Breadth Exam in Numerical Analysis (Fall 2006)

NOTE: This is a two hour closed book exam. Please write on one side of each answer sheet only. Please mark you answer sheets with consecutive numbers. Please do not write your name on any page of the answer sheet, just write your assigned code.

1. Suppose that the partitioned matrix
$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$
is orthogonal, where the submatrices A and C are square. Prove that A and C must be orthogonal, and $B = 0$.

2. How to use Newton's method to approximate the 5th-root of a positive number? Write the general iteration step and suggest a good initial guess to guarantee convergence, and explain your choice.

3. Consider the following two methods for approximating the second derivative of a function f at a point x:

   (a) Evaluate the finite difference quotient
   $$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

   (b) Interpolate f at the points $x - h$, x, and $x + h$ by a quadratic polynomial $p(x)$ and then evaluate $p''(x)$.

   Do these two methods produce the same results? Why?

4. If $Q(f) = \sum_{i=1}^{n} w_i f(x_i)$ is an interpolatory quadrature rule (i.e., based on polynomial interpolation) on the interval $[0, 1]$, then is it true that $\sum_{i=1}^{n} w_i = 1$? Prove your answer.

5. Let A be a tridiagonal matrix of the form
$$\begin{bmatrix} d_1 & c_1 & & & & & & \\ a_1 & d_2 & c_2 & & & & & \\ & a_2 & d_3 & c_3 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & a_{i-1} & d_i & c_i & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & a_{n-2} & d_{n-1} & c_{n-1} \\ & & & & & & a_{n-1} & d_n \end{bmatrix}$$
such that $a_i c_i > 0$ for $1 \le i \le n-1$. Find the general form of the diagonal matrix $D = \mathrm{diag}(\alpha_i)$ with $\alpha_i \neq 0$ such that $D^{-1}AD$ is symmetric. What is the general form of $D^{-1}AD$?