Isovariance Ellipse

Similarly to the idea of an IsoMean line is the idea of an˝IsoVariance ellipse. Here, we choose portfolio weights to provide some target level of risk. ˝However the standard deviation of returns from a three security portfolio is a quadratic˝equation in portfolio weights. We will see below that by choosing weights to give a constant target˝risk traces out an ellipse in portfolio weight space.

The objective of this topic is to construct the set of˝portfolio weights that provide the same portfolio standard deviation for the general three˝security problem and verify that this is indeed an ellipse. To make life simpler, let us initially˝work with the square of the risk, which is the portfolio variance. First, the portfolio variance˝equation is:

$$\sigma_{\alpha}^{2} = \sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_i \alpha_j \sigma_{ij}$$

and again substituting out α_3 for $1 - \alpha_1 - \alpha_2$, the portfolio variance results in an equation of portfolio weights having the following quadratic form.

$$\sigma_{\alpha}^{2} = \alpha_1^2 \sigma_{11} + \alpha_2^2 \sigma_{22} + (1-\alpha_1-\alpha_2)^2 \sigma_{33} + 2\alpha_1\alpha_2\sigma_{12} + 2\alpha_1(1-\alpha_1-\alpha_2)\sigma_{13} + 2\alpha_2(1-\alpha_1-\alpha_2)\sigma_{23}$$

The above equation can be rearranged into the general˝form for an equation of second degree in α_1 and α_2.

The general form for an equation of second degree is:

$$A\alpha_1^2 + B\alpha_1\alpha_2 + C\alpha_2^2 + D\alpha_1 + E\alpha_2 + F = 0$$

Re-arranging the portfolio variance equation yields the˝general form if we let

$$A = \sigma_{11} + \sigma_{33} - 2\sigma_{13}$$
$$B = 2\sigma_{33} + 2\sigma_{12} - 2\sigma_{13} - 2\sigma_{23}$$
$$C = \sigma_{22} + \sigma_{33} - 2\sigma_{23}$$
$$D = 2\sigma_{13} - 2\sigma_{33}$$
$$E = 2\sigma_{23} - 2\sigma_{33}$$
$$F = \sigma_{33} - \text{Target Portfolio Variance}$$

For our 3 firm case these values are:

$$A = .923 + .528 - 2(-.582) = 2.615$$
$$B = 2(.528) + 2(.063) - 2(-.582) - 2(-.359) = 3.064$$

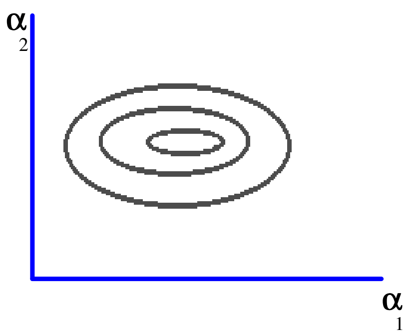
$$C = .923 + .528 - 2(-.359) = 2.169$$
$$D = 2(-.582) - 2(.528) = -2.22$$
$$E = 2(-.359) - 2(.528) = -1.774$$
$$F = .528 - \text{Target Portfolio Variance}$$

By applying the graphical properties of second degree equations if the sign of the discriminant, (B2 - 4AC), is negative then geometrically, then the relationship between the portfolio weights forms an ellipse. The ellipse collapses down to a single point when the target variance is the minimum possible variance.

For the 3 firm case this condition is satisfied:

$$B^2 - 4AC = 3.182^2 - 4(2.777)(2.263) = -15.012$$

Graphically these ellipses are depicted below.



Recall, that in this subject in CAPM Tutor by clicking on different parts of the minimum variance frontier chooses portfolios with different volatilities. As a result, by first clicking on frontier and then clicking on different parts of this frontier allows you to construct graphically the isovariance ellipse in units of portfolio weights.

The next part of this chapter lets you read about characterizing the IsoMean IsoVariance Tangency Points.