Inequalities for the First-Fit chromatic number

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Abstract

The First-Fit (or Grundy) chromatic number of G, written as $\chi_{FF}(G)$, is defined as the maximum number of classes in an ordered partition of $V(G)$ into independent sets so that each vertex has a neighbor in each set earlier than its own.

The well-known Nordhaus-Gaddum Inequality states that the sum of the ordinary chromatic numbers of an n-vertex graph and its complement is at most $n+1$. M. Zaker suggested finding the analogous inequality for the First-Fit chromatic number. We show for $n \geq 10$ that $\lfloor (5n+2)/4 \rfloor$ is an upper bound, and this is sharp. We extend the problem for multicolorings as well and prove asymptotic results for infinitely many cases. We also show that the smallest order of C_4-free bipartite graphs with $\chi_{FF}(G) = k$ is asymptotically $2k^2$ (the upper bound answers a problem of Zaker [9]).

1 Nordhaus-Gaddum for First-Fit chromatic number

A well known inequality [7] relating the chromatic number of an n-vertex graph and its complement is $\chi(G) + \chi(G^c) \leq n + 1$. In fact $\text{col}(G) + \text{col}(G^c) \leq n + 1$ also holds (see for example the proof in [1]) giving a stronger inequality, since $\chi(G) \leq \text{col}(G)$. (Here $\text{col}(G) = 1 + \max\{\delta(H) : H \subseteq G\}$, the *coloring number*, see [3].) Zaker [8] suggested finding the analogous inequality for $\chi_{FF}(G)$, the Grundy or *First-Fit chromatic number* of G, defined as the maximum number of classes in an ordered partition of the vertex set of G into independent sets $A_1, \ldots, A_p$ so that for each $1 \leq i < j \leq p$, and for each $x \in A_j$ there exists a $y \in A_i$ such that x, y are adjacent. We shall refer to such an ordered partition $\mathcal{A} = \{A_1, \ldots, A_p\}$ of $V(G)$ as a First-Fit (or Grundy) partition. In case of $p = \chi_{FF}(G)$ we call $\mathcal{A}$ an *optimal* partition. Clearly, $\chi_{FF}(G)$ and $\text{col}(G)$ are both between $\chi(G)$ and $\Delta(G) + 1$, but they do not relate to each other.

It was conjectured in [8] that the Nordhaus-Gaddum inequality hardly changes for $\chi_{FF}(G)$, namely $\chi_{FF}(G) + \chi_{FF}(G^c) \leq n + 2$. The conjecture was proved for regular graphs and for certain bipartite graphs. We show that it holds for all bipartite graphs (Theorem 1) and it is also true for small graphs with $n \leq 8$ vertices. But it fails in general. In fact, the maximum of $\chi_{FF}(G) + \chi_{FF}(G^c)$ over graphs of $n \geq 10$ vertices is $\lfloor \frac{5n+2}{4} \rfloor$ (Corollary 4).

Theorem 1. *For bipartite graphs $G = [X, Y]$ with n vertices, $\chi_{FF}(G) + \chi_{FF}(G^c) \leq n + 2$.*

Proof: Assume that $G = G[X, Y]$ is a bipartite graph and $\mathcal{A} = \{A_1, \ldots, A_p\}$ and $\mathcal{B}$ are optimal partitions of G and G^c, respectively. Each block of $\mathcal{B}$ spans a complete subgraph in G, therefore its size is at most two. Let $M = \{B_1, B_2, \ldots B_k\}$ be the matching defined by the edges in $\mathcal{B}$, then $\chi_{FF}(G^c) = n - k$. Observe that $Z = V(G) \setminus V(M)$ is an independent set in G.

We have to show that $\chi_{FF}(G) + \chi_{FF}(G^c) = \chi_{FF}(G) + n - k \leq n + 2$, i.e., that $\chi_{FF}(G) \leq k + 2$. Call a set $A_i \in \mathcal{A}$ *type 1* if it has points from both X and Y, moreover it has a nonempty intersection with $V(M)$.

Claim. $V(M) \cap X$ or $V(M) \cap Y$ intersects all type 1 A_i-s.

Indeed, if not, there are $A_i \cap V(M) \subseteq X$ and $A_j \cap V(M) \subseteq Y$, say $i < j$. Since A_j is type 1, it has a vertex $x \in X$, and $x \notin V(M)$ from the assumption. From the property of the partition $\mathcal{A}$, x must be adjacent to some vertex of A_i but it is impossible (no edge from $x \in X$ to $A_i \cap V(M) \subseteq X$ since G is bipartite, no edge from $x \notin V(M)$ to $A_i \setminus V(M)$ because Z is independent) – proving the claim.

There are at most three A_i-s not of type 1 (exceptional), at most one that does not intersect $V(M)$, and at most two that intersect $V(M)$ but not both X, Y. If all the three are present, then – from the claim – either $V(M) \cap X$ or $V(M) \cap Y$ intersects all type 1 A_i-s and one exceptional A_i. Thus k vertices intersect all but at most two A_i-s proving that $\chi_{FF}(G) \le k + 2$. $\Box$

Theorem 2. *Let G be any graph on $n \ge 3$ vertices, $F(G) := \chi_{FF}(G) + \chi_{FF}(G^c)$. Then $F(G) \le \lfloor \frac{5n+2}{4} \rfloor$ for $n \ge 10$, $F(G) \le n + 2$ for $n \le 8$ and $F(G) \le n + 3$ for $n = 9$.*

Proof: In the first part of the proof we establish an upper bound $4F(G) \le 5n + 5$. Then (using Lemma 2) we improve it to $5n + 4$. Then we show that either we can improve it further to $(5n + 2)$ or $F(G) \le n + 3$, finishing the case $n \ge 10$. Finally, we show that $n \le 9$ and $F(G) = n + 3$ imply $n = 9$.

Let $\mathcal{A} = \{A_1, \dots, A_p\}$ and $\mathcal{B}$ be optimal ordered partitions of G and G^c, respectively. Suppose that $\mathcal{A}$ has a_1 sets of size one, a_2 sets of size two and a_3 sets of size at least three. Similarly, $\mathcal{B}$ has b_1 sets of size one, b_2 sets of size two and b_3 sets of size at least three. From the assumption, $\chi_{FF}(G) = a_1 + a_2 + a_3$, $\chi_{FF}(G^c) = b_1 + b_2 + b_3$. From the definitions of a_i and b_i we have $\sum i a_i \le n$ and $\sum i b_i \le n$. To obtain precise upper bounds we write these inequalities in the following form, where $\varepsilon_i \ge 0$ is the *excess*.

$$a_1 + 2a_2 + 3a_3 = n - \varepsilon_1, \tag{1}$$

$$b_1 + 2b_2 + 3b_3 = n - \varepsilon_2, \tag{2}$$

Consider the singletons in the ordered partitions. We may suppose (eventually reorder) that they come last in the orderings. Observe that $K = \{v \in A_i : |A_i| = 1\}$ spans a complete subgraph in G and $L = \{v \in B_j \in \mathcal{B} : |B_j| = 1\}$ spans an independent set in G. Thus $|K \cap L| \le 1$. Note that $|K \cap L| = 1$ implies $\chi_{FF}(G) + \chi_{FF}(G^c) \le n + 1$. Indeed, if $\{x\} = K \cap L \in \mathcal{A} \cap \mathcal{B}$, then there is an edge from x to each other member of $\mathcal{A}$, hence $|\mathcal{A}| \le \deg_G(x) + 1$, and similarly $|\mathcal{B}| \le \deg_{G^c}(x) + 1$. So from now on we may suppose that $K \cap L = \emptyset$.

Let α_i ($i = 2, 3$) be the number of two- and at least three-element blocks of $\mathcal{A}$ contained entirely in L, $\alpha = \sum \alpha_i$, and define similarly β_i and β for $\mathcal{B}$. We have

$$\alpha = \alpha_2 + \alpha_3 \le 1, \quad \beta = \beta_2 + \beta_3 \le 1.$$

Classify the 2-element blocks into 3 groups. There are a_{2t} of them meeting L in exactly t elements. Define b_{2t} analogously (i.e., the number of 2-blocks of $\mathcal{B}$ meeting K in t points). We have

$$a_{22} = \alpha_2, \quad a_2 = a_{20} + a_{21} + a_{22}, \quad b_{22} = \beta_2, \quad b_2 = b_{20} + b_{21} + b_{22}.$$

All but α blocks of $\mathcal{A}$ have points outside L, and (at least) a_{20} of them have two (or more). We obtain that $|\mathcal{A}| - \alpha + a_{20} \le n - |L|$. Again write this (and its analogue for $\mathcal{B}$) in the following form

$$a_1 + a_2 + a_3 + a_{20} + b_1 = n + \alpha - \varepsilon_3, \tag{3}$$
$$b_1 + b_2 + b_3 + b_{20} + a_1 = n + \beta - \varepsilon_4. \tag{4}$$

Consider the a_{21} two-element $\mathcal{A}$-sets $\{u, u'\}$ that intersect L in exactly one vertex, say $u \in L$ and $u' \notin L$. Denote the set of these vertices $u \in L$ by L_1, and the set of vertices $u' \notin L$ by S. Similarly, $K_1 := \{v \in K : \exists v' \notin K$ such that $\{v, v'\} \in \mathcal{B}\}$, and $T := \{v' \notin K : \exists v \in K$ such that $\{v, v'\} \in \mathcal{B}\}$. We have

$$|S| = a_{21}, \quad S \cap (K \cup L) = \emptyset, \quad |T| = b_{21}, \quad T \cap (K \cup L) = \emptyset.$$

Lemma 1. $|S \cap T| \le 1$.

Proof: Assume, on the contrary, that $x_1, x_2 \in S \cap T$. This means that there are $u_1, u_2 \in L$ such that the two-element blocks $\{u_1, x_1\}$ and $\{u_2, x_2\}$ belong to $\mathcal{A}$, and there are $v_1, v_2 \in K$ such that $\{v_1, x_1\}$ and $\{v_2, x_2\} \in \mathcal{B}$. By definition we already know the status of 6 pairs from $\{x_1, x_2, u_1, u_2, v_1, v_2\}$, namely x_1v_1, x_2v_2 and v_1v_2 are edges and x_1u_1, x_2u_2 and u_1u_2 are non-edges. (See Figure 1.). Without loss of generality we may suppose that x_1x_2 is a non-edge (if it is, then replace G with G^c). By symmetry (between $\{u_1, x_1\}$ and $\{u_2, x_2\}$), we may suppose that the order of these blocks is

$$\{u_1, x_1\} <_{\mathcal{A}} \{u_2, x_2\} <_{\mathcal{A}} \{v_1\} <_{\mathcal{A}} \{v_2\}.$$

Then the First-Fit requirements on G between $\{u_1, x_1\}$ and u_2 implies $x_1u_2 \in E(G)$, and $\{u_1, x_1\} <_{\mathcal{A}} x_2$ implies $x_2u_1 \in E(G)$. Considering G^c the block $\{v_1, x_1\}$ precedes $\{u_2\}$ implying $u_2v_1 \in E(G^c)$ and the block $\{v_2, x_2\}$ precedes $\{u_1\}$ implying $u_1v_2 \in E(G^c)$. Then $\{u_1, x_1\} <_{\mathcal{A}} \{v_2\}$ implies $x_1v_2 \in E(G)$, and $\{u_2, x_2\} <_{\mathcal{A}} \{v_1\}$ implies $x_2v_1 \in E(G)$. Then there are there are 3 edges joining $\{v_1, x_1\}$ to $\{v_2, x_2\}$, so no matter how they are ordered in $\mathcal{B}$, either $\{v_1, x_1\}$ and v_2 or $\{v_2, x_2\}$ and v_1 violate the First-Fit requirement in $\mathcal{B}$. Thus $|S \cap T| \ge 2$ is impossible, finishing the proof of the Lemma. $\square$

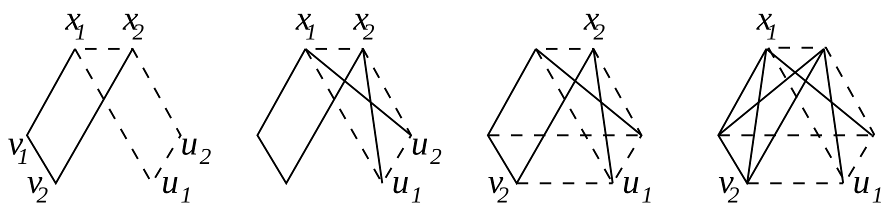

Figure 1. The proof of Lemma 1.

This Lemma is crucial, it shows that the sets K, L, S, T are almost disjoint. Let $\gamma := |S \cap T|$, and denote by $n - \varepsilon_5$ the size of the union of these four sets. We obtain

$$|L \cup K| + |S \cup T| = a_1 + b_1 + a_{21} + b_{21} - \gamma = n - \varepsilon_5. \tag{5}$$

Add the five equalities (1)–(5) we get

$$\begin{aligned} 4(a_1+a_2+a_3+b_1+b_2+b_3) &= 5n+(\alpha+\beta+\gamma)+(\alpha_2+\beta_2)-\sum_{1\le i\le 5}\varepsilon_i \\ &= 5n+s \end{aligned}$$

with some integer s. Notice that $s \le 5$ follows immediately from our assumptions. The rest of the proof is devoted to improve this upper bound.

Improvements.

Lemma 2. *If $\alpha > 0$ then*
(i) there is no block $B \in \mathcal{B}$ with $B \subset S$;
(ii) there is no block $B \in \mathcal{B}$, $B \subset K \cup S$ with $|B \cap S| = |B| - 1$;
(iii) there is no block $A_i \in \mathcal{A}$, $A_i \subset T \cup L$ with $|A_i \cap T| = 1$.
Especially, $\alpha > 0$ implies $\gamma = 0$.

Proof: Indeed, assume that $A_j \subset L$ belongs to $\mathcal{A}$. The first two statements are based on the fact that $G[S, A_j]$ is a complete bipartite graph. Indeed, let $w \in A_j$, $y \in S$. Then there is a $u \in L$ such that $\{y, u\} \in \mathcal{A}$. Since L is independent, the First-Fit requirement between u and A_j implies that u (and its block $\{y, u\}$) precedes A_j in $\mathcal{A}$. Then there is an edge between w and the block $\{y, u\}$, it should be wy.

Now suppose, on the contrary, that $B \subset S$ for $B \in \mathcal{B}$. Take any element $w \in A_j$. In fact, $\{w\} \in \mathcal{B}$, too, and thus there must be a non-edge between w and B, a contradiction.

To prove *(ii)* suppose, on the contrary, that $B \in \mathcal{B}$, $B \subset K \cup S$, and $B \cap K = \{v\}$. Since $\{w\} \in \mathcal{B}$ for all $w \in A_j$, there is a non-edge from w to B, it should be vw. Consider $\{v\} \in \mathcal{A}$ and A_j. There should be an edge vw, $w \in A_j$, a contradiction.

To prove *(iii)* suppose $A_i \cap T = \{x\}$, $(A_i \setminus \{x\}) \subset L$. Notice that $i < j$ otherwise $u \in A_i \cap L$ would violate the First-Fit requirement between u and A_j in $\mathcal{A}$. Then there is an edge from $w \in A_j$ to the block A_i, from this $wx \in E(G)$ follows. By definition of T there is a $v \in K$ such that $\{v, x\} \in \mathcal{B}$. Consider $\{w\}$ and $\{v, x\}$ in $\mathcal{B}$, $vw \in E(G^c)$ follows (for every $w \in A_j$). Then the First-Fit requirement on G is violated between the blocks A_j and $\{v\} \in \mathcal{A}$. □

Similar lemma is true for the case $\beta > 0$, it also implies $\gamma = 0$. Conversely, $\gamma = 1$ implies $\alpha = \beta = 0$, hence $s \le 1$, and we are done. From now on, we suppose that $\gamma = 0$, i.e., $S \cap T = \emptyset$, and $s \le 4$.

We have $s \le 2(\alpha + \beta) - \sum \varepsilon_i$. Hence $s \le 2$ if $\alpha + \beta \le 1$ or $\sum \varepsilon_i \ge 2$, and we are done. From now on, we suppose that $\alpha = 1$, $\beta = 1$ and $\sum \varepsilon_i \le 1$. There exists a block $A' \in \mathcal{A}$, $A' \subset L$ (naturally, it is disjoint to L_1), and there exists a block $B' \in \mathcal{B}$, $B' \subset K$ (and $B' \cap K_1 = \emptyset$). Let $E := V(G) \setminus (K \cup L \cup S \cup T)$, $|E| = \varepsilon_5$.

We claim that there is no block $A \in \mathcal{A}$ contained in $L \cup T$, other than A'. (Similarly, there is no second $\mathcal{B}$-block in $K \cup S$.) Indeed, Lemma 2 (and its analogue for $\beta > 0$) imply that such a block A meets both L and T, and it meets them in at least two-two vertices. If such an A exists then $\varepsilon_1 \ge 1$ in (1). Also, A should be counted twice on the left-hand-side of (3), implying $\varepsilon_3 \ge 1$. These contradict $\sum \varepsilon_i \le 1$.

Consider the case $E = \emptyset$. Then there is no $\mathcal{A}$-block covering the points of T, so T should be empty. Similarly, $S = \emptyset$ follows. Then $V(G) = K \cup L$, hence $F(G) = n+2$, and we are done.

The last case is when $E \neq \emptyset$, $|E| = 1$ and $\varepsilon_1 = \ldots = \varepsilon_4 = 0$. Let A'' be the $\mathcal{A}$-block covering E. There are no more $\mathcal{A}$-blocks in $T \cup (L \setminus L_1) \cup E$ so $|\mathcal{A}| = |K|+|S|+2$. Similarly, $E \in B'' \in \mathcal{B}$ and $|\mathcal{B}| = |L| + |T| + 2$ giving $F(G) = n+3$. Since $n+3 \leq (5n+2)/4$ we are done for $n \geq 10$.

Suppose that $n \leq 9$ and $F(G) = n+3$. We claim that $n = 9$ follows, finishing the proof of the Theorem. Taking the following seven pairwise disjoint sets we get

$$n \geq |A'| + |B'| + |A'' \setminus E| + |K_1| + |B'' \setminus E| + |L_1| + |E|.$$

Here $|A'| \geq 2$, $|B'| \geq 2$, $|E| = 1$. It is easy to see that $|A''\setminus E|+|K_1| \geq 2$ and $|B''\setminus E|+|L_1| \geq 2$. Indeed, $K_1 = \emptyset$ implies $T = \emptyset$ and $A'' \subset L \cup E$. Since $E \notin S$ we get $|A''| \geq 3$. $\square$

Lemma 3. *There is graph G_9 with vertex set $\{1, 2, \ldots, 9\}$ such that $\chi_{FF}(G_9) = \chi_{FF}(G_9^c) = 6$. (See Figure 2).*

The edges form a complete graph on $\{6, 7, 8, 9\}$, the further edges are 14, 15, 18, 19, 27, 36, 38, 47, 49 and 58. Then 123|45|6|7|8|9 and 198|76|5|4|3|2 are grundy partitions for G and its complement, respectively.

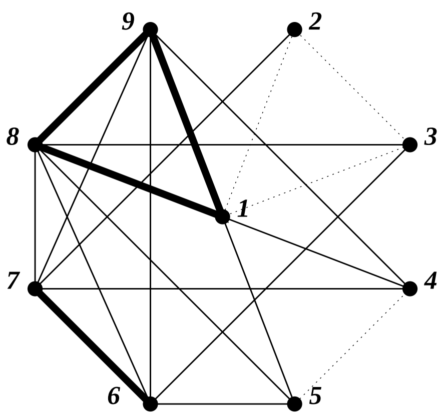

Figure 2.

Theorem 3. *For all $k \geq 1$ there is a graph $G = G_{4k+2}$ with $4k+2$ vertices such that $\chi_{FF}(G) + \chi_{FF}(G^c) \geq 5k+3$.*

Proof: The vertex set of G consists of four disjoint sets P, Q, R and S, such that $|P| = |Q| = k+1$, $|R| = |S| = k$. Label their elements as $P = \{p_1, p_2, \ldots, p_{k+1}\}$, $Q = \{q_1, q_2, \ldots, q_{k+1}\}$, $R = \{r_1, r_2, \ldots, r_k\}$, and $S = \{s_1, s_2, \ldots, s_k\}$. The ordered partition $\mathcal{A}$ on $V(G)$ is composed from k triples $A_i = \{q_i, r_i, s_i\}$ $(1 \leq i \leq k)$ and singletons $A_{k+j} := \{p_j\}$, $(1 \leq j \leq k+1)$ and finally $A_{2k+2} = \{q_{k+1}\}$. The ordered partition $\mathcal{B}$ on $V(G^c)$ is composed from $k+1$ pairs $B_i = \{p_i, q_i\}$ $(1 \le i \le k+1)$ and $2k$ singletons of $R \cup S$ (in arbitrary order).

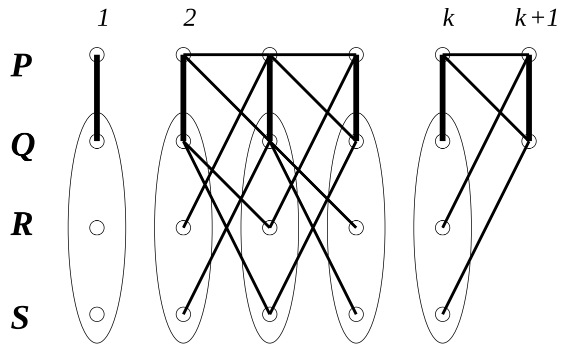

Figure 3. The graph G^{4k+2} with $\chi_{FF}(G) = 2k+2$ and $\chi_{FF}(G^c) = 3k+1$.

We define edges and non-edges of G. Pairs within A_i-s are non-edges, pairs within B_i-s are edges. The pairs within P are edges, the pairs within $R \cup S$ are non-edges. Notice that so far the choices were forced, it is not so in the sequel.

The set $\{p_i, q_i, r_i, s_i\}$ spans only a single edge, $p_i q_i$. The set P spans a complete graph, Q and $R \cup S$ are independent sets.

The spanned bipartite graph $G[P, Q]$ is a so-called half-graph (half complete bipartite) with edge-set $\{p_i q_j : i \le j\}$. $G[P, R]$ is another half-graph, edges going into the another direction, its edge-set is $\{p_i r_j : j < i\}$. $G[P, S]$ has no edge, $E(G[Q, R]) := \{q_i r_j : i < j\}$, finally $G[Q, S]$ is a complete bipartite graph minus an almost perfect matching, $E(G[Q, S]) := \{q_i s_j : i \ne j\}$ (see Figure 3).

It is easy to check that the partition $\mathcal{A}$ and $\mathcal{B}$ are greedy colorings of the graphs G and G^c implying the lower bound stated in the Theorem. $\square$

Observe that if a graph H is extended to H_{new} by adding a new vertex adjacent to all vertices of H, then $\chi_{FF}(H_{new}) = \chi_{FF}(H) + 1$ and $\chi_{FF}(H^c_{new}) = \chi_{FF}(H^c)$. Applying this to G_{4k+2} three times, we get the graphs G_{4k+i} of $4k+i$ vertices $(i = 2, 3, 4, 5)$ with

$$\chi_{FF}(G_{4k+i}) + \chi_{FF}(G^c_{4k+i}) \ge 5k + 1 + i.$$

The path on four vertices shows $F(4) \ge 6$, using Lemma 3, Theorem 3 and combining them with Theorem 2 one obtains

Corollary 4. *Let $F(n) = \max\{\chi_{FF}(G) + \chi_{FF}(G^c)\}$ over all n-vertex graphs. Then*

$$F(n) = \begin{cases} n+2 & \text{for } 4 \le n \le 8 \\ 12 & \text{for } n = 9 \\ \lfloor (5n+2)/4 \rfloor & \text{for } n \ge 10. \end{cases}$$

2 Nordhaus-Gaddum for many colors

The analogue of the Nordhaus-Gaddum inequality for multicolored graphs have been studied recently for many graph parameters in [2]. One of them, the Wilf-Szekeres number, or coloring number, has been investigated earlier in [4]. Here we give bounds on

$$h(n,k) := \max\,\{\chi_{FF}(G_1) + \ldots + \chi_{FF}(G_k)\}$$

where the maximum is taken over all partitions of K_n into k edge disjoint graphs G_i. In Section 1, $h(n,2)$ is determined exactly, but for the next case we know only that $h(n,3) \geq \frac{3n}{2}$. Nevertheless, we determine $h(n,k)$ asymptotically for infinitely many fixed k's (for $k = 5, 13, \ldots$) and give bounds for every n and k.

Theorem 5. *For every n and k,*

$$h(n,k) \leq \left(\frac{1+\sqrt{2k-1}}{2}\right) n + \binom{k}{2} + k. \tag{6}$$

Proof: Consider an optimal decomposition of K_n, i.e. assume that $h(n,k) = \sum_{i=1}^{k} \chi_{FF}(G_i)$ (where the G_i-s decompose K_n). Set $p_i = \chi_{FF}(G_i)$ and let a_i denote the number of one-element classes in the First-Fit partition of G_i into $\chi_{FF}(G_i)$ classes. Since one-element classes in G_i must span a complete subgraph of color i, it follows that

$$0 \leq a_i \leq p_i,\ \sum_{i=1}^{k} a_i \leq n + \binom{k}{2}. \tag{7}$$

Using the First-Fit property one can easily obtain that for each i,

$$|E(G_i)| \geq 2(1+2+\ldots+p_i-a_i-1) + (p_i-a_i) + \ldots + (p_i-1) = \binom{p_i}{2} + \binom{p_i-a_i}{2}$$

and summing that for $1 \leq i \leq k$ we obtain

$$\binom{n}{2} \geq \sum_{i=1}^{k} \left(\binom{p_i}{2} + \binom{p_i-a_i}{2} \right). \tag{8}$$

Use the notations $P := \sum_{i=1}^{k} p_i$, $A := \sum_{i=1}^{k} a_i$, $B := n + \binom{k}{2}$ and assume $P \geq B+k$ (otherwise (6) trivially holds). Apply Jensen's inequality for the convex polynomial $\binom{x}{2} := x(x-1)/2$ and use the fact that $(P-A)/k + (P-B)/k \geq 1$. We get

$$\binom{n}{2} \geq k\binom{P/k}{2} + k\binom{(P-A)/k}{2} \geq k\binom{P/k}{2} + k\binom{(P-B)/k}{2}.$$

Consequently, $0 \geq 2P^2 - 2(B+k)P + (kB + kn + B^2 - kn^2)$, and thus

$$P \leq \frac{B+k}{2} + \frac{1}{2}\sqrt{(2k-1)n^2 - 2kn + k^2 - (B+n)(B-n)}$$

and this easily gives the theorem. □

The following two theorems give lower bounds for special values of k.

Theorem 6. *Assume $k = 2q^2+2q+1$, k divides n and $q = 1$ or a power of a prime. Then*

$$\left(\frac{1+\sqrt{2k-1}}{2}\right) n = (q+1)n \le h(n,k).$$

Proof: Let $m := n/k$ and consider the disjoint m-sets $V_0, V_1, \ldots, V_{k-1}$, their union is V, the vertex set of an n-vertex complete graph K_n.

Define a graph G as follows. Its vertex set is $V(G) := (A_0 \cup A_1 \cup \ldots \cup A_q) \cup (B_1 \cup \ldots \cup B_q)$, $|A_i| = |B_j| = m$, $|V(G)| = (2q+1)m$ and its edge set consists of a complete graph on A_0 and $\binom{q+1}{2}$ complete bipartite graphs $K(A_i, A_j)$ and $\binom{q}{2}$ complete bipartite graphs of $K(B_i, A_j)$ for $1 \le i < j \le q$, finally $G[A_i, B_i]$ is a complete bipartite graph minus a matching (of size m). Obviously, $\chi_{FF}(G) = (q+1)m$. We are going to decompose $E(K_n)$ into graphs $G_0, \ldots, G_{k-1}$ such that $\chi_{FF}(G_i) = (q+1)m$, and each G_i is isomorphic to G (apart from an uninteresting matching of size qm), and the defining sets $A_0, \ldots, B_q$ are selected from the V_i's.

Consider the graph H, the host graph of G, with vertex set $\{a_0, a_1, \ldots, a_q\} \cup \{b_1, \ldots, b_q\}$ with a complete graph of size $q+1$ on $\{a_0, a_1, \ldots, a_q\}$ and additional edges $b_i a_j$ for $i \le j$. The vertex a_0 is called the *special vertex* of H. H has $q^2 + q = (k-1)/2$ edges. We claim that the complete graph K_k with vertex set $\{v_0, v_1, \ldots, v_{k-1}\}$ can be decomposed into k edge-disjoint copies of H, $G = H_0 \cup H_1 \cup \ldots \cup H_{k-1}$, such that their special vertices are all distinct. Then replacing v_i with V_i the t-th copy H_t naturally extends to G_t, a graph isomorphic to G. Finally, the qm matching edges deleted in the definition of G_t can get color $t+1$ (modulo k), (it is easy to see that adding this matching to H_{t+1} does not decrease its χ_{FF}-value) and we obtain the coloring of K_n showing the lower bound in the Theorem.

We identify the vertices of K_k with a vertex set of a regular k-gon (in cyclic order), or rather with the elements of the cyclic group Z_k, and call $\min\{|i-j|, k-|i-j|\}$ the *length* of the edge $v_i v_j$. Since k is odd, the lengths are $1, 2, \ldots, (k-1)/2$. One can get the desired H-decomposition of K_k if there exists a single embedding of H into K_k such that all edges of H has different lengths. The further $k-1$ copies of H are obtained by rotations (see Figure 4). We finish the proof by showing such an embedding of H.

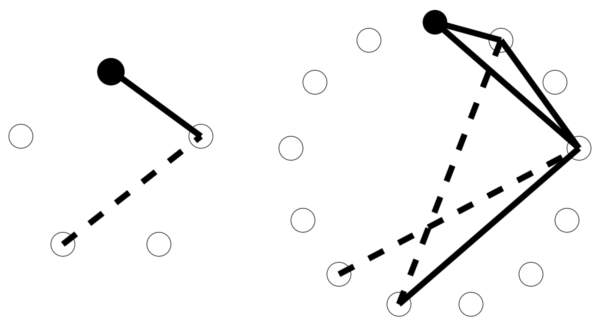

Figure 4. Cyclic k-colorings for $k = 5$ ($q = 1$) and $k = 13$ ($q = 2$).

Singer proved in 1938 (see, e.g., in the textbook [5]) that a $(q^2+q+1, q+1, 1)$ *difference set* exists if and only if $q = 1$ or it is a power of a prime. It means that there exists a set

$D \subset \{0, 1, 2, \dots, q^2 + q\}$, $D = \{d_1, d_2, \dots, d_{q+1}\}$ such that all the differences $d_i - d_j$ for $i \neq j$ are non-zero and are different modulo $q^2 + q + 1$. Suppose $d_1 < d_2 < \dots < d_{q+1}$. Let $a_i := d_{i+1}$ $(0 \leq i \leq q)$ and $b_i := (q^2 + q + 1) + d_i$ $(1 \leq i \leq q)$. Then the lengths between the a's are $\{d_j - d_i : j > i\}$ and the length of $b_i a_j$ $(i \leq j)$ is $(q^2 + q + 1 + d_i) - d_{j+1}$ (now $i < j + 1$). These lengths are all distinct, since they are distinct modulo $q^2 + q + 1$. □

Corollary 7. $h(n, 5) = 2n + O(1)$, $h(n, 13) = 3n + O(1)$, *and for every prime power* q

$$h(n, 2q^2 + 2q + 1) = (q + 1)n + O(q^4).$$

Theorem 8. *Assume* $k = 2(q^2 + q + 1)$, $(q^2 + q + 1)$ *divides* n *and a projective plane of order* q *exists. Then*

$$\left(\frac{1 + \sqrt{2k - 3}}{2}\right) n = (q + 1)n \leq h(n, k).$$

It seems that Theorem 8 is weaker than Theorem 6, it gives the same lower bound using one more color and it is indeed so if q is a power of a prime; however, although widely believed otherwise, there might exist some projective plane of order q that is not a power of a prime.

Proof: For an arbitrary positive integer m set $n = m(q^2 + q + 1)$ and define a k-colored complete graph K_n as follows. Consider disjoint m-sets $V_1, V_2, \dots, V_{q^2+q+1}$, their union is the vertex set of K_n. Consider a finite plane of order q with point set $V_1, V_2, \dots, V_{q^2+q+1}$ and line set $L_1, L_2, \dots, L_{q^2+q+1}$. The König-Hall condition is satisfied for the projective plane, there exists a system of distinct representatives, i.e., we may suppose that $V_t \in L_t$.

Take a fixed line L_t and denote its points by $A_1, A_2, \dots, A_{q+1}$. These A_i's are actually m-sets, and suppose that $A_{q+1} = V_t$. We associate two colors to L_t, colors $2t - 1$ and $2t$ and color some edges of K_n contained in L_t with these colors. For any odd $i < q + 1$ color all edges except a one-factor of $[A_i, A_{i+1}]$ with color $2t - 1$. The m edges of the one-factor are colored with color $2t$. For all j such that $i + 1 < j \leq q + 1$, color also with color $2t - 1$ all edges of $[A_i, A_j]$. Similarly, for any even $i < q + 1$, color all edges except a one-factor of $[A_i, A_{i+1}]$ with color $2t$. The m edges of the one-factor can be colored with color $2t - 1$. For all j such that $i + 1 < j \leq q + 1$, color also with color $2t$ all edges of $[A_i, A_j]$. Finally, the edges within A_{q+1} also colored by $2t$ if q is even and they are colored by $2t - 1$ if q is odd. It is easy to check that for the graphs G_{2t-1}, G_{2t} with edges colored with $2t - 1$, $2t$ respectively, $\chi_{FF}(G_{2t-1}) = \lceil (q + 1)/2 \rceil m$ and $\chi_{FF}(G_{2t}) = \lceil q/2 \rceil m$. This gives a k-coloring of K_n and

$$\sum_{j=1}^{k} \chi_{FF}(G_j) = \frac{k}{2}(q + 1)m = (q + 1)n$$

and the result follows by expressing q as a function of k. □

Combining our results above and using the existence of primes between m and $m + cm^{2/3}$, one can easily get

Corollary 9. *There exists a* $c > 0$ *such that for every* n *and* k

$$\sqrt{\frac{k}{2}} n - ck^{1/3} n - k^2 < h(n, k) < \left(\frac{1 + \sqrt{2k - 1}}{2}\right) n + \binom{k}{2} + k. \tag{9}$$

3 The smallest C_4-free bipartite graph with $\chi_{FF}(G) = k$

It is well known that for every k there are bipartite graphs satisfying $\chi_{FF}(G) = k$, the standard example is obtained from $K_{k,k}$ by removing a perfect matching from it. It is also possible that such a graph has arbitrary large girth, since there are trees with that property, the smallest well-known example is the rooted tree T_k defined recursively by joining the roots of two distinct copies of T_{k-1} and keeping one of the two roots as the new root. Clearly, T_k has 2^{k-1} vertices. In the light of these two extreme examples, it is natural to ask about $c_4(k)$, the smallest order of a C_4-free bipartite graph G with $\chi_{FF}(G) = k$. It is easy to obtain that $c_4(k) \geq (k-1)(k-2) + 2$ and some experience with small graphs show that this is sharp for $2 \leq k \leq 7$ (the proof of this is left to the reader). However, the next theorem shows that the coefficient of k^2 in $c_4(k)$ is two. The upper bound of Theorem 10 answers positively the following problem posed by Zaker in [9]: is it true that $\rho(n) = \Omega(\sqrt{n})$ where

$$\rho(n) = \max\left\{\frac{\chi_{FF}(G)}{\chi(G)} : |V(G)| = n, \mathrm{girth}(G) \geq 5\right\}.$$

Theorem 10.
(i) $c_4(k) \leq 2k^2$ *if there exists a projective plane of order* k*,*
(ii) $c_4(k) \leq 2k^2 - 2k^{3/2} + 2k$ *if* $\sqrt{k}$ *is a prime power,*
(iii) $2k^2 - 8k^{5/3} \leq c_4(k)$*.*

Proof: To prove the upper bound *(i)*, let $G = [P, L]$ be the bipartite graph defined by the incidences of the points and lines of an affine plane of order k. Let L_i denote the i'th parallel class of lines, $i = 0, 1, \ldots, k$. The lines of L_0 partition P into P_i, $i = 1, 2, \ldots, k$, $|P_i| = k$. Set $A_i = P_i \cup L_i$ for $i = 1, 2, \ldots, k$. Consider the bipartite graph G^* obtained from G by removing the vertices of L_0 and all edges within each A_i. Then G^* is a bipartite graph with k^2 vertices in its color classes, G^* is C_4-free and each A_i spans an independent set in G^*. Moreover, for every $i \neq j$ and for every $v \in A_j$ there is (exactly) one vertex of A_i adjacent to v. Thus the sets A_i give a First-Fit partition on G^*.

To prove the upper bound *(ii)* we note that a somewhat smaller graph can be obtained by selecting initially only one vertex v_k from A_k. We are going to use the fact that a Desarguesian affine plane of square order k contains a *Baer subplane* (see, e.g., in [5]). It means, that there exists a subset of vertices B, $|B| = k$, and $\sqrt{k}+1$ special parallel classes such that (1) every line not in a special class meets B in exactly one point, (2) every line in a special class meets B in either $\sqrt{k}$ points or avoids it. Suppose that the special parallel classes are L_0 and $L_{k-\sqrt{k}+1}, \ldots, L_{k-1}, L_k$, and the lines of L_0 are also ordered such a way that $|P_i \cap B| = \sqrt{k}$ for $i > k - \sqrt{k}$. The bipartite graph showing the upper bound *(ii)* is obtained from G^* defined in section *(i)*, such that for $i > k - \sqrt{k}$ one keeps only the lines meeting B, and the points $P_i \cap B$.

To prove the lower bound *(iii)*, let G be a C_4-free bipartite graph with $\chi_{FF}(G) = k$. Consider a First-Fit coloring of G with k colors and let $f(x)$ denote the color of $x \in V = V(G)$. Using the First-Fit rule and that G is bipartite without C_4, it follows easily that the following procedure builds an induced two-level tree T in G. The root of T is a vertex $x \in V$ with $f(x) = k$. Level one of T is defined by selecting vertices $y_1, \ldots, y_{k-1}$ where each y_i is adjacent to the root and $f(y_i) = i$. Level two of G is defined by selecting for each i, $i = 2, 3, \ldots, k-1$ vertices $z_{i,1}, z_{i,2}, \ldots, z_{i,i-1}$ adjacent to y_i such that $f(z_{i,j}) = j$ (for $1 \le j \le i-1$). Note that $|V(T)| = \binom{k}{2} + 1$.

For a suitable r defined later, an r-uniform hypergraph H_r is defined as follows. For each i, j such that $k-2 \ge i-1 \ge j \ge r+1$ a hyperedge $E_{ij} = \{v_{ij1}, v_{ij2}, \ldots, v_{ijr}\}$ is defined by selecting $v_{ijl} \in V \setminus V(T)$ adjacent to $z_{i,j}$ and $f(v_{ijl}) = l$ for $1 \le l \le r$. These vertices exist by the First-Fit property and any two distinct hyperedges intersect in at most one vertex since G has no C_4. There are $m = 1 + 2 + \ldots (k-r-2) = \binom{k-r-1}{2}$ edges in H_r. It is known that any hypergraph of m edges of sizes at least r and pairwise intersections at most one has at least $\frac{r^2 m}{r+m-1}$ vertices (Exercise 13.13 in [6]). Therefore W, the vertex set covered by the hyperedges satisfies

$$|W| \ge \frac{r^2 m}{r+m-1}. \tag{10}$$

Taking $r = k - k^{2/3}$ one gets a lower bound $k^2 - 4k^{5/3}$ for $|W|$. Notice that if x, the root of T is in the first partite class of the bipartite graph G then any vertex in W covered by an edge of H_r is in the second partite class of G. This observation allows to select y_{k-1} in the role of x and repeat the same argument to get a $k^2 - 4k^{5/3}$ lower bound for the first partite class of G. This gives the claimed lower bound on $c_4(k)$. $\square$

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