BREADTH EXAM IN NA, SPRING 2007

PROBLEM 1: Consider the iteration scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)},$$

where $g(x) = \{f[x + f(x)] - f(x)\}/f(x)$, for solving the equation $f(x) = 0$. Using Taylor series, show that $g(x) \approx f'(x)$ if $f(x)$ is small. Compare it with the Newton's iteration scheme regarding the number of function evaluations, and find its convergence rate.

PROBLEM 2: By using Taylor's formula, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(\eta_1)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(\eta_2)$$

Therefore, we can obtain

$$\frac{1}{h^2}[f(x+h) - 2f(x) + f(x-h)] = f''(x) + \frac{h}{6}[f'''(\eta_1) - f'''(\eta_2)]$$

and, hence, the error in this approximation formula for f'' is, modulo a constant, equal to h. Is this analysis correct? Why or why not?

PROBLEM 3: Perform the Gauss elimination with the scaled partial pivoting for the following matrix A. Show the contents of A (mark multipliers) and of the permutation array in all intermediate steps. Finally, write down the corresponding matrices P, L, U ($PLU = A$).

$$A = \begin{bmatrix} 2 & 4 & 4 & 2 \\ 3 & 3 & 3 & 3 \\ 1 & 5 & 0 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

PROBLEM 4: Consider a computer that uses five-decimal-digit mantissa. Let $fl(x)$ denote the floating-point machine number closest to x. Show that if $x = 0.5321487513$ and $y = 0.5321304421$, then the operation $fl(x) - fl(y)$ involves a large relative error. Compute the relative error for this numerical example.

PROBLEM 5:

(i) Derive the formulas for the basic Trapezoid rule for approximating $\int_a^b f(x)\,dx$ and for its error.

(ii) Derive the the formulas for the corresponding composite rule and for its error. Assume that the composite rule uses $n+1$ equally spaced points.