Epipolar Fields on Surfaces

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Abstract. The view lines associated with a family of profile curves of the projection of a surface onto the retina of a moving camera defines a multi-valued vector field on the surface. The integral curves of this field are called epipolar curves and together with a parametrization of the profiles provide a parametrization of regions of the surface. This parametrization has been used in the systematic reconstruction of surfaces from their profiles. We present a complete local investigation of the epipolar curves, including their behaviour in a neighbourhood of a point where the epipolar parametrization breaks down. We also study the epipolar constraint in the image, and the geometry of the view lines. These results give a systematic way of detecting the gaps left by reconstruction of a surface from profiles. They also suggest methods for filling in these gaps.

1 Introduction

Consider a surface M and centres of projection (camera centres) $c(t)$ moving on a curve which lies outside M [2]. For a fixed t, the *critical set* (or **contour generator**) Σ_t is the set of points $\mathbf{r}$ of M where the normal is perpendicular to the line segment ('viewline') from $c(t)$ to $\mathbf{r}$. The critical set is then projected along the visual rays onto a unit sphere centred at $c(t)$ to give the **profile** points $c(t) + p$ in this sphere (the 'image sphere'). Thus $\mathbf{p}$ is regarded as a unit vector giving the direction of the viewline. We have the basic equation

$$r = c(t) + \lambda p \tag{1}$$

where λ is the (positive) distance from $\mathbf{c}$ to $\mathbf{r}$ (the distance from the profile point $c + p$ to $\mathbf{r}$ being $\lambda - 1$). (One may also consider rotated image coordinates $\mathbf{q}$, where $p = Rq$, R being a rotation with $R(0) = \text{identity}$ [2].)

The ideal situation is when M can be parametrized (locally) by t and another variable u, say, such that r_u is along the tangent to the critical set at $\mathbf{r}$ (i.e., 'u parametrizes Σ_t') and r_t is along the viewline at $\mathbf{r}$ (i.e., $r_t || p$). Then we

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say that M is given the **epipolar parametrization**. On M, successive critical sets, for t and $t + \delta t$, are matched by constant u, and, in the image, successive profiles are matched by the **epipolar constraint**. See [2]. In [10], it is shown that the advantage of using the epipolar correspondence for defining a correspondence between points on two or more profiles is that the reconstruction can be transformed readily into an optimal estimation problem.

In this paper we examine what happens in a neighbourhood of points where the epipolar parametrization breaks down. At each point $\mathbf{r}$ of a critical set Σ_t we can draw a tangent vector to M in the direction $r - c$ of the viewline. This gives a (possibly multivalued) vector field on the **visible region** of M, which is swept out by the critical sets as t varies. The integral curves of this vector field are curves tangent to the viewlines and are called **epipolar curves** on M. The vector field is called the **epipolar field** on M. See Fig. 1. In order to examine this vector field, we pass to the **spatio-temporal surface** $\widetilde{M}$ (§2), where the vector field becomes single-valued.

Fig. 1. Two critical sets Σ_1, Σ_2 corresponding to camera centres c_1, c_2, and the epipolar field along them. Note that at the point of intersection, the field is two-valued.

For the epipolar parametrization to be possible, we must firstly have t as one allowable parameter on M (so the $t =$ constant curves are critical sets on M). This says that the critical sets are smooth and do not form an *envelope*.

Definition 1. The envelope of critical sets on M is called the **frontier** of M (relative to the given motion). See Fig. 2. Provided the critical sets are smooth, frontier points $\mathbf{r}$ can also be recognised by the condition $c_t.n = 0$, $\mathbf{n}$ being a non-zero normal to M at $\mathbf{r}$. Assuming c_t is not along the viewline, this is the same as saying that the **epipolar plane**, spanned by c_t and the viewline, is the tangent plane to M at $\mathbf{r}$. See [5].

Note that the frontier is, at least locally, the boundary of the visible region swept out by the critical sets on M.

Secondly, we must have the critical sets transverse (non-tangent) to the epipolar curves, i.e. to the viewlines.

Fig. 2. The frontier F: envelope of critical sets on M (i) the generic case and (ii) the case of a parabolic point of M.

Proposition 2. *The epipolar parametrization can always be used on M except in the following circumstances:*
(i) At frontier points.
(ii) When the profile is singular. *This means that the critical set is tangent to the viewline (and also that the viewline is* asymptotic *at* $\mathbf{r}$*). For an opaque surface the profile appears as an* ending*; for a transparent surface as a* cusp.

Proof. See [5].

We therefore have to consider the cases (i) and (ii) of the above Proposition, including the generic possibility that *both* occur: a frontier point $\mathbf{r}$ can give rise to a singularity of the profile. The latter is by far the most complicated case, and we mention it briefly in §5 below; for full details see [5]. The cases (ii) at non-frontier points are covered in §4, and for (i) we need to introduce an auxiliary surface, the spatio-temporal surface, which we do now.

2 Epipolar curves and the spatio-temporal surface

In order to examine the epipolar curves near the frontier we need to introduce an auxiliary surface (compare [6]).

Definition 3. Let M be a smooth surface, defined locally by a parametrization $(u, v) \rightarrow r(u, v)$. The **spatio-temporal surface** $\widetilde{M}$ is defined to be the surface in R^3 (coordinates u, v, t) given by the equation

$$(r(u, v) - c(t)).n(u, v) = 0, \tag{2}$$

where $n(u, v)$ is a nonzero normal vector at the point $r(u, v)$ of M.

Thus the equation for $\widetilde{M}$ is identical with the equation for the critical sets, except that here we spread out the critical sets in the t direction. The surface $\widetilde{M}$ is smooth unless $\mathbf{r}$ is parabolic *and* $r - c$ is asymptotic *and* $\mathbf{r}$ is a frontier point (the case of 'lips/beaks on the frontier', which is non-generic). See [7, p.458] for information on lips/beaks, and also §4 below.

There is a natural projection π from $\widetilde{M}$ to M, given by

$$\pi : \widetilde{M} \to M, \pi(u, v, t) = r(u, v),$$

and we can 'lift' the critical sets, the frontier and the epipolar curves from M to $\widetilde{M}$:

Lifted critical set $\widetilde{\Sigma}_t$ is given by intersecting $\widetilde{M}$ with the plane $t =$ constant;
Lifted frontier $\widetilde{F}$ is the set of points of $\widetilde{M}$ satisfying $c_t.n(u, v) = 0$;
Lifted epipolar curve is an integral curve of a vector field on $\widetilde{M}$ which associates to $(u, v, t) \in \widetilde{M}$ a vector projecting under π to a nonzero multiple of the viewline vector $r - c$.

Proposition 4. *(i) The lifted critical set $\widetilde{F}$ and the lifted frontier F are tangent at a point of intersection if and only if either c_t is parallel to $r - c$, or $\mathbf{r}$ is a parabolic point of M;*
(ii) An epipolar field on $\widetilde{M}$ has the form

$$\left((c_t.n)\left(\frac{[r-c, r_v, n]}{||n||^3}\right), (-c_t.n)\left(\frac{[r-c, r_u, n]}{||n||^3}\right), -II(r-c, r-c)\right) \qquad (3)$$

Here, II is the second fundamental form of M (see e.g. [7, 8]), and $n = r_u \times r_v$ is normal to M.

The proofs are a matter of examining the equations which give $\widetilde{F}$, $\widetilde{\Sigma}_t$ and $\widetilde{M}$. See [5].

Notes on Proposition 4

1. (i) explains the structure of the critical sets at a parabolic point on the frontier, already sketched in Fig. 2, (ii). A schematic sketch of the projection π from $\widetilde{M}$ to M in this case is shown in Fig. 3, (ii). Note that the lifted critical sets $\widetilde{\Sigma}_t$ move from not meeting $\widetilde{F}$, to touching it, to meeting it twice, corresponding to the critical sets Σ_t not meeting F, then touching it once with high contact, then touching it twice.
2. The condition $c_t || r - c$ appearing in (i) means that the motion of the camera centre is *directly towards* the point $\mathbf{r}$ on M. Such points are automatically frontier points (compare Definition 1). As to the behaviour of the epipolar curves and critical sets, this case is completely analogous to that of a parabolic point on the frontier (Figs. 2, 3).
3. It is a standard fact of surface geometry (see e.g. [2, Eq.(9)]) that $\text{II}(v, v)$, for a tangent vector $\mathbf{v}$, is just the sectional curvature of M in the direction $\mathbf{v}$, scaled by $||v||^2$. Thus, in our case, the term $\text{II}(r - c, r - c)$ in (3) can be rewritten κ^t/λ^2 where κ^t is the 'transverse curvature', i.e. the sectional curvature of M in the direction of viewing. Both quantities here can be measured from the image; see [2, §4].
4. In particular, the last term of (3) is zero precisely for the case of an asymptotic viewing direction.

5. The first two terms of (3) are zero if and only if $\mathbf{r}$ is a frontier point of M. This means that, at frontier points, the epipolar curve of $\widetilde{M}$ has a 'vertical' tangent (parallel to the t-axis), which means that the epipolar curve on M is *singular* at a frontier point. See Fig. 3.
6. *All three* terms in (3) are zero if and only if $\mathbf{r}$ is a frontier point which also gives a singular profile. In that case the epipolar field on $\widetilde{M}$ (and hence on M) actually has a singularity. This makes the analysis of this case much more difficult; see §5.

Fig. 3. Projection from $\widetilde{M}$ to the parameter space of M showing the $\widetilde{\Sigma}_t$ and Σ_t (thin solid lines), F and $\widetilde{F}$ (thick lines), and epipolar curves E on M, $\widetilde{E}$ on $\widetilde{M}$ (dashed lines), (i) at a generic point of the frontier; (ii) at a parabolic point of the frontier

3 Example: the paraboloid

Before going on to examine the exceptional circumstances of Proposition 2 we give a simple example which illustrates many of the ideas above. Full details of the calculations are in [5].

Consider the paraboloid surface M with equation $z = x^2 + y^2$, parametrized by $r(u, v) = (u, v, u^2+v^2)$, so that $n(u, v) = (-2u, -2v, 1)$ is a (non-unit) normal to M. Consider the path of camera centres $c(t) = (1, t, t^2)$. The spatio-temporal surface $\widetilde{M}$ has equation $f(u, v, t) = 0$, from (2), where

$$f(u, v, t) = (u-1)^2 + (v-t)^2 - 1, \tag{4}$$

which is a slanted cylinder whose horizontal ($t =$ constant) cross-sections are all circles. Under the projection to the (u, v) plane, which parametrizes M, these project to circles centred on the u-axis.

The frontier F is the envelope of these circles (we can think of them in the parameter plane or, of course, raised up onto M using the parametrization

$r(u,v))$, which is the two lines $u=0, u=2$ (eliminate t between $f=0$ and $f_t = 0$). The visible region on M is the part parametrized by the strip $0 \le u \le 2$. On $\widetilde{M}$ the lifted frontier $\widetilde{F}$ is the two lines $\{(0,t,t) : t \in IR\}$ and $\{(2,t,t) : t \in IR\}$ 'above' F.

The epipolar field on $\widetilde{M}$ can be taken as

$$((u-1)(v-t), (v-t)^2, 1).$$

To find the epipolar curves on $\widetilde{M}$ we want the solutions of the differential equation

$$\frac{dv}{dt} = (v-t)^2.$$

The solution is $v = t - \tanh(t+k)$ for any constant k. There are two 'exceptional' solutions, namely $v = t \pm 1$, which correspond to '$k = \mp\infty$'. Using the equation $f = 0$ (see (4)), the corresponding solutions for u are $u = 1 \pm sech(t+k)$. The exceptional solutions for v both give $u = 1$. So the epipolar curves on $\widetilde{M}$ are (for any constant k)

$$\begin{aligned} (u,v,t) &= (1 \pm sech(t+k), t - \tanh(t+k), t); \\ (u,v,t) &= (1, t \pm 1, t). \end{aligned} \tag{5}$$

Note that these curves are always nonsingular and are necessarily transverse to the 'lifted critical sets' $\widetilde{\Sigma}_t$, which are given by $t =$ constant. This says that we can always parametrize $\widetilde{M}$ locally with a coordinate grid consisting of the $\widetilde{\Sigma}_t$ and the epipolar curves: 'the epipolar parametrization always works (locally) on $\widetilde{M}$.'

The frontier is given by $c_t.n = 0$, where $c_t = (0,1,2t)$ and $n = (-2u, -2v, 1)$. The epipolar field on M is obtained by projection from $\widetilde{M}$, so of course it becomes zero on the frontier, since $v = t$ there. The epipolar curves on M are obtained by treating the first and second components in (5) as parametrizations with respect to t. For example, consider the curve which, at $t = 0$, passes through $u = v = 0$. This is the curve

$$u = 1 - secht, v = t - \tanh t,$$

which has initial terms in its MacLaurin expansion

$$u = \frac{1}{2}t^2 + \ldots,\ v = -\frac{1}{3}t^3 + \ldots.$$

This curve, like all the epipolar curves on M apart from the 'exceptional' curve $u = 1$, has an ordinary cusp where it meets the frontier. (The exceptional curve does not meet the frontier.) The shape of the epipolar curves in M is shown in Fig. 4. Of course, in this example there are no parabolic points on M, nor singular points on the profile.

Fig. 4. Epipolar curves for the paraboloid example

4 Special non-frontier points

These are the generic cases:

(a) An epipolar direction at $\mathbf{r}$ is asymptotic, making the profile singular. Special cases of this are:
(b) The point $\mathbf{r}$ is parabolic and one of the epipolar directions at $\mathbf{r}$ is asymptotic (creating a 'lips/beaks' transition on the profiles) [7, p.458];
(c) One of the epipolar directions at $\mathbf{r}$ is asymptotic with four-point contact (creating a 'swallowtail' transition on the profiles) [7, p.458].

As the camera moves, the point on the surface which generates the singularity on the profile traces out a curve called the *cusp trajectory* on M. In the general case (a) above (Fig. 5), the critical sets are smooth and transverse to the cusp trajectory. A non-smooth critical set occurs in case (b) and a tangency between a smooth critical set and the cusp trajectory occurs in case (c). In both (b) and (c) two cusps are in the act of appearing or disappearing.

Fig. 5. Local pattern of critical sets (solid lines) and epipolar curves (dashed) in M or $\widetilde{M}$ for the case of a profile with a cusp

The cusp trajectory is a component of the *natural boundary*, which separates the self-occluded points from the rest of the surface. Wheras the frontier separates points which appear in profile on a transparent M from those that do not, the natural boundary separates those points which actually do appear in profile on an opaque surface from those that are obscured by another part of the surface. Thus, this type of boundary can only occur for non-convex objects or configurations of objects. The natural boundary can terminate at lips, beaks and swallowtail transitions.

For the 'lips/beaks' case, the critical set itself is singular, so it cannot be part of a parametrization. However, the epipolar curves are non-singular. Thus, it is necessary to find another family of curves transverse to the epipolar curves. The cusp trajectory is transverse to the epipolar curves, so there is a parametrization such that one family of curves is the epipolar family and the other contains the cusp trajectory which is (locally) the whole natural boundary (see Fig. 6).

A swallowtail point occurs when the tangent ray has order of contact four at a hyperbolic point, i.e. there are nearby tangents intersecting the surface at four points. This occurs along a flecnodal curve on the surface, and the camera center must lie on an asymptotic ray [7, p.448]. In general, the camera trajectory will only intersect the asymptotic developable surface of this flecnodal curve at isolated points. For opaque surfaces, a cusp trajectory and a t-junction trajectory will end at a swallowtail point. These two curves form the natural boundary (see Fig. 7.)

Fig. 6. Lips/beaks: local picture in M or $\widetilde{M}$ of the critical sets (solid), epipolar curves (dashed) and natural boundary NB on M for (i) lips and (ii) beaks transition. Lines on one side of NB are occluded for an opaque surface.

Fig. 7. Swallowtail: local picture in M or $\widetilde{M}$ of critical sets (solid), epipolar curves (dashed) and natural boundary for a swallowtail transition. Here, the natural boundary has two parts, T = locus of T-junctions, and C = locus of cusps. Everything between (say) the left branch of T and the right branch of C is occluded.

5 Frontier points

The pattern of epipolar curves and critical sets, on M and on $\widetilde{M}$, at ordinary and parabolic frontier points has been described in §2 and illustrated in Figures 2, 3. The remaining case from Proposition 2 is that of a frontier point giving a singular profile point, and (see Note 6 on Proposition 4) a zero of the epipolar field on $\widetilde{M}$. We shall not give the full details here (see [5]) but recall that there are three generic possibilities for the local structure of integral curves around such a zero: the **focus, saddle** and **node** (see for example [9, Ch.4] or any book on elementary differential equations). The corresponding pattern of integral curves on surfaces with boundary (such as the visible part of M) was found by Davydov in [3], and in Fig. 8 we show, by way of example, the situation in M for the focus case. The invariant which distinguishes the three cases, and further details, are in [5].

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Fig. 8. Pattern of epipolar curve (dashed) and critical sets (thin solid) round a focus singularity on the frontier (thick solid) (i) in $\widetilde{M}$; (ii) in M.

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