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On edge colorings with at least q colors in every subset of p vertices

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Abstract

For fixed integers p and q an edge coloring of K_n is called a (p,q)-coloring if the edges of K_n in every subset of p vertices are colored with at least q distinct colors. Let $f(n,p,q)$ be the smallest number of colors needed for a (p,q)-coloring of K_n. In [3] Erdős and Gyárfás studied this function, if p and q are fixed and n tends to infinity. They determined for every p the smallest q $(= \binom{p}{2} - p + 3)$ for which $f(n,p,q)$ is linear in n and the smallest q for which $f(n,p,q)$ is quadratic in n. They raised the question whether perhaps this is the only q value which results in a linear $f(n,p,q)$. In this paper we study the behavior of $f(n,p,q)$ between the linear and quadratic orders of magnitude. In particular we show that that we can have at most $\log p$ values of q which give a linear $f(n,p,q)$.

1 Introduction

1.1 Notations and definitions

For basic graph concepts see the monograph of Bollobás [1].
$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. K_n is the complete graph on n vertices. In this paper $\log n$ denotes the base 2 logarithm. $pr(n)$ denotes the parity of the natural number n, so it is 1 if n is odd and 0 otherwise.

1.2 Edge colorings with at least q colors in every subset of p vertices

The following interesting concepts were created by Erdős, Elekes and Füredi (see [2]) and then later studied by Erdős and Gyárfás in [3] (see also [4]). For fixed integers p and q an edge coloring of K_n is called a (p,q)-coloring if the edges of K_n in every subset of p vertices are colored with at least q distinct colors. Let $f(n,p,q)$ be the smallest number of colors needed for a (p,q)-coloring of K_n. It will be always assumed that $p \geq 3$ and $2 \leq q \leq \binom{p}{2}$. We restrict our attention to the case when p and q are fixed and n tends to infinity. The study of $f(n,p,q)$ leads to many interesting and difficult problems. For example determining $f(n,p,2)$ is equivalent to determining classical Ramsey numbers for multicolorings.

Among many other interesting results and problems in [3] Erdős and Gyárfás determined for every p the smallest q ($q_{lin} = \binom{p}{2} - p + 3$) for which $f(n,p,q)$ is linear in n and the smallest q ($q_{quad} = \binom{p}{2} - \lfloor \frac{p}{2} \rfloor + 2$) for which $f(n,p,q)$ is quadratic in n. They raised the striking question whether perhaps q_{lin} is the only q value which results in a linear $f(n,p,q)$. In this paper we study the behavior of $f(n,p,q)$ between the linear and quadratic orders of magnitude, so for $q_{lin} \leq q \leq q_{quad}$. In particular we show that that we can have at most $\log p$ values of q which give a linear $f(n,p,q)$.

In order to state our results, first we need some definitions. We define the following two strictly decreasing sequences a_i and b_j of positive integers. $a_0 = p$. Roughly speaking $a_{i+1} = \lfloor \frac{a_i}{2} \rfloor$ but for every second odd a_i we have to add 1. The other sequence b_j is just the subsequence consisting of the odd a_i-s. More precisely, assume that $a_0, a_1, \ldots, a_i$ are already defined. $b_1, b_2, \ldots, \ldots, b_{i'}$ is just the subsequence of $a_0, a_1, \ldots, a_i$ which contains only the odd a_j-s which are greater than 1. Then we define

$$a_{i+1} = \begin{cases} \lceil \frac{a_i}{2} \rceil & \text{if } a_i = b_j \text{ for an even } j \\ \lfloor \frac{a_i}{2} \rfloor & \text{otherwise} \end{cases}$$

Furthermore if a_{i+1} is odd and greater than 1, then $b_{i'+1} = a_{i+1}$.

Thus we have

$$2a_{i+1} = a_i + \begin{cases} 0 & \text{if } a_i \neq b_j \text{ for any } j \text{ (if } a_i \text{ is even)} \\ 1 & \text{if } a_i = b_j \text{ for an even } j \\ (-1) & \text{if } a_i = b_j \text{ for an odd } j \end{cases} \tag{1}$$

Let l_p be the smallest integer for which $a_{l_p} = 1$. Let l'_p be the number of b_j-s among $a_0, a_1, \ldots, a_{l_p - 1}$. We will need the following simple lemma.

Lemma 1. *For $1 \leq i \leq l_p$, we have*

$$a_i \leq \frac{p}{2^i} + 1 - \frac{1}{2^{i-1}} \quad \left(\leq \frac{p}{2^i} + 1 \right). \tag{2}$$

The simple inductive proof is given in the next section. This lemma immediately gives the bound

$$l_p \leq \lceil \log p \rceil. \tag{3}$$

Our main result is the following.

Theorem 1. *For positive integers p, $1 \leq k \leq l_p$, if $q \geq q_{lin} + a_k + k - 1$, then*

$$f(n,p,q) > \frac{1}{4p^2}\, n^{\frac{2^k}{2^k-1}}.$$

Using Lemma 1, we immediately get the following.

Corollary 2. *For positive integers p, $1 \leq k \leq l_p$, if $q \geq q_{lin} + \frac{p}{2^k} + k$, then*

$$f(n,p,q) > \frac{1}{4p^2}\, n^{\frac{2^k}{2^k-1}}.$$

Note that this is not far from the truth. (In fact, for $k = 1$ it gives the right order of magnitude, namely quadratic.) Indeed, from the general probabilistic upper bound of [3], we get the following.

Theorem 3. *([3]) For positive integers p, $1 \leq k \leq l_p$, if $q \leq q_{lin} + \frac{p}{2^k} - \frac{1}{2^{k-1}}$, then*

$$f(n,p,q) \leq c_{p,q}\, n^{\frac{2^k}{2^k-1}},$$

where $c_{p,q}$ depends only on p and q.

Another corollary of the lower bound in Theorem 1 ($k = l_p$ and we use (3)) is that we can have at most $\log p$ values with a linear $f(n,p,q)$.

Corollary 4. *If $q \geq q_{lin} + \log p$, then*

$$f(n,p,q) > \frac{1}{4p^2}\, n^{\frac{2^{l_p}}{2^{l_p}-1}}.$$

We have roughly a "gap" of size at most k in the values of q between the lower bound of Corollary 2 and the upper bound of Theorem 3. It would be desirable to close this gap. We believe, as is often the case, that the probabilistic upper bound (Theorem 3) is closer to the truth.

First we give some preliminary facts in the next section. Then in Section 3 we prove Theorem 1.

2 Preliminaries

To prove Lemma 1 we use induction on $i = 1, 2, \ldots, l_p$. It is true for $i = 1$. Assume that it is true for i and then for $i + 1$ from the definition of a_{i+1} we get

$$a_{i+1} \leq \frac{a_i + 1}{2} \leq \frac{\frac{p}{2^i} + 1 - \frac{1}{2^{i-1}} + 1}{2} = \frac{p}{2^{i+1}} + 1 - \frac{1}{2^i},$$

and thus proving Lemma 1.

We introduce the following indicator for $0 \le i \le l_p - 1$.

$$\delta_i = \begin{cases} 1 & \text{if } b_{j-1} > a_i \ge b_j \text{ for an odd } j > 1\text{, or if } a_i \ge b_1\text{, or for even } l'_p \text{ if } a_i < b_{l'_p} \\ 0 & \text{otherwise} \end{cases}$$

We will need the following.

Lemma 2. *For any* $0 \le i \le l_p - 1$

$$\sum_{j=0}^{i} a_{l_p-j} = a_{l_p-i-1} - \delta_{l_p-i-1} - pr(l'_p). \tag{4}$$

Proof: We use induction on $i = 0, 1, \ldots, l_p - 1$. (4) is true for $i = 0$, since $a_{l_p} = 1$ and $a_{l_p-1} = 1 + \delta_{l_p-1} + pr(l'_p)$.

Assuming that (4) is true for i, for $i+1$ using (1) we get

$$\sum_{j=0}^{i+1} a_{l_p-j} = \sum_{j=0}^{i} a_{l_p-j} + a_{l_p-i-1} = 2a_{l_p-i-1} - \delta_{l_p-i-1} - pr(l'_p) = a_{l_p-i} - \delta_{l_p-i} - pr(l'_p),$$

proving the lemma.

¿From this we get:

Lemma 3. *For any* $1 \le k \le l_p$

$$\sum_{j=1}^{k} a_j \ge a_0 - a_k - 1 = p - a_k - 1.$$

Proof:

$$\sum_{j=1}^{k} a_j = \sum_{j=0}^{l_p-1} a_{l_p-j} - \sum_{j=0}^{l_p-k-1} a_{l_p-j} = a_0 - \delta_0 - a_k + \delta_k \ge a_0 - a_k - 1.$$

3 Proof of Theorem 1

Let $1 \le k \le l_p$ and $q \ge q_{lin} + a_k + k - 1$. Denote

$$h = h(n,k) = \frac{1}{4p^2}\, n^{\frac{2^k}{2^k-1}}. \tag{5}$$

Assume indirectly that there is a (p,q)-coloring of K_n with at most h colors. From this assumption we get a contradiction.

Consider a fixed (p,q)-coloring of K_n with at most h colors. First we find a sequence of monochromatic matchings $M_1, M_2, \ldots, M_k$ in K_n. For M_1, there is a color class (denoted by C_1) in K_n which contains at least $\frac{\binom{n}{2}}{h}$ edges. In C_1 all the connected components have size at most $p-1$, since otherwise we immediately have a K_p with fewer than q colors, a contradiction. Then in C_1 we can clearly choose a matching M_1 (for example by taking one edge from each component) of even size at least

$$\frac{\binom{n}{2}}{ph}.$$

Partition the vertices spanned by M_1 into A and B, so M_1 is a matching between A and B. Halve the vertices of A arbitrarily and denote one of the halves by A_1. Denote by B_1 the set of vertices in B which are not matched to vertices in A_1 by M_1. Consider the complete bipartite graph between A_1 and B_1 and the color class (denoted by C_2) which contains the most edges in it. Again from these edges in C_2 we can choose a matching M_2 of even size at least

$$\frac{\left(\frac{|M_1|}{2}\right)^2}{ph}.$$

We continue in this fashion. Assume that M_i is already defined. Denote an arbitrary half of the endvertices of M_i in A by A_{i+1}. The set of endvertices of the edges of M_i in B which are not matched to vertices in A_{i+1} is denoted by B_{i+1}. Consider the complete bipartite graph between A_{i+1} and B_{i+1} and the color class (denoted by C_{i+1}) which contains the most edges in it. ¿From these edges in C_{i+1} we can choose a matching M_{i+1} of even size at least

$$\frac{\left(\frac{|M_i|}{2}\right)^2}{ph}.$$

Thus

$$|M_{i+1}| \geq \frac{\left(\frac{|M_i|}{2}\right)^2}{ph}.$$

Then by induction we have

$$|M_i| \geq \frac{n^{2^i}}{(4ph)^{2^i-1}}.$$

Indeed, this is true for $i=1$

$$|M_1| > \frac{n^2}{4ph}.$$

For $i+1$ we get

$$|M_{i+1}| \geq \frac{\left(\frac{|M_i|}{2}\right)^2}{ph} \geq \frac{\left(\frac{n^{2^i}}{2(4ph)^{2^i-1}}\right)^2}{ph} = \frac{n^{2^{i+1}}}{(4ph)^{2^{i+1}-1}}.$$

This and (5) implies that $|M_i| \geq p \geq a_i, 1 \leq i \leq k$ and thus the matchings $M_1, M_2, \ldots, M_k$ can be chosen.

Next using these matchings M_i we choose a K_p such that it contains at most $q-1$ colors, a contradiction. For this purpose we will find another sequence of matchings M_i' such that $M_i' \subset M_i$, $|M_i'| = a_i$ for $1 \leq i \leq k$ and $\left|\cup_{i=1}^k V(M_i')\right| \leq p$.

M_k' is just a set of a_k arbitrary edges from M_k. Assume that $M_k', \ldots, M_{i+1}'$ are already defined and now we define M_i'. We consider the $2a_{i+1}$ vertices in $V(M_{i+1}')$ and the edges of M_i incident to these vertices. We have four cases.

Case 1: If $2a_{i+1} = a_i$ (so we have the first case in (1)), then this is M_i'.

Case 2: If $2a_{i+1} = a_i + 1$ (second case in (1)), so $a_i = b_j$ for an even j, then we remove one of the edges from this set incident to a vertex in $V(M_{i+1}') \cap A$ to get M_i'. Furthermore, we mark this vertex in $V(M_{i+1}') \cap A$ which is not covered by M_i'. This marked vertex is going to be covered only by $M_{i'}'$ if $a_{i'} = b_{j-1}$ (unless $i' = 0$).

Case 3: If $2a_{i+1} = a_i - 1$ (third case in (1)) and there is no marked vertex at the moment, then to get M_i' we add one arbitrary edge of M_i to these $2a_{i+1}$ edges.

Case 4: Finally, if $2a_{i+1} = a_i - 1$ and there is a marked vertex then to get M_i' we add to these $2a_{i+1}$ edges the edge of M_i incident to the marked vertex and we "unmark" this vertex.

We continue in this fashion until $M_k', \ldots, M_1'$ are defined. Then $\left|\cup_{i=1}^k V(M_i')\right| = p$ or $p-1$. Note that it can be $p-1$ only if $a_0 = p = b_1$ is odd, and there is no other odd a_i among $a_1, a_2, \ldots, a_{k-1}$. In this case we add one more arbitrary vertex to get the K_p, otherwise $\cup_{i=1}^k V(M_i')$ is the K_p.

By the above construction this K_p contains a_i edges from the matching M_i (and thus from color class C_i) for $1 \leq i \leq k$.

Now since Lemma 3 implies

$$\sum_{j=1}^{k} (a_j - 1) \geq p - a_k - 1 - k,$$

thus the number of colors used in this K_p is at most

$$\binom{p}{2} - p + a_k + k + 1 \leq q - 1,$$

a contradiction. This completes the proof of Theorem 1.

References

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