On a Turán-type hypergraph problem of Brown, Erdős and T. Sós

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Note

Abstract

We let $G^{(r)}(n,m)$ denote the set of r-uniform hypergraphs with n vertices and m edges, and $f^{(r)}(n,p,s)$ is the smallest m such that every member of $G^{(r)}(n,m)$ contains a member of $G^{(r)}(p,s)$. In this paper we are interested in the growth of $f^{(r)}(n,p,s)$ for fixed values r,p and s. Brown, Erdős and T. Sós ([2]) proved that for $r > k \geq 2$ and $s \geq 3$ we have $f^{(r)}(n, s(r-k)+k, s) = \Theta(n^k)$. This suggests the difficult question whether $f^{(r)}(n, s(r-k)+k+1, s) = o(n^k)$. This was first established for $r = s = 3$ and $k = 2$ by Ruzsa and Szemerédi ([11]). Then for $s = 3$ and $k = 2$ Erdős, Frankl and Rödl ([6]) extended this result for any r, and they conjectured that it also holds for $k = 2$ and any s. In this note we show that

$$f^{(r)}(n, s(r-k)+k+\lfloor \log_2 s \rfloor, s) = o(n^k) \quad \text{for all} \quad k \geq 2.$$

In addition we show that

$$f^{(r)}(n, 4(r-3)+4, s) = o(n^3).$$

1 Introduction

1.1 Notation and definitions

For basic graph and hypergraph concepts see the monograph of Bollobás [1].

A hypergraph $\mathcal{F}$ is called *r-uniform* if $|F| = r$ for every edge $F \in \mathcal{F}$. An r-uniform hypergraph $\mathcal{F}$ on the set X is *r-partite* if there exists a partition $X = X_1 \cup \ldots \cup X_r$ with $|F \cap X_i| = 1$ for every edge $F \in \mathcal{F}$ and $1 \leq i \leq r$. $|\mathcal{F}|$ denotes the number of edges of $\mathcal{F}$. In this paper $\log n$ denotes the base 2 logarithm.

1.2 Turán-type hypergraph problems

We let $G^{(r)}(n,m)$ denote the set of r-uniform hypergraphs with n vertices and m edges, and $f^{(r)}(n,p,s)$ is the smallest m such that every member of $G^{(r)}(n,m)$ contains a member of $G^{(r)}(p,s)$. The determination of $f^{(r)}(n,p,s)$ has been a longstanding open problem. Special cases of this problem appeared in [3], [5]. For more about Turán-type hypergraph results consult the surveys by Füredi [9] and Sidorenko [13]. In this note we are interested in the growth of $f^{(r)}(n,p,s)$ for fixed values r,p and s.

Brown, Erdős and T. Sós ([2]) proved that for $r > k \geq 2$ and $s \geq 3$ we have

$$f^{(r)}(n, s(r-k)+k, s) = \Theta(n^k).$$

This suggests the following difficult question.

Conjecture 1.

$$f^{(r)}(n, s(r-k)+k+1, s) = o(n^k).$$

This was first established for $r = s = 3$ and $k = 2$ by the celebrated result of Ruzsa and Szemerédi ([11]). Then for $s = 3$ and $k = 2$ Erdős, Frankl and Rödl ([6]) extended this result for any r, and they conjectured that it also holds for $k = 2$ and any s. In this direction in [12] we showed that

$$f^{(r)}(n, s(r-2)+2+\lfloor \log s \rfloor, s) = o(n^2).$$

In this note we extend this result for $k > 2$, showing that Conjecture 1 is not far from being true.

Theorem 1. *For all integers $r > k \geq 2$ and $s \geq 3$,*

$$f^{(r)}(n, s(r-k)+k+\lfloor \log s \rfloor, s) = o(n^k).$$

Thus roughly speaking Conjecture 1 is true apart from a $\lfloor \log s \rfloor$ term. However, it still remains open whether one can replace this term with 1 and prove Conjecture 1.

In addition, by using a recent, deep result of Frankl and Rödl ([8]) we show that Conjecture 1 is true for $k = 3$ and $s = 4$.

Theorem 2. *For all integers $r \geq 4$,*

$$f^{(r)}(n, 4(r-3)+4, 4) = o(n^3).$$

In the next section we provide the tools, then we prove the theorems.

2 Tools

We will use a simple but useful result of Erdős and Kleitman ([7], see also on page 1300 in [10]).

Lemma 1. *Every k-uniform hypergraph $\mathcal{F}$ contains a k-partite k-uniform hypergraph $\mathcal{H}$ with*

$$\frac{|\mathcal{H}|}{|\mathcal{F}|} \geq \frac{k!}{k^k}.$$

We will also need a recent result of Frankl and Rödl. Following their notation from [8], let $A_i = \{a_i, b_i\}$ be pairwise disjoint 2-element sets for $1 \leq i \leq k$. Define $F_i = \{a_1, \ldots, a_k, b_i\} \setminus \{a_i\}$ and $\mathcal{F}(k) = \{F_1, \ldots, F_k\}$. Let $ex^*(n, \mathcal{F}(k))$ denote $\max |\mathcal{H}|$ such that $\mathcal{H}$ is a k-partite hypergraph on n vertices that is $\mathcal{F}(k)$-free, and $|H \cap H'| \leq k-2$ holds for all distinct $H, H' \in \mathcal{H}$. In [8] the following deep result is shown.

Lemma 2.

$$ex^*(n, \mathcal{F}(4)) = o(n^3).$$

3 Proof of Theorem 1

Let $r > k \geq 2$, $s \geq 3$, $p = s(r-k) + k + \lfloor \log s \rfloor$ and $l = \lceil \log s \rceil$. For $k = 2$ we showed that the theorem is true in [12]; thus we may assume $k > 2$.

Assume indirectly that there is a constant $c > 0$ such that

$$f^{(r)}(n, p, s) > \lceil cn^k \rceil. \tag{1}$$

From this assumption we will get a contradiction. (1) means that there exists an r-uniform hypergraph $\mathcal{F}$ with

$$f^{(r)}(n, p, s) - 1 \geq \lceil cn^k \rceil \geq cn^k$$

edges that does not contain a member of $G^{(r)}(p, s)$, i.e. a set of p vertices spanning at least s edges. Let us assume that n is sufficiently large.

Using the Erdős-Kleitman theorem (Lemma 1) we find an r-partite subhypergraph $\mathcal{H}$ of $\mathcal{F}$ with at least

$$\frac{r!c}{r^r} n^k$$

edges. Let $X_1, \ldots, X_r$ be the vertex classes of this r-partite hypergraph $\mathcal{H}$. Consider the $(k+1)$-uniform hypergraph $\mathcal{H}^*$ which is defined by the removal of $X_1, \ldots, X_{r-k-1}$ from the vertex set of $\mathcal{H}$ and from all edges of $\mathcal{H}$. If a $(k+1)$-edge of $\mathcal{H}^*$ has multiplicity greater than 1, then we keep only one edge. Note that every $(k+1)$-edge has multiplicity less than s. Indeed, otherwise taking a $(k+1)$-edge with multiplicity at least s and s r-edges of $\mathcal{H}$ containing this edge, we get a set of at most

$$s(r-k-1) + k + 1 \leq s(r-k) + k + \lfloor \log s \rfloor = p$$

vertices that span at least s r-edges, a contradiction. Then if in $\mathcal{H}^*$ we keep only one edge from each multiple $(k+1)$-edge we still have at least

$$\frac{r!c}{r^r s} n^k$$

edges.

Define for every $x_1 \in X_{r-k}, x_2 \in X_{r-k+1}, \ldots, x_{k-2} \in X_{r-3}$ the following hypergraph:

$$\mathcal{H}^*(x_1, \ldots, x_{k-2}) = \{G \setminus \{x_1, \ldots, x_{k-2}\} \mid \{x_1, \ldots, x_{k-2}\} \subset G \in \mathcal{H}^*\}.$$

There are $x_1, \ldots, x_{k-2}$ for which we have

$$|\mathcal{H}^*(x_1, \ldots, x_{k-2})| \geq \tfrac{r!c}{r^r s} n^2.$$

By Theorem 1 for $k = 2$ ([12]), we have a $G^{(3)}(s + 2 + \lfloor \log s \rfloor, s)$ in this 3-uniform $\mathcal{H}^*(x_1, \ldots, x_{k-2})$. Then in the original $\mathcal{H}$ we have a set of at most

$$s(r - (k+1)) + (k-2) + s + 2 + \lfloor \log s \rfloor = s(r-k) + k + \lfloor \log s \rfloor = p$$

vertices spanning at least s r-edges, a contradiction.

This completes the proof of Theorem 1. □

4 Proof of Theorem 2

Let $r \geq 4$ and $p = 4(r-3) + 4$.

Proceeding similarly as above, assume indirectly that there is a constant $c > 0$ such that

$$f^{(r)}(n, p, 4) > \lceil cn^3 \rceil. \quad (2)$$

From this assumption we will get a contradiction. (2) means that there exists an r-uniform hypergraph $\mathcal{F}$ with

$$f^{(r)}(n, p, 4) - 1 \geq \lceil cn^3 \rceil \geq cn^3$$

edges that does not contain a member of $G^{(r)}(p, 4)$, i.e. a set of p vertices spanning at least 4 edges. Let us assume that n is sufficiently large.

Similarly as above, first by using Lemma 1 we find an r-partite subhypergraph $\mathcal{H}$ of $\mathcal{F}$ with at least

$$\frac{r!c}{r^r} n^3$$

edges and with partite sets $X_1, \ldots, X_r$. Then we reduce $\mathcal{H}$ to $\{X_{r-3}, X_{r-2}, X_{r-1}, X_r\}$ to get $\mathcal{H}^*$ with at least

$$\frac{r!c}{r^r s} n^3$$

4-edges.

Now consider an arbitrary 4-edge $H \in \mathcal{H}^*$, and $H_1 \subset H$ with $|H_1| = 3$. There can be at most 3 $H' \in \mathcal{H}^*$ edges with $H \cap H' = H_1$, since otherwise we get a set of at most

$$4(r-4)+7 = 4(r-3)+3 < p$$

vertices spanning at least 4 r-edges, a contradiction.

Since we can choose H_1 in 4 different ways, altogether there can be at most 12 $H' \in \mathcal{H}^*$ edges with $|H \cap H'| = 3$. We remove all these at most 12 H' edges from $\mathcal{H}^*$. In the remaining hypergraph again we consider an arbitrary 4-edge H and we remove all other edges H' for which $|H \cap H'| = 3$. We continue in this fashion until we have no two 4-edges H and H' with $|H \cap H'| = 3$. Denote the resulting hypergraph by $\mathcal{H}^{**}$, then

$$|\mathcal{H}^{**}| \geq \frac{r!c}{13r^r s} n^3. \qquad (3)$$

Furthermore, $\mathcal{H}^{**}$ is $\mathcal{F}(4)$-free, since otherwise we get a set of at most

$$4(r-4)+8 = 4(r-3)+4 = p$$

vertices spanning at least 4 r-edges, a contradiction.

However, then (3) is in contradiction with Lemma 2.

This completes the proof of Theorem 2. □

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