Numerical Analysis Breadth Exam
February 11, 2004 (5:00 - 7:00)

This is a closed book exam lasting no more than two hours. There are five problems, each worth 20 points.

Problem 1. Consider a straightforward algorithm for computing

$$F(\vec{a}, \vec{b}) = \frac{\sum_{i=1}^{k} a_i}{\sum_{i=1}^{k} b_i}$$

in a *fl*-arithmetic with unit round-off u. Estimate its relative error and discuss when it is large and when it is small. Of course, assume that there is no over- nor under-flow. Is this algorithm numerically stable (explain briefly)?

Problem 2. Explain how to use Newton's method to find a positive number r such that $e^{-r} = \sqrt{r}$. The solution should contain the choice of a function f to which the method is to be applied and a good initial approximation x_0 (with explanation).

Problem 3. Consider approximating $\int_a^b f(t)\,dt$ by the quadrature $Q_{x_0}(f) = c_0 f(x_0) + c_1 f'(x_0)$ that is exact for polynomials of as high the degree as possible.

(i) Derive the coefficients c_0 and c_1. (Of course they depend on x_0.)

(ii) Provide an error formula for Q_{x_0}.

(iii) If the point x_0 could be selected, which one would you choose and why?

Problem 4. Consider solving $Ax = b$ when A is a tridiagonal matrix of order n and x and b are n-vectors.

(i) Define a tridiagonal matrix.

(ii) Develop an $O(n)$ in time and space algorithm to solve $Ax = b$.

(iii) What are the constants in the $O(n)$ expression for time and space?

(iv) How does your algorithm in (ii) differ in time and space with respect to a general matrix factorization and solution method?

Problem 5. The Lagrange interpolation problem is the following: Given certain distinct points $(x_1, x_2, ..., x_n)$ and values at the points $(f_1, f_2, ..., f_n)$, find a polynomial of degree $n-1$ such that

$$P(x_i) = f_i.$$

Define $P(x)$ either as a function or through an algorithm for the following:

(i) When there are only two points x_1 and x_2.

(ii) When there are only three points x_1, x_2, and x_3.

(iii) When there are n points (the general case).