

# Multipartite Ramsey numbers for odd cycles <sup>\*</sup>

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## Abstract

In this paper we study multipartite Ramsey numbers for odd cycles. We formulate the following conjecture: Let  $n \geq 5$  be an arbitrary positive odd integer, then in any two-coloring of the edges of the complete 5-partite graph  $K((n-1)/2, (n-1)/2, (n-1)/2, (n-1)/2, 1)$  there is a monochromatic  $C_n$ ,

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a cycle of length  $n$ . This roughly says that the Ramsey number for  $C_n$  (i.e.  $2n-1$ ) will not change (somewhat surprisingly) if four large “holes” are allowed. Note that this would be best possible as the statement is not true if we delete from  $K_{2n-1}$  the edges within a set of size  $(n+1)/2$ . We prove an approximate version of the above conjecture.

# 1 Introduction

## 1.1 Ramsey numbers for odd cycles

If  $G_1$  and  $G_2$  are graphs, then the Ramsey number  $R(G_1, G_2)$  is the smallest positive integer  $r$  such that if the edges of a complete graph  $K_r$  are partitioned into 2 disjoint color classes giving graphs  $H_1$  and  $H_2$ , then one of the subgraphs  $H_i$  ( $i = 1, 2$ ) has a subgraph isomorphic to  $G_i$ . The existence of such a positive integer is guaranteed by Ramsey’s original paper [17]. The number  $R(G_1, G_2)$  is called the Ramsey number for the graphs  $G_1$  and  $G_2$ . The determination of these numbers has turned out to be remarkably difficult in certain cases (see eg. [5] or [16] for results and problems). In this paper we consider the case when each  $G_i$  is a cycle  $C_n$  on  $n$  vertices, where  $n$  is odd. A theorem obtained independently by Rosta [18] and Faudree and Schelp [2] (see also a new simple proof in [8]) states that for any  $n \geq 5$  odd positive integer

$$R(C_n, C_n) = 2n - 1. \tag{1}$$

Recently there has been some interest to see what happens to the Ramsey numbers when we allow fixed edge deletions from the complete graph  $K_r$ , in particular if we delete complete subgraphs from  $K_r$ . One result of this type appeared in [7], where we gave a tripartite version of the Gerencsér-Gyárfás Theorem, i.e. we showed that the Ramsey number for a path is about the same when two-colorings of a complete graph or a balanced complete tripartite graph are considered. Another result of this type appeared in [15], where it was shown for any odd  $n \geq 5$  that if we delete the edges of a complete subgraph of order  $(n-1)/2$  from the complete graph of order  $2n-1$  and we two-color the rest, we can still guarantee a monochromatic  $C_n$ .

In this paper along these lines we consider a multipartite version of (1). We formulate the following conjecture.

**Conjecture 1.** *Let  $n \geq 5$  be an arbitrary positive odd integer, then in any two-coloring of the edges of the complete 5-partite graph  $K(\frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}, 1)$ , there is a monochromatic  $C_n$ .*

Again this roughly says that the Ramsey number for  $C_n$  will not change (somewhat surprisingly) if four large “holes” are allowed. Note that this would be best possible

as the statement is already not true if we have an independent set of size  $\frac{n+1}{2}$  (so even one hole of size  $\frac{n+1}{2}$  is not allowed). Indeed, let us remove all the edges spanned by the set  $A$ , where  $|A| = \frac{n+1}{2}$ . Divide the vertices  $V \setminus A$  into two sets  $B$  and  $C$  with  $|B| = \frac{n-1}{2}$  and  $|C| = n - 1$ . Let the first color be all the edges within  $B$ , within  $C$  and between  $A$  and  $B$ . The second color is the remaining edges. Then it is easy to see that there is no monochromatic  $C_n$ .

Conjecture 1 holds for  $n = 5$ , but is open in general. It is the purpose of this paper to give an approximate result which gives further evidence to the truth of this conjecture. More precisely we prove the following theorem.

**Theorem 1.** *For all  $0 < \eta < 1/2$  there exists an  $n_0 = n_0(\eta)$  with the following properties. For any  $n \geq n_0$  positive odd integer, in any two-coloring of the edges of the complete 5-partite graph of order  $(2 + \eta)n$  with 5 parts of size  $g(1), g(2), g(3), g(4)$  and  $g(5)$ , where we have*

$$n/2 \geq g(1) \geq g(2) \geq g(3) \geq g(4) \geq g(5) \geq \eta n,$$

*there is a monochromatic  $C_n$ .*

We note that recently there has been some interest in multipartite versions of classical results, see e.g. the result of Magyar and Martin [14], a tripartite version of the Corrádi-Hajnal Theorem, or our result in [7], a tripartite version of the Gerencsér-Gyárfás Theorem.

In the proof of Theorem 1 the notion of an **odd connected matching** plays a central role; this is a matching  $M$  in a graph  $G$  such that all edges of  $M$  are in the same non-bipartite connected component of  $G$ , such a component is called an **odd component**. This is related to the concept of a connected matching that was introduced by Łuczak [13] and applied e.g. in [3], [6] and [7].

Sections 2 and 3 provide our main tools including the Regularity Lemma. Then in Section 4 we prove our main lemma (Lemma 6) which states that in any two-coloring of a  $(1 - \varepsilon)$ -dense 5-partite graph with the right parameters there is a sufficiently large monochromatic odd connected matching. Finally in Section 5 we show how Lemma 6 implies Theorem 1.

## 1.2 Notation and definitions

For basic graph concepts see the monograph of Bollobás [1]. Disjoint union of sets will be sometimes denoted by  $+$ .  $V(G)$  and  $E(G)$  denote the vertex-set and the edge-set of the graph  $G$ . Usually  $G_n$  is a graph with  $n$  vertices,  $G(n_1, \dots, n_k)$  is a  $k$ -partite graph with classes containing  $n_1, \dots, n_k$  vertices.  $(A, B, E)$  denotes a bipartite graph  $G = (V, E)$ , where  $V = A + B$ , and  $E \subset A \times B$ .  $K_n$  is the complete graph

on  $n$  vertices,  $K(n_1, \dots, n_k)$  is the complete  $k$ -partite graph with classes containing  $n_1, \dots, n_k$  vertices,  $P_n$  ( $C_n$ ) is the path (cycle) with  $n$  vertices. For a graph  $G$  and a subset  $U$  of its vertices,  $G|_U$  is the restriction to  $U$  of  $G$ .  $\Gamma(v)$  is the set of neighbors of  $v \in V$ . Hence the size of  $\Gamma(v)$  is  $|\Gamma(v)| = \deg(v) = \deg_G(v)$ , the degree of  $v$ .  $\delta(G)$  stands for the minimum, and  $\Delta(G)$  for the maximum degree in  $G$ . For a vertex  $v \in V$  and set  $U \subset V - \{v\}$ , we write  $\deg(v, U)$  for the number of edges from  $v$  to  $U$ . A graph  $G_n$  is  $\gamma$ -**dense** if it has at least  $\gamma \binom{n}{2}$  edges. The  $(A, B, E)$  bipartite graph is  $\gamma$ -**dense** if it has at least  $\gamma|A||B|$  edges. The  $G(n_1, \dots, n_k)$   $k$ -partite graph is  $\gamma$ -**dense** if all the bipartite graphs between two classes are  $\gamma$ -dense. When  $A, B$  are disjoint subsets of  $V(G)$ , we denote by  $e_G(A, B)$  the number of edges of  $G$  with one endpoint in  $A$  and the other in  $B$ . For non-empty  $A$  and  $B$ ,

$$d_G(A, B) = \frac{e_G(A, B)}{|A||B|}$$

is the **density** of the graph between  $A$  and  $B$ .

**Definition 1.** *The bipartite graph  $G = (A, B, E)$  is  $(\varepsilon, G)$ -**regular** if*

$$X \subset A, Y \subset B, |X| > \varepsilon|A|, |Y| > \varepsilon|B| \quad \text{imply} \quad |d_G(X, Y) - d_G(A, B)| < \varepsilon,$$

*otherwise it is  $\varepsilon$ -irregular.*

## 2 The Regularity Lemma

In the proof a two-color version of the Regularity Lemma plays a central role.

**Lemma 1 (Regularity Lemma [19]).** *For every positive  $\varepsilon$  and positive integer  $m$  there are positive integers  $M$  and  $n_0$  such that for  $n \geq n_0$  the following holds. For all graphs  $G_1$  and  $G_2$  with  $V(G_1) = V(G_2) = V$ ,  $|V| = n$ , there is a partition of  $V$  into  $l + 1$  classes (clusters)*

$$V = V_0 + V_1 + V_2 + \dots + V_l$$

*such that*

- $m \leq l \leq M$
- $|V_1| = |V_2| = \dots = |V_l|$
- $|V_0| < \varepsilon n$
- *apart from at most  $\varepsilon \binom{l}{2}$  exceptional pairs, the pairs  $\{V_i, V_j\}$  are  $(\varepsilon, G_s)$ -regular for  $s = 1, 2$ .*

For an extensive survey on different variants of the Regularity Lemma see [11]. Note also that if we apply the Regularity Lemma for a multipartite graph  $G$  with big enough partite classes we can guarantee that for each cluster that is not  $V_0$ , all vertices of the cluster belong to the same partite class of  $G$  (see eg. [14]).

We will also use the following simple property of  $(\varepsilon, G)$ -regular pairs.

**Lemma 2.** *Let  $G$  be a bipartite graph with bipartition  $V(G) = V_1 \cup V_2$  such that  $|V_1| = |V_2| = m \geq 45$ . Furthermore, let  $e_G(V_1, V_2) \geq m^2/4$  and the pair  $\{V_1, V_2\}$  be  $(\varepsilon, G)$ -regular for  $0 < \varepsilon < 0.01$ . Then for every  $l, 1 \leq l \leq m - 5\varepsilon m$  and for every pair of vertices  $v' \in V_1, v'' \in V_2$ , where  $\deg(v'), \deg(v'') \geq m/5$ ,  $G$  contains a path of length  $2l + 1$  connecting  $v'$  and  $v''$ .*

This lemma is used by Łuczak in [13]. Lemma 2 (with somewhat weaker parameters) also follows from the much stronger Blow-up Lemma (see [9] and [10]).

### 3 Further graph theory tools

A set  $M$  of pairwise disjoint edges of a graph  $G$  is called a matching. The size  $|M|$  of a maximum matching is the matching number,  $\nu(G)$ . A key notion in our approach is the notion of an odd connected matching. A matching  $M$  is an odd connected matching in  $G$  if all edges of  $M$  are in the same non-bipartite connected component of  $G$ , such a component is called an odd component. For a multipartite graph  $G$ , we shall work with its *multipartite complement*,  $\overline{G}$ , defined as the graph we obtain from the usual complement after deleting all edges within the partite classes. The next lemmas collect some simple properties of multipartite graphs of high density.

**Lemma 3.** *Assume  $\Delta(\overline{G}_n) < \sqrt{\varepsilon}n$  and  $H = [A, B]$  is a bipartite subgraph of  $G_n$  with  $2\sqrt{\varepsilon}n < |A| \leq |B|$ . Then  $H$  is a connected subgraph of  $G_n$  and contains a matching of size at least  $|A| - \sqrt{\varepsilon}n$ . Moreover, if only  $2\sqrt{\varepsilon}n < |B|$  and  $A \neq \emptyset$  is assumed then there is a subgraph  $H'$  which is connected and covers  $A$  and all but at most  $\sqrt{\varepsilon}n$  vertices of  $B$ .*

**Proof:** Two vertices in  $A$  ( $B$ ) have a common neighbor in  $B$  ( $A$ ). Also if  $a \in A, b \in B$  then any neighbor of  $a$  and  $b$  have a common neighbor in  $A$ . Thus  $H$  is a connected subgraph. Moreover any maximum matching  $M$  misses fewer than  $\sqrt{\varepsilon}n$  vertices of  $A$ . The statement about  $H'$  follows by fixing a vertex  $a \in A$  and  $H'$  is obtained by deleting from  $B$  the vertices nonadjacent to  $A$ .  $\square$

**Lemma 4.** *Assume that  $G$  is an  $r$ -partite graph with  $N$  vertices such that  $r \geq 3$ ,  $\Delta(\overline{G}) < \rho N$  where  $\rho < \frac{1}{r(r-1)}$ . Suppose that the largest partite class of  $G$  has at most as many vertices as the sum of the orders of the other color classes. Then  $G$  has a matching covering at least  $(1 - 2\rho)N$  vertices of  $G$ .*

**Proof:** Assume that  $V_1, \dots, V_r$  are the vertex classes of  $G$  in nondecreasing order. Continue with the following procedure until the number of vertices  $U$  uncovered in  $V_1 \cup \dots \cup V_{r-1}$  is smaller than the number of vertices in  $V_r$ . Note that by the assumption of the lemma, the stopping condition is not present at the beginning.

Take the edges of a largest matching from  $V_1$  to  $V_2$ , one by one. Take the edges of a largest matching from the unmatched vertices of  $V_2$  to  $V_3$ , one by one, etc. The last step is to take a largest matching from the unmatched vertices of  $V_{r-2}$  to  $V_{r-1}$ , one by one.

This procedure covers two vertices at a time and by Lemma 3 leaves at most  $\rho N$  vertices uncovered in each of the first  $r - 1$  partite classes. If the stopping condition never occurs then

$$(r - 1)\rho N \geq |V_r| \geq \frac{N}{r},$$

contradicting the assumption of the lemma. Thus the procedure eventually stops - notice that we have at least  $|V_r| - 2$  uncovered vertices at this time because each step covered two points. Now we finish by taking a largest matching from the set  $U$  to  $V_r$ . At this point by Lemma 3 we leave at most  $\rho N$  uncovered points in  $V_r$  and also in  $V(G) \setminus V_r$ , proving the lemma.  $\square$

**Lemma 5.** *Let  $0 < \varepsilon, \eta < 1/2$ . Assume that  $G = G(g(1), g(2), g(3), g(4), g(5))$  is a  $(1 - \varepsilon)$ -dense 5-partite graph on  $(2 + \eta)n = g(1) + g(2) + g(3) + g(4) + g(5)$  vertices, where we have*

$$n/2 \geq g(1) \geq g(2) \geq g(3) \geq g(4) \geq g(5) \geq \eta n.$$

*Then  $G$  has a 5-partite subgraph  $H = H(h(1), h(2), h(3), h(4), h(5))$  with  $h(i) \geq (1 - \frac{5\sqrt{\varepsilon}}{8\eta^2})g(i)$  for all  $1 \leq i \leq 5$  such that  $\Delta(\overline{H}) < 4\sqrt{\varepsilon}g(1)$ .*

**Proof:** If  $G$  has  $p(i)$  vertices in the same partite class of  $g(i)$  vertices with degree at least  $4\sqrt{\varepsilon}g(i)$  in  $\overline{G}$ , then  $\overline{G}$  has at least  $p(i)4\sqrt{\varepsilon}g(i)$  edges. Therefore  $p(i)4\sqrt{\varepsilon}g(i) \leq 10\varepsilon g(1)^2$ , implying  $p(i) \leq \frac{5}{2}\sqrt{\varepsilon}\frac{g(1)^2}{g(i)}$ . Removing these  $p(i)$  vertices from the partite class with  $g(i)$  vertices for each  $1 \leq i \leq 5$ , the remaining vertices induce the subgraph  $H$ . Clearly  $\Delta(\overline{H}) < 4\sqrt{\varepsilon}g(1)$ . We also have

$$h(i) \geq g(i) - \frac{5}{2}\sqrt{\varepsilon}\frac{g(1)^2}{g(i)} = \left(1 - \frac{5}{2}\sqrt{\varepsilon}\frac{g(1)^2}{g(i)^2}\right)g(i) \geq \left(1 - \frac{5\sqrt{\varepsilon}}{8\eta^2}\right)g(i). \quad \square$$

## 4 Large monochromatic odd connected matchings in dense 5-partite graphs

In our main lemma we show that we can find large monochromatic odd connected matchings in dense 5-partite graphs.

**Lemma 6.** For all  $0 < \varepsilon \leq \frac{\eta^6}{100^2}$ ,  $0 < \eta < \frac{1}{2}$ , there exists an  $n_0 = n_0(\varepsilon, \eta)$  with the following properties. For any  $n \geq n_0$  positive integer, assume that  $G = G(g(1), g(2), g(3), g(4), g(5))$  is a  $(1-\varepsilon)$ -dense 5-partite graph on  $(2+\eta)n = g(1)+g(2)+g(3)+g(4)+g(5)$  vertices, where we have

$$n/2 \geq g(1) \geq g(2) \geq g(3) \geq g(4) \geq g(5) \geq \eta n.$$

Then for each two-coloring of  $G$  there is a monochromatic odd connected matching covering at least  $n$  vertices.

**Proof:** Let us apply Lemma 5 first for the 5-partite graph  $G$  to find a 5-partite subgraph  $H = H(h(1), h(2), h(3), h(4), h(5))$  of  $G$  with  $h(i) \geq (1 - \frac{5\sqrt{\varepsilon}}{8\eta^2})g(i)$  for all  $1 \leq i \leq 5$  such that  $\Delta(\overline{H}) < 4\sqrt{\varepsilon}g(1) \leq 2\sqrt{\varepsilon}n$ . Thus each vertex in any partite set of  $H$  is adjacent to almost all (all but  $2\sqrt{\varepsilon}n$ ) vertices in the remaining four partite sets of  $H$ . This is used throughout without special mention and observe that with  $\rho = 2\sqrt{\varepsilon}$  any subgraph  $F$  of  $H$  with at least  $n$  vertices satisfies the condition  $\Delta(\overline{F}) < \rho n \leq \rho|V(F)|$  in Lemma 4. Here we use the assumption  $\rho = 2\sqrt{\varepsilon} < \frac{1}{5 \times 4} = \frac{1}{20}$ .

We consider only the two-coloring (red/blue) of  $E(H)$  induced by the coloring of  $E(G)$ . In this two-coloring of  $H$  select a maximal monochromatic (say red) connected odd component  $C$ . In fact we need only that the component we select is not proper part of another odd component in the other color. It is easy to check that there exists monochromatic odd components since  $H$  is a 5-chromatic graph.

For  $1 \leq i \leq 5$ , let  $V_i$  denote the partite sets of  $H$ , so  $|V_i| = h(i)$ , let  $X_i = V_i \setminus V(C)$ ,  $Y_i = V(C) \cap V_i$ . Call an  $X_i$  large if  $|X_i| > 4\sqrt{\varepsilon}n$ .

We will distinguish two cases:

**Case 1:** At least two of the  $X_i$ -s are large.

By Lemma 3 (since all edges of  $H$  from a large  $X_i$  to  $C$  are blue) the blue subgraph of  $H$  is a spanning connected subgraph. Moreover, any blue edge between two  $Y_i$ -s would make the blue subgraph odd - this contradicts the choice of  $C$ . Similarly, any blue edge between a pair of large  $X_i$ -s would contradict the choice of  $C$ . Thus  $C$  and the union of the large  $X_i$ -s both are almost complete red partite graphs. The larger, denote it by  $F$ , have at least

$$\frac{1}{2} \left( 2 + \eta - (2 + \eta) \frac{5\sqrt{\varepsilon}}{8\eta^2} - 12\sqrt{\varepsilon} \right) n = (1 + \alpha)n$$

vertices. The condition  $\varepsilon \leq \frac{\eta^6}{100^2}$  ensures that

$$(1 - 2\rho)(1 + \alpha) \geq 1, \tag{2}$$

in particular  $\alpha$  is positive here. Since each partite class of  $F$  has at most  $\frac{n}{2}$  vertices, the assumption of Lemma 4 about the size of the largest partite class holds and it is

also ensured that  $F$  is  $r$ -partite with  $r \geq 3$ . Thus the red matching  $M$  obtained by Lemma 4 is odd and  $M$  covers at least  $(1 - 2\rho)(1 + \alpha)n \geq n$  vertices because of (2).

**Case 2:** At most one  $X_i$  - say  $X_1$  (if there is one) - is large.

If there is no blue edge in the subgraph induced by  $S = V(C) \setminus V_1$  then we can apply Lemma 4 to the red subgraph of  $H$  induced by  $S$  and we have a red odd matching in  $C$  covering almost all points of  $S$  - this is much larger than we need since  $|S| > \frac{3n}{2}$ . Thus we may assume that there is at least one blue edge in  $S$ . If  $X_1$  is large then this ensures - since all edges of  $H$  from  $X_1$  to  $S$  are blue - a blue odd component  $C_1$  of  $H$  covering  $X_1$  and all vertices of  $S$ . Let  $Z$  denote the part of  $V_1$  that is not covered by  $C_1$ . Since all edges of  $H$  from  $Z$  to  $V(H) \setminus V_1$  are red (from the definition of  $C_1$ ) we can cover  $Z$  by a red matching  $M_R$ . Similarly we can cover  $X_1$  with a blue matching  $M_B$  that is disjoint from  $M_R$ . One can easily complete the matching  $M_R \cup M_B$  to a matching  $M$  covering almost all vertices of  $V(H)$  (here we can apply Lemma 4 again for the 2-colored partite graph spanned by  $V(H) \setminus (V(M_R) \cup V(M_B))$ ). Selecting the majority color from the edges of  $M$  we have the required monochromatic large odd matching.

If  $X_1$  is not large then  $C$  covers almost all vertices of  $H$ . Select the largest red matching  $M$  in  $C$ . If  $M$  covers less than  $n$  vertices then - since all edges of  $H$  in the subgraph induced by  $T = V(C) \setminus V(M)$  are blue - Lemma 4 ensures again that we have a blue matching covering almost all vertices of the odd component induced by  $T$ . This matching is large enough.  $\square$

## 5 Proof of Theorem 1

We will assume that  $n$  is a sufficiently large odd natural number. Let  $0 < \eta < 1/2$  be arbitrary and choose

$$\varepsilon = \frac{\left(\frac{\eta}{2}\right)^6}{100^2}. \quad (3)$$

Let  $G$  be the complete 5-partite graph of order  $(2 + \eta)n$  with 5 parts of size  $g(1), g(2), g(3), g(4)$  and  $g(5)$ , where we have

$$n/2 \geq g(1) \geq g(2) \geq g(3) \geq g(4) \geq g(5) \geq \eta n.$$

We need to show that each 2-edge coloring of  $G$  leads to a monochromatic  $C_n$ . Consider a 2-edge coloring  $(G_1, G_2)$  of  $G$ . Let  $V_i$  denote the partite classes, so  $|V_i| = g(i)$ . Apply the two-color 5-partite version of the Regularity Lemma (Lemma 1), with  $\varepsilon$  as in (3) and (by using the remark after the lemma) we can get a partition for  $i = 1, \dots, 5$  of  $V_i = V_i^0 + V_i^1 + \dots + V_i^{l_i}$ , where  $|V_i^j| = m$ ,  $1 \leq j \leq l_i$ ,  $1 \leq i \leq 5$  and  $|V_i^0| < \varepsilon n$ ,  $1 \leq i \leq 5$ . We define the following **reduced graph**  $G^r$ : The vertices of  $G^r$  are  $p_i^j$ ,  $1 \leq j \leq l_i$ ,  $1 \leq i \leq 5$ , and we have an edge between vertices  $p_{i_1}^{j_1}$  and  $p_{i_2}^{j_2}$ ,  $1 \leq j_1 \leq l_{i_1}$ ,



$1 \leq j_2 \leq l_{i_2}$ ,  $1 \leq i_1, i_2 \leq 5$ ,  $i_1 \neq i_2$ , if the pair  $\{V_{i_1}^{j_1}, V_{i_2}^{j_2}\}$  is  $(\varepsilon, G_s)$ -regular for  $s = 1, 2$ . Thus we have a one-to-one correspondence  $f : p_i^j \rightarrow V_i^j$  between the vertices of  $G^r$  and the non-exceptional clusters of the partition. Then  $G^r$  is a  $(1 - \varepsilon)$ -dense 5-partite graph on  $l = l_1 + \dots + l_5$  vertices, where note again that  $l$  is a constant (it does not depend on  $n$ ). Define a 2-edge coloring  $(G_1^r, G_2^r)$  of  $G^r$  in the following way. The color of the edge between the clusters  $V_{i_1}^{j_1}$  and  $V_{i_2}^{j_2}$  is the majority color in the pair  $\{V_{i_1}^{j_1}, V_{i_2}^{j_2}\}$ . Let

$$l' = \frac{l}{4 + \eta} \quad (4)$$

(assume for simplicity that this is an integer).

Using (3), (4) and Lemma 6 with  $\eta/2$  instead of  $\eta$  implies that in such a 2-coloring of  $G^r$  we can find a monochromatic odd connected matching  $M = \{e_1, e_2, \dots, e_{l'}\}$  covering  $2l'$  vertices of  $G^r$ . Assume that  $M$  is in  $G_1^r$ . Thus we have

$$\left| \bigcup_{i=1}^{l'} \cup_{p \in e_i} f(p) \right| \geq \frac{2 + \eta}{2 + \frac{\eta}{2}} (1 - \varepsilon)n \geq \left(1 + \frac{\eta}{8}\right)n, \quad (5)$$

i.e. the total number of vertices of  $G$  in the clusters covered by  $M$  is significantly more than  $n$ , a fact that will be important later. Furthermore, define  $f(e_i) = (C_1^i, C_2^i)$  for  $1 \leq i \leq l'$  where  $C_1^i, C_2^i$  are the clusters assigned to the end points of  $e_i$ . In the remainder from this odd connected matching  $M$  in  $G_1^r$  we will construct a cycle  $C_n$  in  $G_1$ .

Since  $M$  is a connected matching in  $G_1^r$  we can find a connecting path  $P_i^r$  in  $G_1^r$  from  $f^{-1}(C_2^i)$  to  $f^{-1}(C_1^{i+1})$  for every  $1 \leq i \leq l' - 1$ . Note that these paths in  $G_1^r$  may not be internally vertex disjoint. The last connecting path  $P_{l'}^r$  in  $G_1^r$  from  $f^{-1}(C_2^{l'})$  to  $f^{-1}(C_1^1)$  will be used to guarantee the right parity. Since  $M$  is an odd connected matching in  $G_1^r$  we can find an odd cycle  $C$  in the component of  $G_1^r$  containing  $M$ . For the construction of the last connecting path  $P_{l'}^r$  let us take first an arbitrary connecting path  $P'$  in  $G_1^r$  from  $f^{-1}(C_2^{l'})$  to an arbitrary cluster  $C'$  on the cycle  $C$ , and then another connecting path  $P''$  in  $G_1^r$  from another cluster  $C''$  on the cycle  $C$  to  $f^{-1}(C_1^1)$ . On the cycle  $C$  there are two paths,  $Q'$  and  $Q''$ , of different parity connecting  $C'$  and  $C''$ . As a first try construct the last connecting path  $P_{l'}^r$  as  $(P', Q', P'')$ . Now compute the parity of the length of the closed trail that consists of all the connecting paths  $P_i^r$ ,  $1 \leq i \leq l'$  and where we connect each  $C_1^i$  and  $C_2^i$  with an arbitrary path of odd length. If this parity is even, then we change the construction (and thus the parity) of the last connecting path  $P_{l'}^r$  to  $(P', Q'', P'')$ . Thus we may assume that this parity is always odd and note that it remains odd in the remainder when we “blow-up” parts of the path.

From these paths  $P_i^r$  in  $G_1^r$  we can construct vertex disjoint connecting paths  $P_i$  in  $G_1$  connecting a typical vertex  $v_2^i$  of  $C_2^i$  to a typical vertex  $v_1^{i+1}$  of  $C_1^{i+1}$ . More

precisely we construct  $P_1$  with the following simple greedy strategy. Denote  $P_1^r = (p_1, \dots, p_t), 2 \leq t \leq l$ , where according to the definition  $f(p_1) = C_2^1$  and  $f(p_t) = C_1^2$ . Let the first vertex  $u_1 (= v_2^1)$  of  $P_1$  be a vertex  $u_1 \in C_2^1$  for which  $\deg_{G_1}(u_1, f(p_2)) \geq m/4$  and  $\deg_{G_1}(u_1, C_1^1) \geq m/4$ . By  $\varepsilon$ -regularity most of the vertices satisfy this in  $C_2^1$ . The second vertex  $u_2$  of  $P_1$  is a vertex  $u_2 \in (f(p_2) \cap N_{G_1}(u_1))$  for which  $\deg_{G_1}(u_2, f(p_3)) \geq m/4$ . Again by regularity most vertices satisfy this in  $f(p_2) \cap N_{G_1}(u_1)$ . The third vertex  $u_3$  of  $P_1$  is a vertex  $u_3 \in (f(p_3) \cap N_{G_1}(u_2))$  for which  $\deg_{G_1}(u_3, f(p_4)) \geq m/4$ . We continue in this fashion, finally the last vertex  $u_t (= v_1^2)$  of  $P_1$  is a vertex  $u_t \in (f(p_t) \cap N_{G_1}(u_{t-1}))$  for which  $\deg_{G_1}(u_t, C_2^2) \geq m/4$ .

Then we move on to the next connecting path  $P_2$ . Here we follow the same greedy procedure, we pick the next vertex from the next cluster in  $P_2^r$ . However, if the cluster has occurred already on the paths  $P_1^r$  or  $P_2^r$ , then we just have to make sure that we pick a vertex that has not been used on  $P_1$  or  $P_2$ .

We continue in this fashion and construct the vertex disjoint connecting paths  $P_i$  in  $G_1$ ,  $1 \leq i \leq l'$ . These will be parts of the final cycle  $C_n$  in  $G_1$ . We remove the internal vertices of these paths from  $G_1$ . Note that the total number of removed vertices, denoted by  $C$ , is still a constant of size at most  $l^2 \ll n$ , if  $n$  is sufficiently large. By doing this we may create some discrepancies in the cardinalities of the clusters of this odd connected matching. We remove at most  $l^2$  vertices from each cluster of the matching to assure that now we have the same number of vertices left in each cluster of the matching. Assume without loss of generality that  $\lfloor \frac{n-C}{l'} \rfloor$  is odd (otherwise take  $\lfloor \frac{n-C}{l'} \rfloor - 1$ ). By applying Lemma 2 for  $1 \leq i \leq l' - 1$ , find a path of length  $\lfloor \frac{n-C}{l'} \rfloor$  in  $G_1|_{f(e_i)}$  connecting  $v_1^i$  and  $v_2^i$ . Indeed, (3) and (5) imply that the conditions of Lemma 2 are satisfied, since in each  $f(e_i)$  we still have  $(1 + \frac{\eta}{9})\frac{n}{l'}$  vertices available after the removals if  $n$  is sufficiently large. Finally apply Lemma 2 one more time to find a path of the right length in  $G_1|_{f(e_{l'})}$  connecting  $v_1^i$  and  $v_2^i$  so that the overall length of the cycle is exactly  $n$ . This completes the proof of Theorem 1.  $\square$

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