASSUMPTION: SOUNDNESS (CONSISTENCY)

IF A PROOF EXISTS,  
THEN THE STATEMENT IS TRUE.

### THEOREM

The set of provable statements  
in Number Theory  
 $\text{Th}(\mathbb{N}, +, *)$   
is Turing Recognizable.

So we can enumerate all  
the PROVABLE statements

### PROOF

- Finite Set of Axioms.
- Finite Set of Rules-of-inference.
- Each proof is finite in length.
- Each formula is finite in length.
- Just start listing them all out.  
→ Enumerate all Proofs.

## THEOREM

Some statement  $\psi$   
is true but  
is not provable!

[Some statement in  $\text{Th}(\mathbb{N}, +, \times)$   
has no proof.]

"Simple arithmetic may contains  
truths which are inaccessible."

"Number Theory is way-deep."



## Proof (By Contradiction)

Assume all true statements are provable.

Here is an algorithm to DECIDE the truth of a statement:

Look for a proof of  $\phi$ .

Look for a proof of  $\neg\phi$ .

Do these searches simultaneously, in parallel.

One or the other will be true.

Eventually we'll find a proof of either  $\phi$  or  $\neg\phi$ .

But we know that Number Theory is Undecidable!  
(We can't decide the TRUTH of statements.)

CONTRADICTION!

IDEA:

"This sentence is not provable."

$\psi_{\text{UNPROVABLE}}$  = an encoding of  
this sentence into a statement  
in Number Theory.

Is it true?

Find a proof  $\Rightarrow$  contradiction!

Therefore, it must not be  
provable.

Therefore it is true!

PROBLEM:

The statement contains  
"this sentence".

SOLUTION:

The RECURSION THEOREM!