Examination "Breadth" – Theory

September 18, 2007

SOLVE EACH PROBLEM ON A SEPARATE PAGE. WRITE ONLY ON ONE SIDE OF A PAGE. WRITE YOUR CODE (see the last page) AT THE TOP OF EACH ANSWER SHEET. IF YOU HAVE ANY QUESTIONS/CLARIFICATION REQUESTS PLEASE ASK.

1. (a) Define SAT problem (a well-known NP complete problem).

   (b) Suppose, we have a polynomial time algorithm A for Boolean satisfiability: Given a Boolean formula φ, $A(\varphi)$ returns a 1 if φ is satisfiable, else returns a 0.

   Give an algorithm for finding a satisfying assignment for a formula φ that makes at most $|\varphi|$ calls to A.

2. Give a language L that is accepted by a pushdown automaton or generated by a context-free grammar, but is not regular. Prove that:

   (a) L is accepted by a pushdown automaton (or generated by a context-free grammar);

   (b) L is not regular.

3. (a) Describe how to implement a max-heap in an array. Where are the parent and children of a node located?

   (b) Give an efficient algorithm which, given a list of integers $a_1, \cdots, a_n$ builds a max-heap containing exactly these integers. Provide a narrative description and pseudocode. Analyze the running time (time complexity); prove that your algorithm achieves this time complexity.

4. Design an efficient algorithm for the following problem.

   Instance: A sequence $A = (a_1, \ldots, a_n)$ of n distinct real numbers.

   Question: What is the permutation $P = (p_1, \ldots, p_n)$ such that $a_{p_1} < a_{p_2} < \ldots < a_{p_n}$.

   Example: For $A = (6.81, 15.8, 7.22, 1.97, 11.56)$, $P = (4, 1, 3, 5, 2)$.

Provide a narrative description, pseudocode, justify correctness, analyze the running time, give a small example.

Note: Algorithms with the running time of an order $\Omega(n^2)$ will get only a small partial credit.

5. Given a graph $G = \langle V, E \rangle$, an edge coloring is an assignment of a color to each edge of G. Let G be a complete graph K_n on n vertices whose edges are colored *red* or *blue*. A set of three vertices with all edges between them colored *red* is called a *red* triangle. Similarly we define a *blue* triangle.

   (a) Prove that in any edge coloring of K_6 there is a red triangle, or a blue triangle.

   (b) Is the same true for K_5? Or, is there a coloring of K_5 such that there are neither *red* nor *blue* triangles? Justify.

SELECT ONE OF PROBLEMS. SOLVE ONLY ONE OF THESE. WRITE ON YOUR SELECTION. DO NOT INCLUDE SOLUTIONS OF MORE THAN ONE PROBLEM, ONLY THE PROBLEM LISTED ON PAGE ONE WILL COUNT.

(The only requested topic is *Graph Algorithms.*)

6. In the Dijkstra's shortest path algorithm, when a vertex u is analyzed and moved to vertices with the known shortest distances, the algorithm relaxes every outgoing edge (u, v). Describe this RELAX function, i.e., explain what it does and provide pseudocode.

   Show how to modify/augment RELAX so for each vertex v in the graph we can learn the length of the longest edge in the shortest path from the source to v.

   Provide narrative description and pseudocode. Justify that your solution is correct. Show a small example.

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your code:__________$F2007$

your name (print):____________________