Determination of Uncertainties for Temperature Fixed Points & Their ITS-90 Subranges

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Abstract

The International Temperature Scale of 1990 (ITS-90) defines temperature values over the range of –259.3467 to 961.78 °C at fourteen fixed points. ITS-90 temperature values between these fixed points are defined using a Standard Platinum Resistance Thermometer (SPRT) as an interpolation device. In order to determine the uncertainty of the values produced by an SPRT between the fixed points an uncertainty propagation analysis must be performed. Because of the natural variations between SPRTs (non-uniqueness), the uncertainty propagation will be different for each SPRT. This paper shows data for fixed points ranging from the Triple Point of Argon (TPAr) (-189.3442 °C) to the Freezing Point of Aluminum (FPAl)(660.323 °C). The uncertainties at the fixed points are then used to determine the uncertainties of the applicable subranges. The method for determining the uncertainty propagation of a typical SPRT is explained in detail. Examples are given for ITS-90 subranges 4, 7, and 8. The method described may be applied to other ITS-90 subranges.

1. Fixed Point Uncertainty Analysis

1.1. Between –259.3467 and 961.78 °C the ITS-90 is defined in terms of ratios derived from an SPRT's resistance at various fixed points divided by its resistance at the Triple Point of Water (TPW). Temperatures between the fixed points are defined by reference functions and deviation functions using resistance ratios. An SPRT's resistance ratio at a given temperature is expressed as $W(T_{90})$, where W is the ratio and (T_{90}) is the thermodynamic temperature in Kelvin. Alternately t_{90} is sometimes used to express the thermodynamic temperature in degrees Celsius. The deviation between the SPRT's resistance ratio and the reference function is expressed as $\Delta W(T_{90})$.

A TPW and four other fixed points are required to calibrate an SPRT for use as an ITS-90 interpolation device over the range of –200 °C to 500 °C. The additional four points are, the Triple Point of Argon (TPAr), the Triple Point of Mercury (TPHg), the Freezing Point of Tin (FPSn), and the Freezing Point of Zinc (FPZn). The Freezing Point of Aluminum (FPAl), can be added to extend the range to 661 °C. These six fixed points as well as their ITS-90 temperatures and subranges are illustrated in Figure 1.

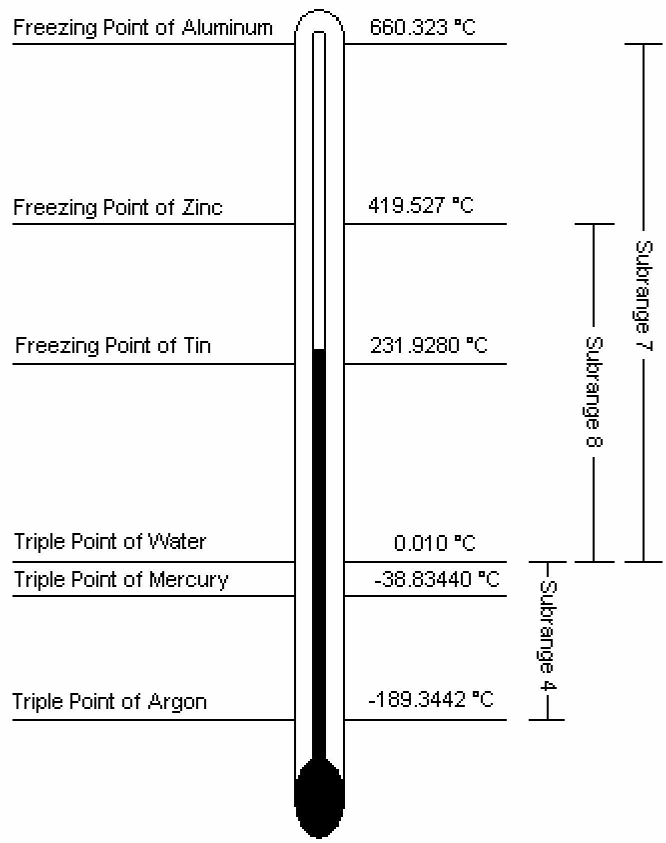

Figure 1

Proper analysis of the uncertainty of an SPRT calibration must consider not only the uncertainty of each of the fixed point measurements but also the effects of the interaction of the fixed point measurement uncertainties. The general characteristics of SPRTs must be considered, as well as the variations between individual SPRTs.

1.2. The process begins by determining the uncertainty of the measurements at each fixed point. The uncertainty of the resistance measurement at the TPW is considered before the others due to its involvement in the determination of the W value at each of the other fixed points. Major contributors to the uncertainty of the TPW include possible impurities in the cell, cell reproducibility, and errors in the resistance-measuring device. A sample analysis of uncertainties at the TPW is shown in Table 1.

Table 1. TPW measurement uncertainty.

Component	Value mK	Distribution Type	Divisor Value	Adjusted Component	Component Square	Notes
Type A						
Reproducibility	**0.100**	**Normal**	**1.0**	**0.100**	**0.0100**	**Mfg. Spec.**
Type B						
Bridge Error	**0.360**	**Rectangular**	**1.732**	**0.208**	**0.0432**	**Bridge error of (+/- 1ppm of ind. + 0.0001 ohms).**
Cell Specification	**0.040**	**Rectangular**	**1.732**	**0.023**	**0.0005**	**Confirmed by NIST Study.**
				Sum	**0.0537**	
			U=	**Square Root**	**0.2318**	
			Expanded U=	**(For k=2)**	**0.4636**	**Use 0.47 mK**

The uncertainties of the measurements at the remaining fixed points are considered next. This determination includes the plotting of the fixed point cell's change of state plateau. A check thermometer is used during calibration to ensure that the fixed point is in plateau during the period of the test. A sample TPHg plateau is illustrated in Figures 2.

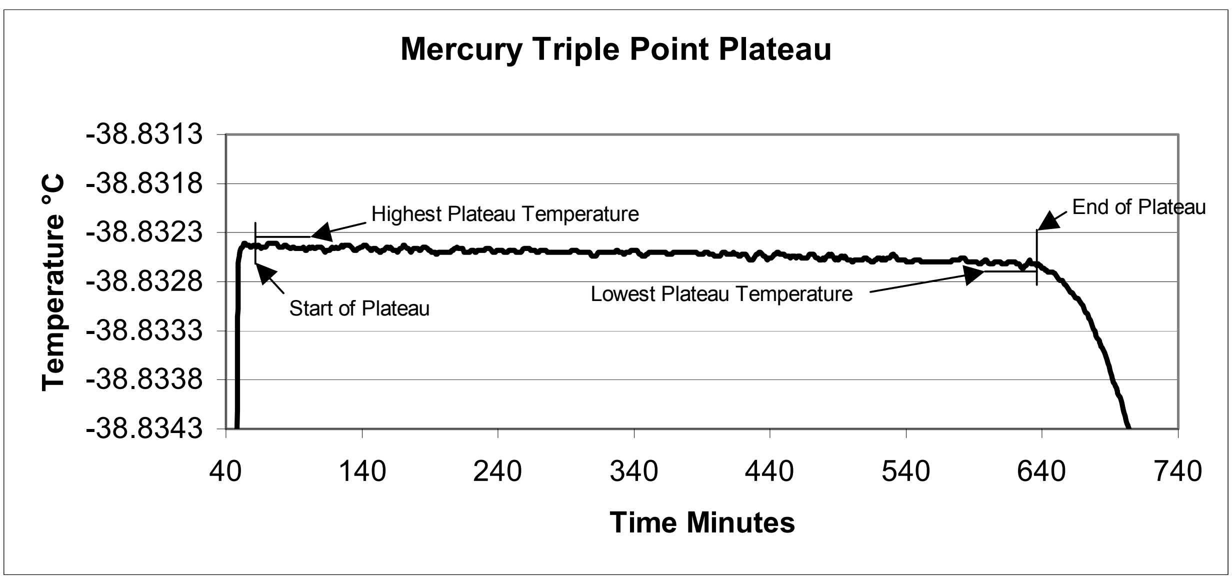

Figure 2. Mercury Triple Point Plateau.

A reasonable estimate can be made of the uncertainty of a fixed point cell's plateau using graphs like the one above. The duration of a fixed point's plateau may be calculated mathematically by determining the period over which the short-term temperature drift does not significantly exceed the instability of the resistance-measuring device. In most cases, however, it is easier to graph the data and determine the plateau duration visually. The uncertainty of the plateau is then calculated by subtracting the lowest reading from the highest reading within the plateau. Using this method, the uncertainty of the Mercury Triple Point graphed above can be conservatively estimated to be 0.3 mK.

The uncertainty of the overall measurement process at each fixed point is then calculated. The analysis includes the uncertainties of the fixed point cell, resistance measurement uncertainties, and the effects on W of the uncertainties at the TPW.

The contribution of the uncertainty at the TPW can be estimated by first calculating a typical W for the temperature under consideration. The W value is then recalculated using a TPW value that has been adjusted by the amount of the TPW uncertainty. The two W values are then converted to temperatures using ITS-90 equations. The difference in the two temperatures is the effect of the TPW uncertainty at the test temperature.

Table 2 illustrates a fixed point uncertainty analysis using the TPHg as an example. The uncertainty attributed to the resistance bridge is not based on the absolute accuracy of the resistances reported. Bridge uncertainty is based on the bridge linearity over the range between the TPW and the fixed point. Since the measurement being considered is a ratio of the two resistance readings, bridge linearity is the only uncertainty component that needs to be considered. Cell specifications include an analysis of the purity of the cell sample.

Table 2. Triple Point of Mercury uncertainty analysis.

Component	Value mK	Distribution Type	Divisor Value	Adjusted Component	Component Square	Notes
Type A						
Plateau Drift	**0.200**	**Normal**	**1.0**	**0.200**	**0.0400**	From Graph of Plateau Drift
Type B						
Bridge Error	**0.310**	**Rectangular**	**1.732**	**0.179**	**0.0320**	Bridge error of (+/- 1ppm of ind. + 0.0001 ohms).
Cell Specification	**0.150**		**2.000**	**0.075**	**0.0056**	1999 NIST study. Value (k=2).
TP H_2O Ratio Error	**0.380**	**Rectangular**	**1.732**	**0.219**	**0.0481**	W w/o TPW error minus W w/TPW error.
				Sum	**0.1258**	
			U=	**Square Root**	**0.3547**	
			Expanded U=	**(For k=2)**	**0.7094**	**Use 0.71 mK**

2. Analysis of Uncertainty Propagation

2.1. Once the uncertainties of the individual fixed points have been calculated they must be combined to determine the SPRT's overall uncertainty. This could be accomplished using a simple root sum square analysis, similar to those shown above, using the uncertainties of the fixed points, the resistance measurement uncertainty, and the TPW ratio error. A more thorough method is to analyze the uncertainty propagation of the SPRT using ITS-90 equations.

The ITS-90 coefficients for a given range are calculated using actual calibration data for the SPRT in question. The coefficients are then re-calculated with the temperature data adjusted to reflect the worst-case error at one of the fixed points. A separate set of coefficients is generated to reflect the worst-case error at each of the other fixed points involved in the calibration.

Table 3 illustrates an uncertainty propagation based on an ITS-90 subrange 4 calibration (-190 to 0 °C). Subrange 4 coefficients (a_4 and b_4) are defined by the ITS-90 deviation equation, Mangum [1], as:

$$\Delta W_4(T_{90}) = a_4[W(T_{90}) - 1] + b_4[W(T_{90}) - 1]\ln W(T_{90}).$$

Table 3. Uncertainty propagation from –200 to 0 °C.

U @ Ar =	0.0021	°C					
U @ Hg =	0.00071	°C					
	t_{90}	Ω	Ω TPW				
Argon (Ar):	-189.34420	5.5485533	25.5067000	a_4:	4.83725E-03	**Original**	
Mercury (Hg):	-38.83440	21.5151198	25.5067500	b_4:	4.57314E-03	**Coefficients**	
Ar + U @ Ar:	-189.34630	5.5485533	25.5067000	a_4 + U @ Ar:	4.83871E-03	**Coefficients**	
Hg:	-38.83440	21.5151198	25.5067500	b_4 + U @ Ar:	4.58174E-03	**w/U of Ar**	
Ar:	-189.34420	5.5485533	25.5067000	a_4 + U @ Hg:	4.81663E-03	**Coefficients**	
Hg + U @ Hg:	-38.83511	21.5151198	25.5067500	b_4 + U @ Hg:	4.55963E-03	**w/U of Hg**	
Nominal	**Nominal**	**t_{90} With**	**t_{90} w/U**	**t_{90} w/U**	**Δt w/U**	**Δt w/U**	**Total**
t_{90}	**W**	**No U**	**@ Argon**	**@ Hg**	**@ Ar mK**	**@ Hg mk**	**U mK**
-200	0.1724019	-199.999958	-200.002585	-199.999353	2.627	-0.605	2.696
-175	0.2788005	-175.000067	-175.001654	-175.000626	1.587	0.559	1.683
-150	0.3850042	-149.999964	-150.000939	-150.001080	0.976	1.116	1.483
-125	0.4898617	-124.999935	-125.000505	-125.001273	0.570	1.338	1.454
-100	0.5935443	-100.000076	-100.000374	-100.001411	0.297	1.335	1.368
-75	0.6962833	-75.000026	-75.000148	-75.001194	0.123	1.168	1.175
-50	0.7982181	-49.999905	-49.999929	-50.000780	0.024	0.875	0.875
-25	0.8994280	-25.000011	-24.999997	-25.000491	-0.014	0.480	0.480
0	0.9999601	0.000047	0.000047	0.000047	0.000	0.000	0.000

An evaluation of the contents of Table 3 and the process used to generate it are given below. Details of the process that are not contained in the main text are included in Appendix A. Corresponding Appendix A formula numbers have been included in braces, {}, for reference purposes.

The W value for the TPHg (W_{Hg}) and the W value for the TPAr (W_{Ar}) are calculated. {A1}

$$W_{Hg} = 21.5151198/25.50675 = 0.843506907$$
$$W_{Ar} = 5.5485533/25.5067 = 0.21753317$$

The ITS-90 reference W values for the TPHg (W_{rHg}) and the TPAr (W_{rAr}) are calculated. {A2 through A6}

$$y_{Hg} = 0.897740814$$
$$y_{Ar} = 0.212296342$$

$$W_{rHg} = 0.844142105$$
$$W_{rAr} = 0.215859752$$

The reference W values are subtracted from the measured W values to determine the SPRT's deviation from the reference function at the fixed points (ΔW).

$$\Delta W_{Hg} = 0.843506907 - 0.844142105 = -0.000635198$$
$$\Delta W_{Ar} = 0.21753317 - 0.215859752 = 0.001673418$$

A matrix is then established using the data generated above. {A7 through A14} The results are:

$$\text{Determinant} = -0.1659471$$
$$D_{Hg} = 7.588998E-4$$
$$D_{Ar} = -8.0272687E-4.$$

The matrix is then solved to determine the values of a_4 and b_4. {A15} So:

$$a_4 = 4.83724E-3$$
$$b_4 = 4.57314E-3$$

Noticeable but insignificant differences may occur between hand calculations and computer calculations due to rounding errors.

A group of temperatures is then generated for each set of coefficients based on selected nominal W values. The temperatures are generally evenly spaced across the range under examination. The differences between the temperatures generated by the original coefficients and those generated by the adjusted coefficients indicate the effect of the propagation of the uncertainties.

The example in Table 3 begins with temperatures (column 1) at intervals of 25 °C beginning at 0 °C and ending at –200 °C.

The original ITS-90 coefficients are used to convert the column 1 temperatures to nominal W values (column 2) for the selected temperatures {A19}. Using –100 °C as an example the result would be, W = 0.59354431.

The original and adjusted coefficients are then used to generate temperatures based on the nominal W values (columns 3, 4, and 5) {A1, A6, and A16 through A18}. Using the original coefficients and the W value from column as an example, the resulting temperature equals -100.000076 °C. Notice that the temperature generated by the original coefficients is not identical to the initial nominal temperature. This is due to the imperfection of the iteration used and the difference between the reference function and the approximate inverse reference function.

The differences between the temperatures generated by the original coefficients and the temperatures generated by the adjusted coefficients are then calculated (columns 6 and 7).

$$\Delta t \text{ w/U @ Argon} = -100.000076\ °C + 100.000374\ °C = 0.000298\ °C = 0.298 \text{ mK}$$

$$\Delta t \text{ w/U @ Mercury} = -100.000076\ °C + 100.001411\ °C = 0.001335\ °C = 1.335 \text{ mK}$$

The root sum squares of the differences are then calculated. The result (column 8) is the total SPRT uncertainty for the nominal temperature being considered.

$$\text{Total SPRT uncertainty @ } -100\ °C = \text{Sqrt}(0.298^2 + 1.335^2) = 1.368 \text{ mK}$$

When the data are represented graphically, it is easy to see the effects of the uncertainty of each individual fixed point and the compound effect of the combined uncertainties. Figure 6 was generated using standard graphing tools and the data from columns 3, 6, 7, & 8 of Table 3.

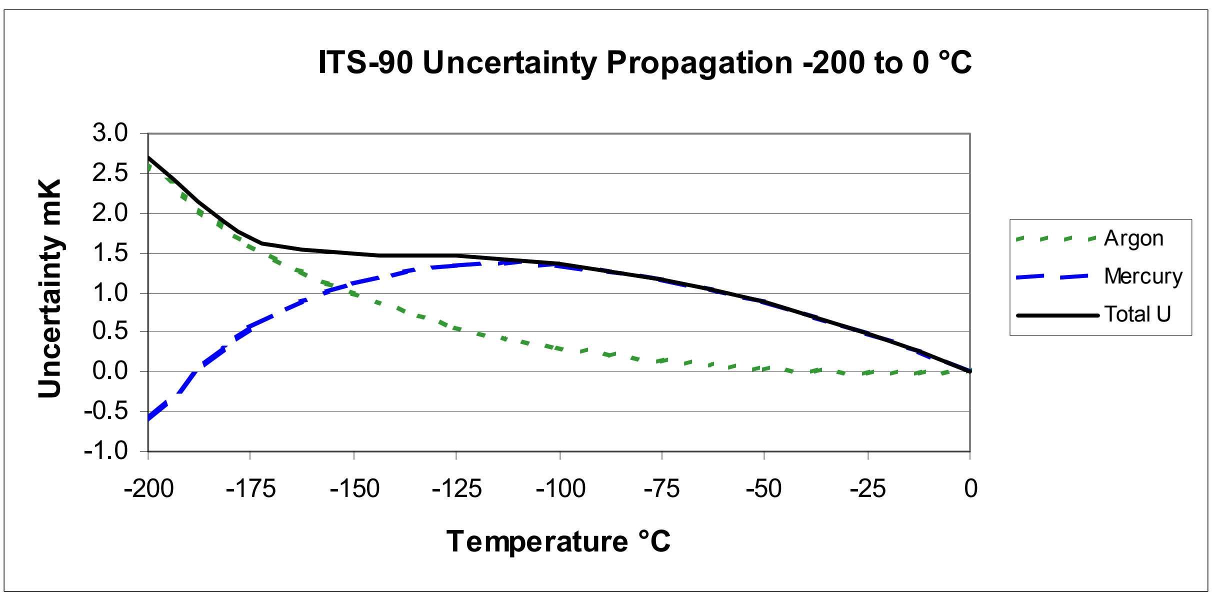

Figure 6. Uncertainty propagation from –200 to 0 °C.

An analysis based on a simple root sum square of the components will yield a result similar to the largest value determined using the uncertainty propagation method. For example, a simple root sum square of the components of the calibration used in Figure 6 estimated an uncertainty of 2.34mK. This compares favorably to the 2.736mK estimate illustrated in Figure 6.

The advantage of calculating the uncertainty propagation is that the result more accurately reflects the true uncertainty of the SPRT in normal use. This is particularly true when considering the uncertainty of individual temperatures within the range as opposed to the uncertainty of the full range. Also, each SPRT will generate a somewhat different uncertainty propagation based on its unique coefficients. Therefore, while the simple root sum square method gives an accurate estimate of the overall process uncertainty at full scale, the uncertainty propagation method gives a more accurate estimate of the uncertainty of a particular SPRT calibration over it's entire range.

2.2. Table 4 illustrates an uncertainty propagation analysis for an SPRT calibrated for ITS-90 subrange 8 (0 to 420 °C). Except for the ITS-90 equations involved, the process is identical to the one used for subrange 4. Subrange 8 coefficients (a_8 and b_8) are defined by the ITS-90 deviation equation, Mangum [1], as:

$$\Delta W_8(T_{90}) = a_8[W(T_{90}) - 1] + b_8[W(T_{90}) - 1]^2.$$

See Appendix A for details regarding subrange 8 calculations.

Table 4. Uncertainty propagation from 0 to 500 °C.

U @ Sn =	0.0017	°C					
U @ Zn =	0.0027	°C					
	t_{90}	**Ω**	**Ω TPW**				
Tin (Sn):	232.000000	48.2530300	25.4914000	a_8:	-1.606189E-04	**Original**	
Zinc (Zn):	420.000000	65.5204200	25.4914000	b_8:	-9.399133E-06	**Coefficients**	
Sn + U:	231.998300	48.2530300	25.4914000	a_8 + U @ Sn:	-1.442330E-04	**Coefficients**	
Zn:	420.000000	65.5204200	25.4914000	b_8 + U @ Sn:	-1.983405E-05	**w/U of Sn**	
Sn:	232.000000	48.2530300	25.4914000	a_8 + U @ Zn:	-1.685400E-04	**Coefficients**	
Zn + U:	419.997300	65.5204200	25.4914000	b_8 + U @ Zn:	-5.281072E-07	**w/U of Zn**	
Nominal	**Nominal**	**t_{90} With**	**t_{90} w/U**	**t_{90} w/U**	**Δt w/U**	**Δt w/U**	**Total**
t_{90}	**W**	**No U**	**@ Sn**	**@ Zn**	**@ Sn mK**	**@ Zn mk**	**U mK**
0	0.999960	0.000000	0.000000	0.000000	0.000	0.000	0.000
100	1.392708	100.000009	99.998762	100.000460	1.248	-0.450	1.326
200	1.773533	200.000057	199.998342	200.000275	1.715	-0.218	1.729
300	2.142644	300.000061	299.998658	299.999365	1.403	0.697	1.567
400	2.500179	399.999990	399.999678	399.997693	0.312	2.297	2.318
500	2.846068	500.000012	500.001575	499.995422	-1.562	4.590	4.849

Figure 7 is a graphic representation of the data in Table 4.

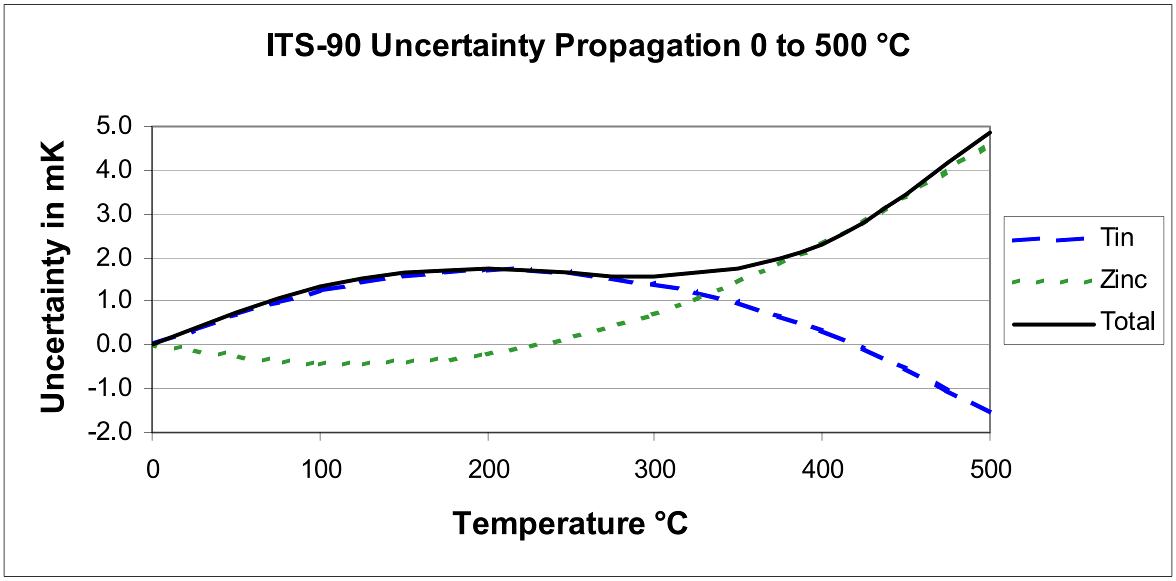

Figure 7. Uncertainty propagation from 0 to 500 °C.

2.3. The process for determining the uncertainty propagation for subrange 7 (0 to 661 °C) is like the one used for subranges 4 and 8 except that a third set of equations is added for the third fixed point. Table 5 illustrates an analysis of a subrange 7 calibration. Subrange 7 coefficients (a_7, b_7, and c_7) are defined by the ITS-90 deviation equation, Mangum [1], as:

$$\Delta W_7(T_{90}) = a_7[W(T_{90}) - 1] + b_7[W(T_{90}) - 1]^2 + c_7[W(T_{90}) - 1]^3.$$

See Appendix A for details regarding subrange 7calculations.

Table 5. Uncertainty propagation from 0 to 661 °C.

U @ Sn =	0.00024	°C							
U @ Zn =	0.00041	°C							
U @ Al =	0.00064	°C							
	t90 °C	**T. I. Ohms**	**T. I. TPW**						
(Sn):	232.0000	48.2802	25.5067	a_7:	-2.43039E-04	**Original Coefficients**			
(Zn):	420.0000	65.5565	25.5067	b_7:	-6.75756E-06				
(Al):	661.0000	86.1501	25.5067	c_7:	-1.02624E-06				
Sn+U @ Sn:	231.9998	48.2802	25.5067	a_7+U @ Sn:	-2.39334E-04	**Coefficients w/U of Sn**			
Zn:	420.0000	65.5565	25.5067	b_7+U @ Sn:	-1.06752E-05				
Al:	661.0000	86.1501	25.5067	c_7+U @ Sn:	-3.38497E-08				
Sn:	232.0000	48.2802	25.5067	a_7+U @ Zn:	-2.46582E-04	**Coefficients w/U of Zn**			
Zn+U @ Zn:	419.9996	65.5565	25.5067	b_7+U @ Zn:	-1.30017E-06				
Al:	661.0000	86.1501	25.5067	c_7+U @ Zn:	-2.69497E-06				
Sn:	232.0000	48.2802	25.5067	a_7+U @ Al:	-2.42030E-04	**Coefficients w/U of Al**			
Zn:	420.0000	65.5565	25.5067	b_7+U @ Al:	-8.52975E-06				
Al+U @ Al:	660.9994	86.1501	25.5067	c_7+U @ Al:	-3.06717E-07				
Nominal	**Nominal**	**t_{90} With**	**t_{90} W/U**	**t_{90} W/U**	**t_{90} W/U**	**Δt w/U @**	**Δt w/U @**	**Δt w/U @**	**Total**
t_{90}	**W**	**No U**	**@ Sn**	**@ Zn**	**@ Al**	**Sn mK**	**Zn mk**	**Al mK**	**U mK**
0	0.99996012	0.00000	0.00000	0.00000	0.00000	0.000	0.000	0.000	0.000
100	1.392676272	100.00001	99.99977	100.00018	99.99997	0.235	-0.168	0.043	0.293
200	1.773470805	200.00006	199.99979	200.00012	200.00004	0.262	-0.066	0.014	0.270
300	2.142552252	300.00006	299.99990	299.99990	300.00009	0.165	0.162	-0.024	0.232
400	2.500057861	399.99999	399.99996	399.99961	400.00000	0.026	0.379	-0.013	0.380
500	2.845918856	500.00001	500.00009	499.99955	499.99991	-0.079	0.459	0.103	0.477
600	3.179852091	600.00008	600.00016	599.99980	599.99971	-0.080	0.282	0.375	0.476
661	3.377548252	661.00006	661.00006	661.00006	660.99942	0.000	0.000	0.640	0.640

Figure 8 represents the Table 5 data graphically.

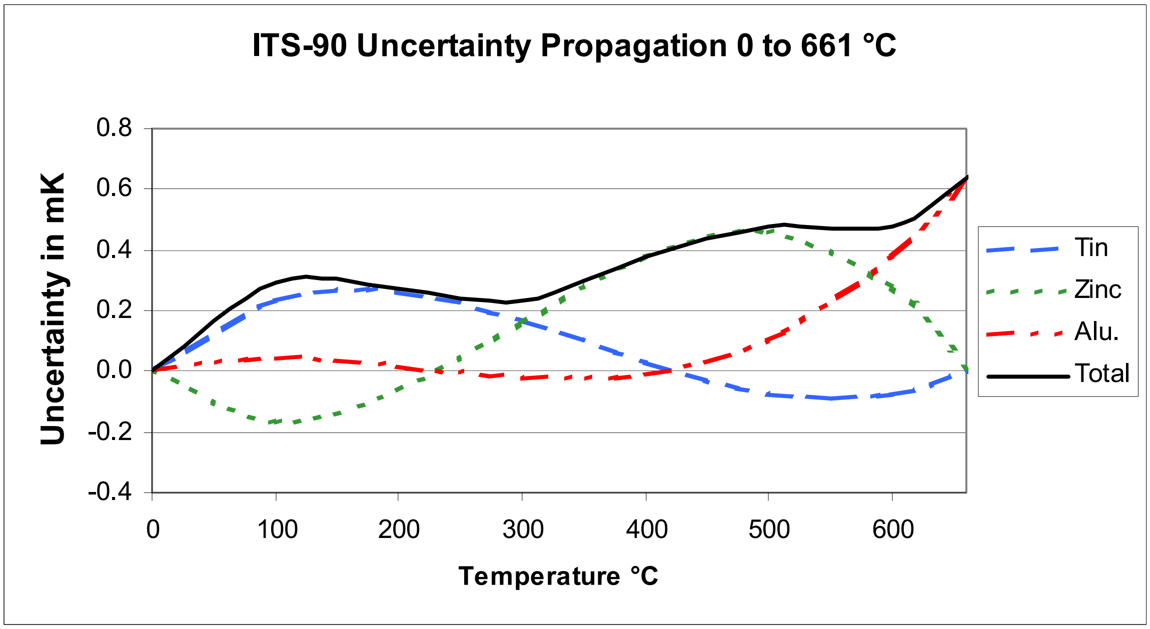

Figure 8. Uncertainty propagation from 0 to 661 °C.

3. Process Validation

The process outlined above was used to develop an uncertainty propagation analysis based on a particular set of equipment and a specific calibration method. Two NIST calibrated SPRTs were then calibrated using the identified equipment and method.

The two SPRTs were calibrated a total of four times over a period of three months. After each calibration the difference between the local calibration and the NIST calibration was calculated.

Figure 9 is a graph of the differences that were calculated. The calibration of the first SPRT is represented by the letter A. Three calibrations of the second SPRT are represented as B1, B2, and B3. U+ and U- represent the results of the uncertainty propagation analysis.

The NIST calibrations are being used as the 0 mK error line on the graph (y axis 0). Therefore, the uncertainty of the NIST calibration (NIST U) was included to show the possible effect of those uncertainties on the data.

The local calibration data are, to the extent possible, graphed in the positive direction. The NIST uncertainty data are graphed in the negative direction. This was done to make the graph easier to read.

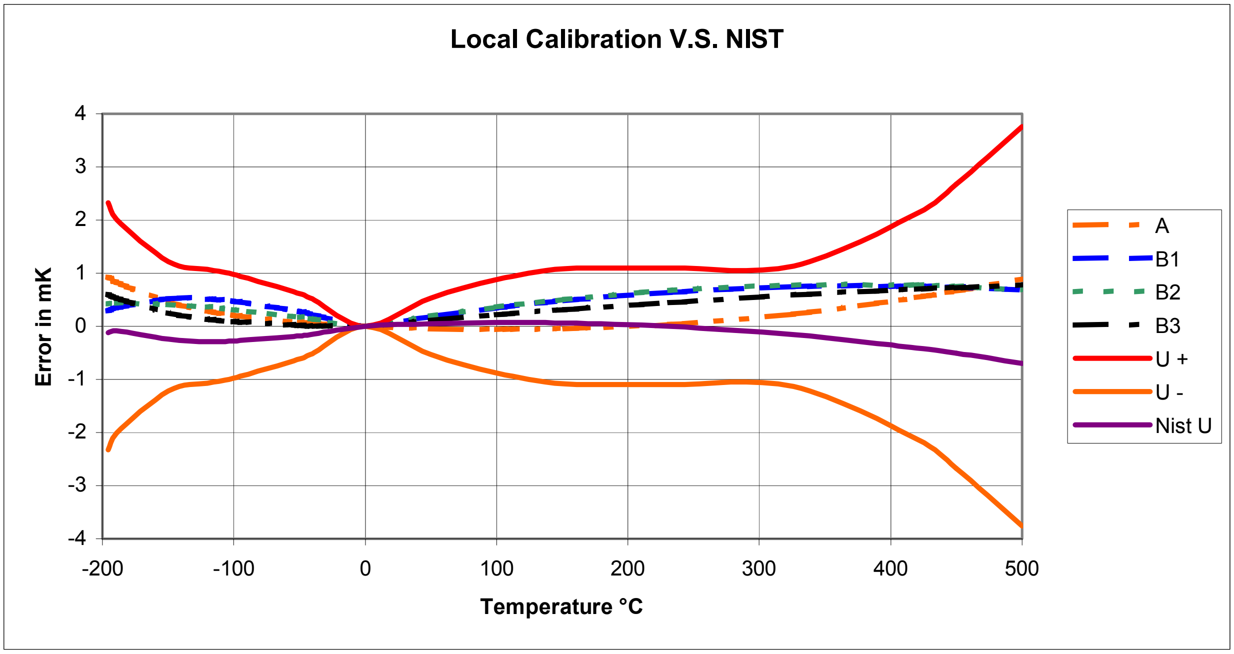

Figure 9.

4.0. Conclusion

SPRTs are used as the ITS-90 interpolating device for all temperatures between the fixed points. Since the SPRT is the interpolating device there is no way to directly measure the uncertainty of temperatures indicated by the SPRT between the fixed points. The most accurate method for estimating the uncertainty of these reported temperatures is to perform an uncertainty analysis that includes an uncertainty propagation analysis. The uncertainty propagation analysis may be accomplished with reasonable effort using ITS-90 equations, a spreadsheet, and graphing tools. The efficacy of this method can be demonstrated through the use of empirical data.

5.0. Acknowledgements

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6.0. References

1. B. W. Mangum, G. T. Furukawa, Guidelines for Realizing the International Temperature Scale of 1990 (ITS-90), NIST Technical Note 1265, pp. 1-18, 1990.

Appendix A

Notes for calculations of ITS-90 subranges 4, 7, and 8

All calculations presented below are based on the details for ITS-90 as outlined in reference [1], Mangum, et.al., Guidelines for Realizing the International Temperature Scale of 1990, pages 1-18.

SUBRANGE 4

Calculating a_4 and b_4

Definition of terms

- t = an ITS-90 temperature in °C.
- Rt = The resistance of the thermometer at an ITS-90 temperature t.
- $R_{0.01}$ = The resistance of the thermometer at 0.01 °C, the Triple Point of Water.

Begin with the following measurement data:

- Rt_{Hg}, The Resistance of the thermometer at the Triple Point of Mercury.
- Rt_{Ar}, The resistance of the thermometer at the Triple Point of Argon.
- $R_{0.01Hg}$, The resistance of the thermometer at the Triple Point of Water recorded directly after the Rt_{Hg} reading.
- $R_{0.01Ar}$, The resistance of the thermometer at the Triple Point of Water recorded directly after the Rt_{Ar} reading.

Calculate the W value for both temperatures.

$$W = Rt / R_{0.01} \qquad \text{(A1)}$$

$$W_{Hg} = Rt_{Hg} / R_{0.01Hg} \quad \text{and} \quad W_{Ar} = Rt_{Ar} / R_{0.01Ar}$$

Determine the reference W for both test points.

Convert the temperature in °C (t) to the temperature in K (T),

$$T = t + 237.15 \qquad \text{(A2)}$$

Define the variable y as

$$y = [\ln(T/273.16) + 1.5] / 1.5 \qquad \text{(A3)}$$

Calculate the reference W.

$$\begin{aligned} Z = \ln(W_r) = & -2.13534729 + 3.1832472 \cdot y - 1.80143597 \cdot y^2 \\ & + 0.71727204 \cdot y^3 + 0.50344027 \cdot y^4 - 0.61899395 \cdot y^5 \\ & - 0.05332322 \cdot y^6 + 0.28021362 \cdot y^7 + 0.10715224 \cdot y^8 \\ & - 0.29302865 \cdot y^9 + 0.04459872 \cdot y^{10} + 0.11868632 \cdot y^{11} \\ & - 0.05248134 \cdot y^{12} \end{aligned} \qquad \text{(A4)}$$

$$W_r = e^Z \qquad \text{(A5)}$$

If the temperature is being determined by a calibrated SPRT, another method of calculating the reference W is

$$W_r = W_m - a_4 \cdot (W_m - 1) - b_4 \cdot (W_m - 1) \cdot \ln(W_m) \qquad \text{(A6)}$$

Where W_m = the measured W value of the standard SPRT at temperature T, and a_4 and b_4 are the ITS-90 coefficients for the standard SPRT

Calculate the differences between the measured Ws and the reference Ws. $\Delta W = W_m - W_r$

Solve for the two unknowns (a_4 and b_4) in the deviation function

$$a_4 [W(T_{90}) - 1] + b_4 [W(T_{90}) - 1] \ln W(T_{90}) = \Delta W_4(T_{90}) \qquad \text{(A7)}$$

Use the mercury point such that

$$a_4 \cdot [W_{Hg} - 1] + b_4 \cdot [W_{Hg} - 1] \cdot \ln(W_{Hg}) = \Delta W_{4Hg} = \Delta W_{Hg} \qquad (A8)$$

Let

$$a_{11} = [W_{Hg} - 1], \quad a_{12} = [W_{Hg} - 1] \cdot \ln(W_{Hg}) \text{ and } c_1 = \Delta W_{Hg}$$

Likewise for the argon point

$$a_4 \cdot [W_{Ar} - 1] + b_4 \cdot [W_{Ar} - 1] \cdot \ln(W_{Ar}) = \Delta W_{4Ar} = \Delta W_{Ar} \qquad (A9)$$

Let

$$a_{21} = [W_{Ar} - 1], \quad a_{22} = [W_{Ar} - 1] \cdot \ln(W_{Ar}), \text{ and } c_2 = \Delta W_{Ar}$$

In matrix notation these two equations are

$$\left(\frac{a_{11}\ a_{12}}{a_{21}\ a_{22}} \right) \left(\frac{a_4}{b_4} \right) = \left(\frac{c_1}{c_2} \right) \qquad (A10)$$

Of which the determinant (D) is

$$D = |A| = \left| \frac{a_{11}\ a_{12}}{a_{21}\ a_{22}} \right| = a_{11} \cdot a_{22} - a_{21} \cdot a_{12} \qquad (A11)$$

From this, we have

$$a_{11} \cdot a_4 + a_{12} \cdot b_4 = c_1$$

and (A12)

$$a_{21} \cdot a_4 + a_{22} \cdot b_4 = c_2$$

as well as

$$a_4 = D_1/D \text{ and } b_4 = D_2/D \qquad (A13)$$

where D_1 = The determinant of the matrix A with c_2 replacing a_{22} and c_1 replacing a_{12} and
D_2 = The determinant of the matrix A with c_1 replacing a_{11} and c_2 replacing a_{21}

The resulting determinants are then

$$D = [W_{Hg}-1] \cdot [(W_{Ar}-1)\cdot\ln(W_{Ar})] - [W_{Ar}-1] \cdot [(W_{Hg}-1)\cdot\ln(W_{Hg})]$$

$$D_{Hg} = (W_{Hg}-1) \cdot (\Delta W_{Ar}) - (W_{Ar}-1) \cdot (\Delta W_{Hg}) \qquad (A14)$$

$$D_{Ar} = (\Delta W_{Hg}) \cdot [(W_{Ar}-1)\cdot\ln(W_{Ar})] - (\Delta W_{Ar}) \cdot [(W_{Hg}-1) \cdot \ln(W_{Hg})]$$

With the solution

$$a_4 = D_{Ar} / D \qquad (A15)$$

$$b_4 = D_{Hg} / D$$

Calculating a temperature in °C (t) from a W value:

Calculate the W value using equation (A1).

Use equation (A6) to determine the reference W

$$W_r = W_m - a_4 \cdot (W_m - 1) - b_4 \cdot (W_m - 1) \cdot \ln(W_m)$$

Where W_m = the measured W value of the SPRT at temperature T, and a_4 and b_4 are the ITS-90 coefficients for the SPRT.

Define the variable x as

$$x = (W_r^{1/6} - 0.65) / 0.35 \qquad (A16)$$

Calculate the temperature in Kelvin (T)

$$T = 273.16 \cdot (0.183324722 + 0.240975303 \cdot x + 0.209108771 \cdot x^2 + 0.190439972 \cdot x^3 + 0.142648498 \cdot x^4 + 0.077993465 \cdot x^5 + 0.012475611 \cdot x^6 - 0.032267127 \cdot x^7 - 0.075291522 \cdot x^8 - 0.05647067 \cdot x^9 + 0.076201285 \cdot x^{10} + 0.123893204 \cdot x^{11} - 0.029201193 \cdot x^{12} - 0.091173542 \cdot x^{13} + 0.001317696 \cdot x^{14} + 0.026025526 \cdot x^{15}) \quad \text{(A17)}$$

Convert the temperature in Kelvin (T) to °C (t)

$$t = T - 273.15 \quad \text{(A18)}$$

Approximating a W value from a temperature in °C (t).

Calculate the reference W using equations (A2) through (A5).

Using equation (A6) find the approximate W value by repeating the following iteration until the difference between W_g (see below) and the reference W (W_r) is less than $5x10^{-10}$. To reduce the number of iterations required, the initial value should be a number reasonably close to the final answer. Setting the initial W to the value of the reference W works well.

$$W_g = W - a_4 \cdot (W - 1) - b_4 \cdot (W - 1) \cdot \ln(W) \quad \text{(A19)}$$
$$W(new) = W + (Wr - W_g)$$

SUBRANGE 7

Calculating a_7, b_7, and c_7

Begin with the following measurement data

Rt_{Al}, The resistance of the thermometer at the Freezing Point of Aluminum.
Rt_{Zn}, The resistance of the thermometer at the Freezing Point of Zinc.
Rt_{Sn}, The resistance of the thermometer at the Freezing Point of Tin.
$R_{0.01Al}$, The resistance of the thermometer at the Triple Point of Water recorded directly after the Rt_{Al} reading.
$R_{0.01Zn}$, The resistance of the thermometer at the Triple Point of Water recorded directly after the Rt_{Zn} reading.
$R_{0.01Sn}$, The resistance of the thermometer at the Triple Point of Water recorded directly after the Rt_{Sn} reading.

Calculate the W value for all three temperatures using equation (A1)

$$W_{Al} = Rt_{Al} / R_{0.01Al}, \quad W_{Zn} = Rt_{Zn} / R_{0.01Zn}, \quad \text{and } W_{Sn} = Rt_{Sn} / R_{0.01Sn}$$

Determine the reference W for all three test points.

Convert the temperature in Kelvin to °C (t), using equation (A2).

$$T = t + 237.15$$

Define the variable y as

$$y = (T - 754.15)/481 \quad \text{(A20)}$$

Calculate the reference W

$$W_r = 2.78157254 + 1.64650916 \cdot y - 0.1371439 \cdot y^2 - 0.00649767 \cdot y^3 - 0.00234444 \cdot y^4 + 0.00511868 \cdot y^5 + 0.00187982 \cdot y^6 - 0.00204472 \cdot y^7 - 0.00046122 \cdot y^8 + 0.00045724 \cdot y^9 \quad \text{(A21)}$$

If the temperature is being determined by a calibrated SPRT, another method of calculating the reference W is

$$W_r = W_m - a_7 \cdot (W_m - 1) - b_7 \cdot (W_m - 1)^2 - c_7 \cdot (W_m - 1)^3 \tag{A22}$$

Where W_m = the measured W value of the standard SPRT at temperature T, and a_7, b_7, and c_7 are the ITS-90 coefficients for the standard SPRT.

Calculate the differences between the measured Ws and the reference Ws. $\Delta W = W_m - W_r$.
Solve for the three unknowns (a_7, b_7, and c_7) in the deviation function

$$\Delta W_7(T_{90}) = a_7[W(T_{90}) - 1] + b_7[W(T_{90}) - 1]^2 + c_7[W(T_{90}) - 1]^3 \tag{A23}$$

Use the aluminum point such that

$$a_7 \cdot (W_{Al} - 1) + b_7 \cdot (W_{Al} - 1)^2 + c_7 \cdot (W_{Al} - 1)^3 = \Delta W_{Al} \tag{A24}$$

Let

$$a_{11} = (W_{Al} - 1),\ a_{12} = (W_{Al} - 1)^2, \text{ and } a_{13} = (W_{Al} - 1)^3$$

Likewise for the zinc point

$$a_7 \cdot (W_{Zn} - 1) + b_7 \cdot (W_{Zn} - 1)^2 + c_7 \cdot (W_{Zn} - 1)^3 = \Delta W_{Zn} \tag{A25}$$

Let

$$a_{21} = (W_{Zn} - 1),\ a_{22} = (W_{Zn} - 1)^2, \text{ and } a_{23} = (W_{Zn} - 1)^3$$

And the tin point

$$a_7 \cdot (W_{Sn} - 1) + b_7 \cdot (W_{Sn} - 1)^2 + c_7 \cdot (W_{Sn} - 1)^3 = \Delta W_{Sn} \tag{A26}$$

Let

$$a_{31} = (W_{Sn} - 1),\ a_{32} = (W_{Sn} - 1)^2, \text{ and } a_{33} = (W_{Sn} - 1)^3$$

In matrix notation these three equations are

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_7 \\ b_7 \\ c_7 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{A27}$$

Of which the determinant (D) is

$$D = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = \tag{A28}$$

$$a_{11} \cdot (a_{22} \cdot a_{33} - a_{32} \cdot a_{23}) - a_{21} \cdot (a_{12} \cdot a_{33} - a_{32} \cdot a_{13}) + a_{31}(a_{12} \cdot a_{23} - a_{22} \cdot a_{13})$$

From this we have

$$a_{11} \cdot a_7 + a_{12} \cdot b_7 + a_{13} \cdot c_7 = c_1$$

and

$$a_{21} \cdot a_7 + a_{22} \cdot b_7 + a_{23} \cdot c_7 = c_2 \tag{A29}$$

and

$$a_{31} \cdot a_7 + a_{32} \cdot b_7 + a_{33} \cdot c_7 = c_3$$

as well as

$$a_4 = D_1/D,\ b_4 = D_2/D,\ c_4 = D_3/D \tag{A30}$$

where D_1 = The determinant of the matrix A with c_1 replacing a_{11}, c_2 replacing a_{21}, and c_3 replacing a_{31}.
D_2 = The determinant of the matrix A with c_1 replacing a_{12}, c_2 replacing a_{22}, and c_3 replacing a_{32}.

D_3 = The determinant of the matrix A with c_1 replacing a_{13}, c_2 replacing a_{23}, and c_3 replacing a_{33}.

The resulting determinants are then

$$\begin{aligned} D = \; & (W_{Al}-1) \cdot [(W_{Zn}-1)^2 \cdot (W_{Sn}-1)^3 - (W_{Sn}-1)^2 \cdot (W_{Zn}-1)^3] + \\ & (W_{Zn}-1) \cdot [(W_{Sn}-1)^2 \cdot (W_{Al}-1)^3 - (W_{Al}-1)^2 \cdot (W_{Sn}-1)^3] + \\ & (W_{Sn}-1) \cdot [(W_{Al}-1)^2 \cdot (W_{Zn}-1)^3 - (W_{zn}-1)^2 \cdot (W_{Al}-1)^3] \end{aligned}$$

$$\begin{aligned} D_{al} = \; & (\Delta W_{Al}) \cdot [(W_{Zn}-1)^2 \cdot (W_{Sn}-1)^3 - (W_{Sn}-1)^2 \cdot (W_{Zn}-1)^3] + \\ & (\Delta W_{Zn}) \cdot [(W_{Sn}-1)^2 \cdot (W_{Al}-1)^3 - (W_{Al}-1)^2 \cdot (W_{Sn}-1)^3] + \\ & (\Delta W_{Sn}) \cdot [(W_{Al}-1)^2 \cdot (W_{Zn}-1)^3 - (W_{Zn}-1)^2 \cdot (W_{Al}-1)^3] \end{aligned}$$

$$\begin{aligned} D_{Zn} = \; & (W_{Al}-1) \cdot [(\Delta W_{Zn}) \cdot (W_{Sn}-1)^3 - (\Delta W_{Sn}) \cdot (W_{Zn}-1)^3] + \\ & (W_{Zn}-1) \cdot [(\Delta W_{Sn}) \cdot (W_{Al}-1)^3 - (\Delta W_{Al}) \cdot (W_{Sn}-1)^3] + \\ & (W_{Sn}-1) \cdot [(\Delta W_{Al}) \cdot (W_{Zn}-1)^3 - (\Delta W_{Zn}) \cdot (W_{Al}-1)^3] \end{aligned} \qquad \text{(A31)}$$

$$\begin{aligned} D_{Sn} = \; & (W_{Al}-1) \cdot [(W_{Zn}-1)^2 \cdot (\Delta W_{Sn}) - (W_{Sn}-1)^2 \cdot (\Delta W_{Zn})] + \\ & (W_{Zn}-1) \cdot [(W_{Sn}-1)^2 \cdot (\Delta W_{Al}) - (W_{Al}-1)^2 \cdot (\Delta W_{Sn})] + \\ & (W_{Sn}-1) \cdot [(W_{Al}-1)^2 \cdot (\Delta W_{Zn}) - (W_{Zn}-1)^2 \cdot (\Delta W_{Al})] \end{aligned}$$

With the solution

$$\begin{aligned} a_7 &= D_{Al} / D \\ b_7 &= D_{Zn} / D \\ c_7 &= D_{Sn} / D \end{aligned} \qquad \text{(A32)}$$

Calculating a temperature in °C (t) from a W value

Calculate the W value using equation (A1).

Use equation (22) to determine the reference W

$$W_r = W_m - a_7 \cdot (W_m - 1) - b_7 \cdot (W_m - 1)^2 - c_7 \cdot (W_m - 1)^3$$

Where W_m = the measured W value of the SPRT at temperature T, and a_7, b_7, and c_7 are the ITS-90 coefficients for the SPRT.

Define the variable x as

$$x = (W_r - 2.64) / 1.64) \qquad \text{(A33)}$$

Calculate the temperature in °C (t).

$$\begin{aligned} t = (& 439.932854 + 472.41802 \cdot x + 37.684494 \cdot x^2 + 7.472018 \cdot x^3 \\ & + 2.920828 \cdot x^4 + 0.005184 \cdot x^5 - 0.963864 \cdot x^6 - 0.188732 \cdot x^7 \\ & + 0.191203 \cdot x^8 + 0.049025 \cdot x^9) \end{aligned} \qquad \text{(A34)}$$

Approximating a W value from a temperature in °C (t)

Convert the temperature in °C (t) to the temperature in K (T) using equation (A2):

$$T = t + 237.15.$$

Define the variable y as

$$y = (T - 754.15)/ 481 \qquad \text{(A35)}$$

Calculate the reference W

$$\begin{aligned} W_r = \; & 2.78157254 + 1.64650916 \cdot y - 0.1371439 \cdot y^2 - 0.00649767 \cdot y^3 \\ & - 0.00234444 \cdot y^4 + 0.00511868 \cdot y^5 + 0.00187982 \cdot y^6 - 0.00204472 \cdot y^7 \\ & - 0.00046122 \cdot y^8 + 0.00045724 \cdot y^9 \end{aligned} \qquad \text{(A36)}$$

Using equation (A22) find the approximate W value by repeating the following iteration until the difference between W_g (see below) and the reference W (Wr) is less than $5x10^{-10}$. To reduce the number of iterations required, the initial value should be a number reasonably close to the final answer. Setting the initial W to the value of the reference W works well.

$$W_g = W - a_7 \cdot (W - 1) - b_7 \cdot (W - 1)^2 - c_7 \cdot (W - 1)^3 \quad (A37)$$

$$W(new) = W + (Wr - W_g)$$

SUBRANGE 8

Calculating a_8 and b_8

Begin with the following measurement data:

Rt_{Zn}, The resistance of the thermometer at the Freezing Point of Zinc.
Rt_{Sn}, The resistance of the thermometer at the Freezing Point of Tin.
$R_{0.01Zn}$, The resistance of the thermometer at the Triple Point of Water recorded directly after the Rt_{Zn} reading.
$R_{0.01Sn}$, The resistance of the thermometer at the Triple Point of Water recorded directly after the Rt_{Sn} reading.

Calculate the W value for both temperatures using equation (A1)

$$W_{Zn} = Rt_{Zn}/ R_{0.01Zn}, \text{ and } W_{Sn} = Rt_{Sn}/ R_{0.01Sn}$$

Determine the reference W for both test points.
Convert the temperature in Kelvin to °C (t), using equation (A18)

$$T = t + 237.15$$

Define the variable y as

$$y = (T - 754.15)/ 481 \quad (A38)$$

Calculate the reference W

$$\begin{aligned} W_r &= 2.78157254 + 1.64650916 \cdot y - 0.1371439 \cdot y^2 \\ &- 0.00649767 \cdot y^3 - 0.00234444 \cdot y^4 + 0.00511868 \cdot y^5 \\ &+ 0.00187982 \cdot y^6 - 0.00204472 \cdot y^7 - 0.00046122 \cdot y^8 \\ &+ 0.00045724 \cdot y^9 \end{aligned} \quad (A39)$$

If the temperature is being determined by a calibrated SPRT, another method of calculating the reference W is

$$W_r = W_m - a_8 \cdot (W_m - 1) - b_8 \cdot (W_m - 1)^2 \quad (A40)$$

Where W_m = the measured W value of the standard SPRT at temperature T, and a_8, and b_8 are the ITS-90 coefficients for the standard SPRT.
Note: This formula is the same formula as subrange 7 with the third term removed.
Calculate the differences between the measured Ws and the reference Ws. $\Delta W = W_m - W_r$.

Solve for the two unknowns (a_8 and b_8) in the deviation function (using the same method used for subrange 4)

$$\Delta W_8(T_{90}) = a_8[W(T_{90}) - 1] + b_8[W(T_{90}) - 1]^2 \quad (A41)$$

The resulting determinants are then

$$\begin{aligned} D &= (W_{Zn} -1) \cdot (W_{Sn} -1)^2 - (W_{Sn} -1) \cdot (W_{Zn} -1)^2 \\ D_{Zn} &= (\Delta W_{Zn}) \cdot (W_{Sn} -1)^2 - (\Delta W_{Sn}) \cdot (W_{Zn} -1)^2 \\ D_{Sn} &= (W_{Zn} -1) \cdot (\Delta W_{Sn}) - (W_{Sn} -1) \cdot (\Delta W_{Zn}) \end{aligned} \quad (A42)$$

With the solution

$$a_8 = D_{Zn} / D \qquad (A43)$$

$$b_8 = D_{Sn} / D$$

Calculating a temperature in °C (t) from a W value.
Calculate the W value using equation (A1).
Use equation (A33) to determine the reference W:

$$W_r = W_m - a_8 \cdot (W_m - 1) - b_8 \cdot (W_m - 1)^2$$

Note: This formula is the same formula as paragraph subrange 7 with the third term removed.

Define the variable x as

$$x = (W_r - 2.64) / 1.64) \qquad (A44)$$

Calculate the temperature in °C (t).

$$t = (439.932854 + 472.41802 \cdot x + 37.684494 \cdot x^2 + 7.472018 \cdot x^3 + 2.920828 \cdot x^4 + 0.005184 \cdot x^5 - 0.963864 \cdot x^6 - 0.188732 \cdot x^7 + 0.191203 \cdot x^8 + 0.049025 \cdot x^9) \qquad (A45)$$

Approximating a W value from a temperature in °C (t)
Convert the temperature in °C (t) to the temperature in K (T), using equation (A2)

$$T = t + 237.15.$$

Define the variable y as

$$y = (T - 754.15)/481 \qquad (A46)$$

Calculate the reference W

$$W_r = 2.78157254 + 1.64650916 \cdot y - 0.1371439 \cdot y^2 - 0.00649767 \cdot y^3 - 0.00234444 \cdot y^4 + 0.00511868 \cdot y^5 + 0.00187982 \cdot y^6 - 0.00204472 \cdot y^7 - 0.00046122 \cdot y^8 + 0.00045724 \cdot y^9 \qquad (A47)$$

Using equation (A33) find the approximate W value by repeating the following iteration until the difference between W_g (see below) and the reference W (W_r) is less than $5x10^{-10}$. To reduce the number of iterations required, the initial value should be a number reasonably close to the final answer. Setting the initial W to the value of the reference W works well.

$$W_g = W - a_8 \cdot (W - 1) - b_8 \cdot (W - 1)^2 \qquad (A48)$$

$$W(new) = W + (W_r - W_g)$$