Breadth Exam in Numerical Analysis (Fall 2004)

NOTE: This is a two hour closed book exam. Please write on one side of each answer sheet only, mark you answer sheets with consecutive numbers, and do not write your name on any page of the answer sheet, just write your assigned code. Each problem is worth 20 points. No calculator is allowed.

1. Suppose we wish to evaluate $\sqrt{x}$, but the machine representation of x is $\mathrm{fl}(x) = x(1+\epsilon)$ where the relative error ϵ is small.

   (a) Show that if the machine routine SQRT is exact, i.e., $\mathrm{SQRT}(y) = \sqrt{y}$ for a machine number y, then $\sqrt{x}$ is obtained with a small relative error.

   (b) If the routine SQRT is such that $\mathrm{SQRT}(y) = \sqrt{y}(1+\delta)$ where δ is small, can $\sqrt{x}$ still be obtained with a small relative error? why?

2. Approximate the function $f(x)$ by an interpolating polynomial of degree 2, and use the approximation to establish the formula

$$f''(x) \approx \frac{2}{h^2}\left[\frac{f(x_0)}{(1+\alpha)} - \frac{f(x_1)}{\alpha} + \frac{f(x_2)}{\alpha(\alpha+1)}\right]$$

   by using the unevenly spaced points $x_0 < x_1 < x_2$, where $x_1 - x_0 = h$ and $x_2 - x_1 = \alpha h$.

3. For a continuously differentiable function $f(x)$, show that the root-finding formula

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)},$$

   where $g(x) = \{f[x+f(x)] - f(x)\}/f(x)$, converges quadratically.

4. Let $\mathbf{z}$ be a computed solution of $A\mathbf{x} = \mathbf{b}$, where A is a nonsingular square matrix. Suppose that the components of A and $\mathbf{b}$ are of "reasonable" size, but $\mathbf{z}$ has large components. Please analyze the following two situations (with your justifications)

   (a) if $\mathbf{x}$ has no "large" components, what can be said about $\mathbf{z}$? why?

   (b) if $\mathbf{z}$ is accurate, what can be said about A? why?

5. Consider a (basic) quadrature $Q(f) = C_0 f(c) + C_1 f'(c) + C_2 f''(c)$ for approximating $\int_a^b f(x)\,dx$ that is exact for polynomials of degree at most 2. Derive the coefficients C_i and the error of Q.

   Next, derive a formula for the error of the corresponding composite quadrature that uses n-subintervals of equal lengths.