Long monochromatic Berge cycles in colored 4-uniform hypergraphs

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Abstract

Here we prove that for $n \geq 140$, in every 3-coloring of the edges of $K_n^{(4)}$ there is a monochromatic Berge cycle of length at least $n-10$. This result sharpens an asymptotic result obtained earlier. Another result is that for $n \geq 15$, in every 2-coloring of the edges of $K_n^{(4)}$ there is a 3-tight Berge cycle of length at least $n-10$.

1 Introduction

Let $\mathcal{H}$ be an r-uniform hypergraph (a family of some r-element subsets of a set). The *shadow graph* of $\mathcal{H}$ is defined as the graph $\Gamma(\mathcal{H})$ on the same vertex set, where two vertices are adjacent if they are covered by at least one edge of $\mathcal{H}$. A coloring of the edges of an r-uniform hypergraph $\mathcal{H}$, $r \geq 2$, induces a multicoloring on the edges of the shadow graph $\Gamma(\mathcal{H})$ in a natural way; every edge e of $\Gamma(\mathcal{H})$ receives the color of all hyperedges containing e. We shall denote by $c(x,y)$ the color set of the edge xy in $\Gamma(\mathcal{H})$. A subgraph of $\Gamma(\mathcal{H})$ is *monochromatic* if the color sets of its edges have a nonempty intersection. Let $K_n^{(r)}$ denote the complete r-uniform hypergraph on n vertices.

In any r-uniform hypergraph $\mathcal{H}$ for $2 \leq t \leq r$ we define an r-uniform t-tight Berge-cycle of length ℓ, denoted by $C_\ell^{(r,t)}$, as a sequence of distinct vertices $v_1, v_2, \ldots, v_\ell$, such that for each set $(v_i, v_{i+1}, \ldots, v_{i+t-1})$ of t consecutive vertices on the cycle, there is an edge e_i of $\mathcal{H}$ that contains these t vertices and the edges e_i are all distinct for $i, 1 \leq i \leq \ell$ where $\ell + j \equiv j$. This notion was introduced in [5] and for $t = 2$ we get ordinary Berge-cycles ([1]) and for $t = r$ we get the tight cycle (see e.g. [11] or [15]). A Berge-cycle of length n in a hypergraph of n vertices is called a Hamiltonian Berge-cycle. It is important to keep in mind that, in contrast to the case $r = t = 2$, for $r > t \geq 2$ a Berge-cycle $C_\ell^{(r,t)}$, is not determined uniquely, it is considered as an arbitrary choice from many possible cycles with the same triple of parameters.

In this paper, continuing investigations from [5], [6], [8] and [9], we study long Berge-cycles in hypergraphs. In [5] (by generalizing an earlier conjecture from [6]) the following conjecture was formulated.

Conjecture 1.1. *For any fixed $2 \leq c, t \leq r$ satisfying $c + t \leq r + 1$ and sufficiently large n, if we color the edges of $K_n^{(r)}$ with c colors, then there is a monochromatic Hamiltonian t-tight Berge-cycle.*

In [5] it was proved that if the conjecture is true it is best possible, since for any values of $2 \leq c, t \leq r$ satisfying $c + t > r + 1$ the statement is not true. The conjecture was proved for $r = 3$ in [6]. The asymptotic form of the conjecture was proved for $r = 4$ and $t = 2$ in [6] and for every r and $t = 2$ in [9] - in both papers the Regularity Lemma was used. In this paper we apply an elementary approach and we study the $r = 4$ case. We prove the conjecture in both cases ($c = 3$, $t = 2$ and $c = 2$, $t = 3$) with a constant error term.

Theorem 1.2. *Suppose that an 3-coloring is given on the edges of $K_n^{(4)}$, where $n \geq 140$. Then there is a monochromatic Berge-cycle of length at least $n - 10$.*

This sharpens the asymptotic result obtained earlier for $r = 4$ in [6].

Theorem 1.3. *Suppose that an 2-coloring is given on the edges of $K_n^{(4)}$, where $n \geq 15$. Then there is a monochromatic 3-tight Berge-cycle of length at least $n - 10$.*

2 Proofs

Proof of Theorem 1.2. Suppose that c is a 3-coloring on the edges of $\mathcal{K} = K_n^{(4)}$, where $n \geq 140$. Color $i \in c(x, y)$ on the edge xy of $G = \Gamma(\mathcal{K})$ is a *good color* if at least 3 edges of color i contain $\{x, y\}$ in $\mathcal{K}$. We consider G with a new coloring c^* where $c^*(x, y) \subseteq c(x, y)$ is the set of good colors on xy. Assuming that $\binom{n-2}{2} > 6$, i.e. $n > 6$, every edge of $\mathcal{K}$ has at least one color in c^*.

Suppose first that some edge xy of $G = \Gamma(\mathcal{K})$ is colored (under c^*) with a single color, say with color 1. We claim that there is a Hamiltonian Berge cycle in $\mathcal{K}$ in color 1. Indeed, the definition of xy implies that under c^* at most four edges of $H = G \setminus \{x, y\}$ are not colored with 1. Since for $n > 10$ we have $n - 6 > (n - 2)/2$, the color 1 subgraph of H satisfies Dirac's condition (see [13]), and thus one can easily find a Hamiltonian path $P = \{y_1, \ldots y_{n-2}\}$ of color 1 in H such that there are two extra edges $y_1 y_p$ and $y_{n-2} y_k$ of color 1 from the endpoints of P with $2 < p, k < n - 3$. Now the cyclic ordering $x, y_1, y_2, \ldots, y_{n-2}, y$ defines a Hamiltonian Berge-cycle in color 1 with the following edge assignments. For x, y_1 assign $e_n = \{x, y_1, y_p, y\}$. For y_j, y_{j+1} $(1 \leq j \leq n - 3)$ assign $e_j = \{x, y, y_j, y_{j+1}\}$, for y_{n-2}, y assign $e_{n-2} = \{y_{n-2}, y, y_k, x\}$, and finally for x, y we can assign e_{n-1} as any edge of color 1 containing x, y and different from all other e_i-s.

Now we may assume that c^* colors all edges of G with one of the four color sets: $12, 13, 23, 123$.

Lemma 2.1. *Assume that there is a monochromatic Hamiltonian cycle C in G under coloring c^*. Then there is a Hamiltonian Berge-cycle in $\mathcal{K}$ under coloring c.*

Proof. Assume that $C = x_1, x_2, \ldots, x_n$ is a Hamiltonian cycle of G in color 1 (under c^*). Then, following the cyclic order of vertices on C, let A_j be the set of edges of $\mathcal{K}$ in color 1 containing x_j, x_{j+1}. Since each A_j has at least three elements and no element of A_j covers more than three consecutive pairs of C, Hall's theorem ensures a one-to one correspondence from the consecutive pairs to the sets A_j. This clearly defines the required Hamiltonian Berge-cycle. $\square$

We need some observations on the structure of the coloring c^*. Let x be an arbitrary vertex, define $U_{12}(x), U_{13}(x), U_{23}(x), U_{123}(x)$ as the sets to which x is connected in color sets $12, 13, 23, 123$ respectively. Define

$$B_i = \{x \in V(G) | U_{ij} = U_{ik} = \emptyset, U_{jk} \neq \emptyset\},$$

where i, j, k are the elements of $\{1, 2, 3\}$ in some order. Observe that the B_i-s are pairwise disjoint, within the B_i-s every edge of G has color set $\{j, k\}$

or 123, and for $j \neq i$, an edge of G from B_i to B_j has color set 123. Set $B_4 = \{x \in V(G) | |U_{123}(x)| \geq n/2\}$.

Lemma 2.2. *Suppose that $\cup_{i=1}^4 B_i = V(G)$. Then there is a Hamiltonian cycle G in the coloring c^*.*

Proof. Suppose w.l.o.g that $|B_1| \leq |B_2| \leq |B_3|$. We show that there is a Hamiltonian cycle in color 1. Denoting the degree of a vertex v in color i by $d_i(v)$, we have that $d_1(v) \geq |B_2|+|B_3| \geq |B_2|+|B_1|$ if $v \in B_1$, $d_1(v) = n-1$ if $v \in B_2 \cup B_3$ and $d_1(v) \geq \frac{n}{2}$ if $v \notin \cup_{i=1}^3 B_i$ (since in the latter case $v \in B_4$). These conditions immediately imply - through either Pósa's or Chvátal's condition (see [13]) that there is a Hamiltonian cycle. $\square$

Thus we may assume that there exists $x \in V(G) \setminus \cup_{i=1}^4 B_i$ (otherwise Lemmas 2.1 and 2.2 would finish the proof). Set $U = V(G) \setminus (\{x\} \cup U_{123})$ and assume w.l.o.g. $|U_{23}| \leq |U_{12}| \leq |U_{13}|$. Since $x \notin B_2$ we have $U_{12} \neq \emptyset$ and $x \notin B_4$ implies that $|U| \geq \lfloor n/2 \rfloor$.

We show that $|U_{23}| \leq 1$. Indeed, otherwise we may select two two-element sets $A_{23} \subseteq U_{23}, A_{12} \subseteq U_{12}$ and a five-element set $A_{13} \subseteq U_{13}$. (The condition $|U| \geq \lfloor n/2 \rfloor$ implies that $|U_{13}| \geq \frac{\lfloor n/2 \rfloor}{3} \geq 5$ so A_{13} can be defined.) For every fixed $u_{23} \in A_{23}$ there are at most two edges of color 1 among the edges of $\mathcal{K}$ in the form $\{x, u_{23}, x_{12}, x_{13}\}$ where $x_{12} \in A_{12}, x_{13} \in A_{13}$ are arbitrary. Repeating this argument for fixed u_{12}, u_{13} we get that there are at most $4 + 4 + 10 = 18$ edges of $\mathcal{K}$ in the form $\{x, x_{23}, x_{12}, x_{13}\}$. However, there are $2 \times 2 \times 5 = 20$ such edges giving a contradiction.

Now we fix $y \in U_{12}, z \in U_{13}$ and define a graph H on the vertices of $V(G) \setminus (U_{23} \cup \{x, y, z\})$ as follows. Let $uv \in E(H)$ be an edge of H in the following cases: (i) $u \in U_{13}, c(\{x, y, u, v\}) = 1$, in this case the edge is called an xy-edge; (ii) $u \in U_{12}, c(\{x, z, u, v\}) = 1$, now the edge is called an xz-edge. Set $|V(H)| = N$ and note that $N \geq n - 4$.

Lemma 2.3. *The graph H has a cycle C of length at least $N - 6$ in color* 1.

Proof. Set

$$T_{12} = U_{12} \cap V(H), T_{13} = U_{13} \cap V(H), T = U \cap V(H), T_{123} = U_{123}.$$

Consider an arbitrary vertex $u \in T_{12} \cup T_{13}$. Set $w = z$ if $u \in T_{12}$ otherwise set $w = y$. Apart from at most four choices of $v \in V(H)$ the edge $\{x, u, w, v\}$ of $\mathcal{K}$ is of color 1. Thus every vertex of $T \subseteq V(H)$ has degree at least $N - 5$ in H. Consider the set $S \subseteq T_{123}$ of vertices whose degrees are at most 11 in the bipartite subgraph $[T, T_{123}]$ of H. Observe that

$$|T|(|T_{123}| - 5) \leq |E[T, T_{123}]| \leq (|T_{123}| - |S|)|T| + 11|S|$$

implying that $|S| \leq 6$ if $66 \leq |T|$ and this is true since $|T| > \lfloor n/2 \rfloor - 4 > 65$. Now consider the subgraph F of H induced by $T \cup (T_{123} \setminus S)$. In fact, we may assume that $|S| = 6$ since deleting $6-|S|$ vertices does not influence the following observation: each vertex $v \in T$ has degree at least $N - 11$ in F and each vertex

$v \in T_{123} \setminus S$ has degree more than 11. Now we can apply Chvátal's condition (see [13]) to prove that there is a Hamiltonian cycle in $F \subset H$. Indeed, with $M = |V(F)|$, we have to show that $d_k \le k < \frac{M}{2}$ implies that $d_{M-k} \ge M - k$ where $d_1 \le d_2 \le \cdots \le d_M$ is the degree sequence of F. This is immediate because the number of vertices with possibly small degrees (i.e. $v \in T_{123} \setminus S$) is at most

$$|U_{123}| - 6 \le \left\lfloor \frac{n}{2} \right\rfloor - 6 \le \left\lfloor \frac{N+4}{2} \right\rfloor - 6 = \left\lfloor \frac{M+10}{2} \right\rfloor - 6 = \left\lfloor \frac{M}{2} \right\rfloor - 1. \qquad (1)$$

Indeed, let us take a k for which $d_k \le k < \frac{M}{2}$. $11 < d_k \le k$ implies that $k > 11$. But then from (1) we get

$$d_{M-k} \ge d_{\lceil \frac{M}{2} \rceil} \ge N - 11 \ge M - 11 > M - k,$$

as desired. □

To finish the proof of Theorem 1.2, observe that the cycle C obtained from Lemma 2.3 defines a Berge-cycle if its xy-edges and xz-edges are extended (with $\{x, y\}$ or with $\{x, z\}$ to edges of $\mathcal{K}$. Thus we have a Berge-cycle of length $N - 6 \ge n - 10$ as required. □

Proof of Theorem 1.3. Suppose that a 2-coloring c is given on the edges of $\mathcal{K} = K_n^{(4)}$. Let V be the vertex set of $\mathcal{K}$ and observe that c defines a 2-multicoloring on the complete 3-uniform hypergraph $\mathcal{T}$ with vertex set V by coloring a triple T with the colors of the edges of $\mathcal{K}$ containing T. We say that $T \in \mathcal{T}$ is *good in color* i if T is contained in at least two edges of $\mathcal{K}$ of color i ($i = 1, 2$).

Lemma 2.4. *Every edge $xy \in E(G)$ is in at least $n-4$ good triples of the same color.*

Proof. Consider an edge xy in G. Coloring c induces a 2-coloring c' on $W = V \setminus \{x, y\}$. Applying a result of Bollobás and Gyárfás, [2], there exists a subgraph H with at least $|W| - 2 = n - 4$ vertices such that H is 2-connected and monochromatic under c', say in color 1. In particular, every vertex of H has degree at least two in color 1. Thus, for every vertex z of H, $\{x, y, z\}$ is a good triple in color 1. □

Using Lemma 2.4, we can define a 2-coloring c^* on the shadow graph $G = \Gamma(\mathcal{K})$ by coloring $xy \in E(G)$ with the color of the (at least $n - 4$) good triples containing xy. Using a well-known result about the Ramsey number of even cycles ([4], [14]) there is a monochromatic even cycle C of length $2t$ where $2t = \lceil \frac{2n}{3} \rceil - 6$ or $2t = \lceil \frac{2n}{3} \rceil - 7$. (In fact there is a bit longer cycle, but that is too long for our purposes.) Assume that C is in color 1. Label the edges of C as $e_j = \{p_j, p_{j+1}\}$, $j = 1, 2, \ldots, 2t$. We use here index arithmetic $\mod 2t$.

We shall find a large Berge-cycle in color 1 with the following greedy procedure. By Lemma 2.4, for each $i \in [2t]$ there is a set $A_i \subset V$ such that $|A_i| \ge n-4$ and the triple $T_i = \{p_i, p_{i+1}, x\}$ is good in color 1 for every $x \in A_i$. We claim that we can find a set $\{v_j \in A_{2j-1} \setminus V(C)\}$ for $j \in [t]$ with the following property: for every $j \in [t]$,

$$v_j \in A_{2j-2} \cap A_{2j-1} \cap A_{2j}.$$

Assume that for $j \leq h < t$ we have this property and there are at least seven vertices in $S = V \setminus (V(C) \cup \{\cup_{j=1}^{h} v_j\})$. Indeed, if $|S| \geq 7$, then - because each of the three sets intersects S in at least five elements - $U = S \cap A_{2h} \cap A_{2h+1} \cap A_{2h+2} \neq \emptyset$ so we can select $v_{h+1} \in U$. Now we only have to observe that during the whole process

$$|S| \geq n - 3t \geq n - \frac{3}{2}(\lceil \frac{2n}{3} \rceil - 6) \geq 7,$$

and thus the claim is proved.

Now we finish the proof by claiming that the cyclic permutation $P = p_1, v_1, p_2, p_3, v_2, p_4, \ldots, p_{2t-1}, v_t, p_1$ determines a Berge-cycle. Indeed, from the definition of v_j, every triple of three consecutive vertices on P is good in color 1. Therefore at least two edges $\mathcal{K}$ of color 1 are available to cover a consecutive triple. However, no edge of $\mathcal{K}$ can cover more than two consecutive triples of P. Thus, by Hall's theorem, there is a matching from the consecutive triples of P to the set of color 1 edges of $\mathcal{K}$ containing them. The length of this Berge-cycle is $3t \geq \frac{3}{2}(\lceil \frac{2n}{3} \rceil - 7) \geq n - 10$. $\Box$

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