Examination “Breadth” – Theory

August 31, 2006

SOLVE EACH PROBLEM ON A SEPARATE PAGE. WRITE ONLY ON ONE SIDE OF A PAGE. WRITE YOUR CODE (see the last page) AT THE TOP OF EACH ANSWER SHEET.

1. Provide definitions of recursively enumerable and recursive classes of languages. Give two examples of languages in each of these classes. What is the relationship between these classes?

2. Provide the pumping lemmas for regular languages and for context-free languages. Demonstrate how they can be used to prove that the language $L = \{a^n b^n c^{2n} \;:\; n \geq 1\}$ is not context-free.

3. The search sequence $s(x)$ for a key value x in a Binary Search Tree T is a list of labels along the search path (in the order of the visited nodes) when T is searched for x using the standard search algorithm for Binary Search Tree.

   Design a procedure that given a sequence $y_1, \ldots, y_k, x$ of integers decides if it is a viable (i.e., possible) search sequence for some Binary Search Tree. Provide narrative description, pseudocode, proof of the correctness and the running time analysis.

4. Let T be a binary tree with $n = 2^k - 1$ nodes including exactly 2^{k-1} leaf nodes, with all the nodes labeled with integer numbers.

   Design an algorithm that copies the labels of T into array $B[1..n]$ in such a way that the label of the root is placed in $B[1]$, the children of the root (from left to right) are placed in $B[2]$ and $B[3]$, the third level of T, from left to right, is placed in $B[4]$, $B[5]$, $B[6]$ and $B[7]$, etc.

   (a) Provide narrative description, pseudocode, the running time analysis and justification of the correctness. The grade will depend on the presentation, correctness and the running time of the algorithm.

   (b) Answer the following question: If the label of a node is stored in $B[i]$, where are the labels of its children stored?

5. Let $G = (V, E)$ be an undirected graph, with no loops. For every vertex $v_i \in V$ we assign a propositional variable p_i and form the following propositional formula F_G

$$\bigwedge_{\{v_i, v_j\} \notin E,\, v_i \neq v_j} \neg p_i \vee \neg p_j.$$

(Observe that we take conjunction over the edges of the *complement graph* of G.) Now, given a set $S \subseteq V$ we define a propositional valuation a_S by the following formula

$$a_S(p_i) = 1 \text{ if } v_i \in S \text{ and } 0 \text{ otherwise.}$$

Prove the following: If $S \subseteq V$ is a clique in G then a_S satisfies F_G. (Note: the implication in the opposite direction is also true.)

Write a definition of a clique in a graph. Include this definition as the first part of your solution.

The below statement is not a part of the problem statement - it simply recalls a definition of a valuation. Recall that a valuation is a mapping of the set of propositional atoms into Boolean, i.e. $\{0, 1\}$. A literal is a propositional atom or negated propositional atom. A valuation v satisfies a literal l if l is an atom p and $v(p) = 1$, or l is negated atom, $\neg q$, and $v(q) = 0$. A valuation v satisfies the clause $l_1 \vee \ldots \vee l_k$ if for some j, $1 \leq j \leq k$, v satisfies l_j. Furthermore, a valuation satisfies the conjunction of clauses if it satisfies each clause.

SELECT ONE OF PROBLEMS. SOLVE ONLY ONE OF THESE. WRITE ON YOUR SELECTION. DO NOT INCLUDE SOLUTIONS OF MORE THAN ONE PROBLEM, ONLY THE PROBLEM LISTED ON PAGE ONE WILL COUNT.

(The only requested topic is *Graph Algorithms*.)

6. Let $G = (V, E)$, be an undirected connected graph with no loops represented as the adjacency list for its vertices. Graph G has a special property: it has exactly one cycle. Design an efficient algorithm that prints all vertices that form this cycle in G. Provide a narrative description and pseudocode for your algorithm, justify its correctness and analyze the running time of your solution in terms of the size of G's representation. Show a small example. The grade will depend on the correctness, efficiency and presentation of your algorithm.

Theory Breadth Exam - August 31, 2006

your code:______________$F2006$

your name (print):________________________