An Approximation Method For Finding Emissivity and Temperature of a Blackbody

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Abstract: The determination of a blackbody's emissivity and temperature can be found by comparing the measured spectral radiance of a test blackbody to the measured spectral radiance of a standard of known emissivity and temperature. Using Planck's law, calculating the temperature and emissivity of the blackbody can be accomplished by finding the temperature that minimizes the standard deviation of emissivity measurements. This can require specialized programming to do iterative calculations. In this paper, an approximation method is employed that uses simple statistical functions to arrive at results without iteration.

1. Introduction

Characterizing temperature sources is required for maintaining a temperature scale. One particular method that finds the temperature and emissivity of blackbodies is discussed. The difficulty in implementing this method lies in the calculation required to minimize the standard deviation of the emissivity. This paper discusses a way to greatly simplify these calculations by making three approximations that are valid for high emissivity calibrations. The measurement setup is described. The approximations are derived. A procedure that uses these approximations is given. Two examples are completed and the results are compared to the exact method.

2. Description of the Calibration Method

In [1], the test and standard blackbodies are set to within 0.5 K of each other and allowed to stabilize. In the calibration setup, the standard and test blackbodies use largely the same optical path. A turning mirror is used to alternately direct the radiant energy of each blackbody along the common optical path ultimately to the detector. At a series of pre-selected wavelengths the detector signal is recorded for the test and standard blackbodies. Since the same detector takes readings close together in time and the two blackbodies are very close in temperature, we assume that the detected signal for the standard and test have the same proportionality to the spectral radiant intensity, i.e.,

$$S_{test} = \alpha_{test} I_{test}, S_{std} = \alpha_{std} I_{std}, \alpha_{std} = \alpha_{test} . \tag{1}$$

Planck's Law states the following [2]:

$$I(\varepsilon, \lambda, T) = \frac{\varepsilon \cdot c_1}{\lambda^5 \cdot (\exp(c_2 / \lambda T) - 1)} \text{ (W/cm}^2\ \mu\text{m}^{-1}\text{) is the spectral radiant intensity,} \tag{2}$$

where ε is the emissivity of the blackbody, λ is the wavelength in µm, and T is the absolute temperature in K, c_1 is 37417.719 W µm^4/cm^2, c_2 is 14387.752 µm K [3]. Using Planck's Law, the ratio of the two signals is

$$R(\lambda) = \frac{S_{test}}{S_{std}} = \frac{\alpha_{test} I_{test}}{\alpha_{std} I_{std}} = \frac{\varepsilon_{test}\left(\exp(c_2/\lambda T_{std}) - 1\right)}{\varepsilon_{std}\left(\exp(c_2/\lambda T_{test}) - 1\right)}. \tag{3}$$

Then the test emissivity at each wavelength is given by

$$\varepsilon_i = \varepsilon_{test}(\lambda_i) = R(\lambda_i)\varepsilon_{std} \frac{\exp(c_2/\lambda_i T_{test}) - 1}{\exp(c_2/\lambda_i T_{std}) - 1}. \tag{4}$$

We know the wavelength, the standard temperature and the standard emissivity. We want to solve for the test temperature to minimize the standard deviation of the test emissivities. For N measurements (one measurement at each predetermined wavelength), minimizing the standard deviation of the emissivities is equivalent to minimizing

$$\min_{T_{test}}\left(\sum_{i=1}^{N}\left(\varepsilon_i - \varepsilon_{avg}\right)^2\right). \tag{5}$$

This test temperature can be calculated iteratively by refining guesses to find the minimum of this function. This may become a cumbersome calculation, particularly if there is a large set of data to be analyzed. In order to find an alternative method to find this solution, three approximations are used.

3. Approximation Method

3.1. Approximation 1

The first approximation uses Wien's Law for both the test and standard blackbodies in the expression for the test emissivities.

$$\varepsilon_i = R(\lambda_i)\varepsilon_{std} \frac{\exp(c_2/\lambda_i T_{test}) - 1}{\exp(c_2/\lambda_i T_{std}) - 1} \approx R(\lambda_i)\varepsilon_{std} \exp\left[\left(c_2/\lambda_i\right)\left(1/T_{test} - 1/T_{std}\right)\right] \tag{6}$$

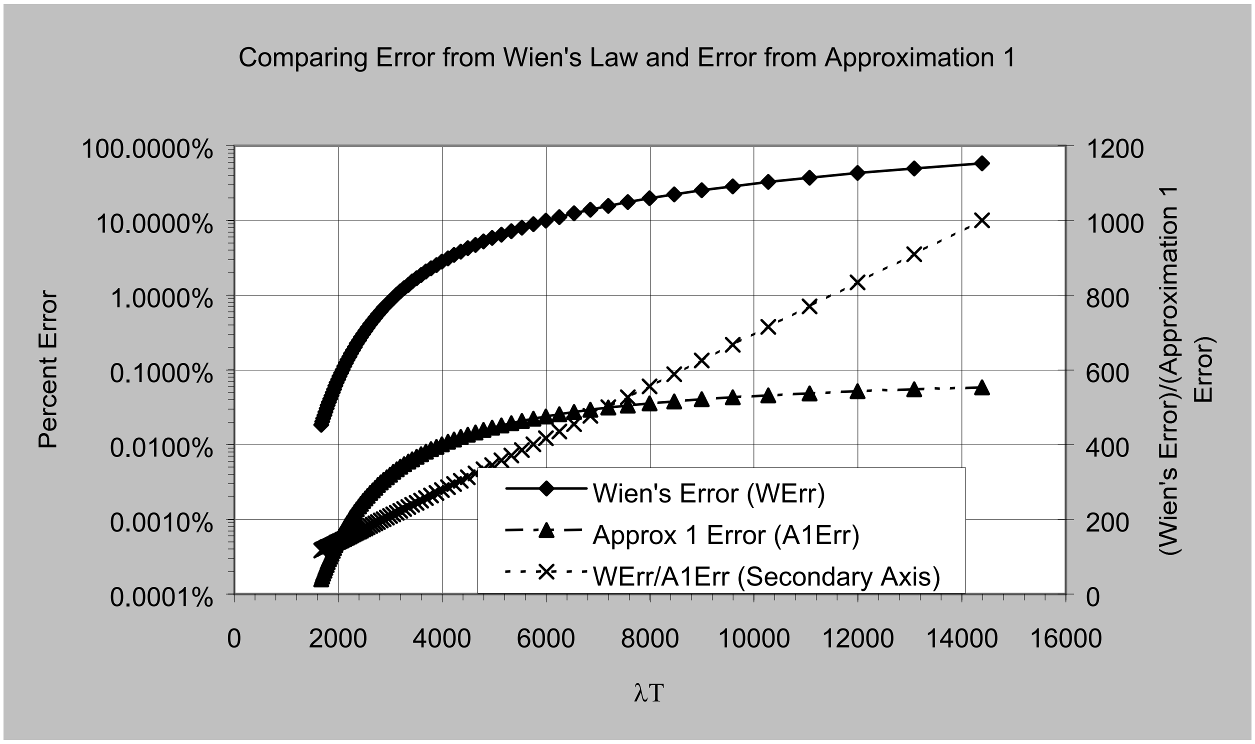

Figure 1. Graph showing the relative error for Wien's approximation and Approximation 1.

The relative error for this approximation is about $\frac{\Delta\varepsilon}{\varepsilon} = \frac{Q^{\Delta T/T} - 1}{Q}, Q = \exp\left(c_2/_{\lambda T}\right)$. Figure 1 graphs the ratio of the error of Wein's approximation to the error of Approximation 1 when the test blackbody temperature is 0.1% greater than the standard blackbody temperature. We see that even when Wien's Law may not be appropriate, this approximation is very accurate.

3.2. Approximation 2

Rather than minimizing the standard deviation of the test emissivities, we minimize the standard deviation of the natural logarithm of the test emissivities. Since these are high emissivity measurements ($\varepsilon_{test} \rightarrow 1$), this should be a good approximation. The natural logarithm can be constructed by a series, as $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$. As x gets small, $\ln(1+x) \approx x$. Let $\varepsilon_i = 1 + x$, so $\ln(\varepsilon_i) \approx \varepsilon_i - 1$. Then,

$$\varepsilon_i - \varepsilon_{avg} \approx \ln(\varepsilon_i) - \ln(\varepsilon_{avg}) \tag{7}$$

The error in the natural logarithm estimate is $x - \ln(1+x) \approx \frac{1}{2}x^2 = \frac{1}{2}(\varepsilon_i - 1)^2$. This error clearly gets smaller as the emissivity gets closer to unity.

3.3. Approximation 3

The third approximation uses the geometric mean of the emissivities instead of the arithmetic mean. For two numbers, k and $k(1 + b)$, the difference between the arithmetic mean(A) and geometric mean(G) is less than $kb^2/8$.

$$A = k(1 + b/2)$$

$$G = k\sqrt{1+b}$$

$$\sqrt{1+b} = 1 + \frac{1}{2}b - \frac{1}{2\cdot 4}b^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}b^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}b^4 \pm \cdots$$

Since these emissivities are close to each other, we can say that the arithmetic mean and geometric mean of the emissivities are approximately equal. As b approaches zero, $G \approx A$, therefore,

$$\varepsilon_{avg} = (1/N)\sum \varepsilon_j \approx \left(\prod \varepsilon_j\right)^{1/N} \tag{8}$$

3.4. Full approximation method derivation

The above approximations can all be used to find the T_{test} that minimizes the standard deviation of the test emissivities. From (5) and (7), we see that

$$\min_{T_{test}}\left(\sum_{i=1}^{N}\left(\varepsilon_i - \varepsilon_{avg}\right)^2\right) \approx \min_{T_{test}}\left(\sum_{i=1}^{N}(\ln\varepsilon_i - \ln\varepsilon_{avg})^2\right). \tag{9}$$

From (6),

$$\ln \varepsilon_i = \ln\,[R(\lambda_i)] + \ln[\varepsilon_{std}] + [(c_2/\lambda_i)(1/T_{test} - 1/T_{std})] \tag{10}$$

Letting

$$w_i = c_2/\lambda_i \tag{11}$$

$$L_i = ln[R(\lambda_i)] \tag{12}$$

$$v = 1/T_{test} - 1/T_{std}, \tag{13}$$

then (10) becomes

$$\ln \varepsilon_i = L_i + \ln[\varepsilon_{std}] + w_i v \tag{14}$$

From (8) and (14),

$$\ln(\varepsilon_{avg}) = \tfrac{1}{N}\sum_{k=1}^{N}\ln(\varepsilon_k) = \tfrac{1}{N}\sum_{k=1}^{N}\left(L_i + \ln[\varepsilon_{std}] + w_i v\right) = \tfrac{1}{N}\sum_{k=1}^{N}L_i + \tfrac{v}{N}\sum_{k=1}^{N}w_i + \ln[\varepsilon_{std}] \tag{15}$$

$$\ln(\varepsilon_{avg}) = \mu_L + v\mu_w + \ln[\varepsilon_{std}] \tag{16}$$

where $\mu_L = \tfrac{1}{N}\sum_{k=1}^{N}L_i$ and $\mu_w = \tfrac{1}{N}\sum_{k=1}^{N}w_i$.

Then,

$$\ln[\varepsilon_i] - \ln[\varepsilon_{avg}] = (L_i - \mu_L) + v(w_i - \mu_w) = a_i + b_i v \tag{17}$$

where $a_i = L_i - \mu_L$ and $b_i = w_i - \mu_w$.

Then, the minimization statement becomes,

$$\min_{T_{test}}\left(\sum_{i=1}^{N}\left(\varepsilon_i - \varepsilon_{avg}\right)^2\right) \approx \min_{v}\left(\sum_{i=1}^{N}\left[a_i + b_i v\right]^2\right) \tag{18}$$

To find the minimum of (18), differentiate with respect to v and set the expression to zero, as

$$\sum_{i=1}^{N}2b_i\left[a_i + b_i v\right] = 0.$$

Solving for v gives

$$v = \frac{-\sum a_i b_i}{\sum b_i^2} \tag{19}$$

Solving (13) for T_{test},

$$T_{test} = \frac{T_{std}}{1 + vT_{std}} \tag{20}$$

By (17) and (19), we see that

$$\sum a_i b_i = \sum (L_i - \mu_L)(w_i - \mu_w) = N \operatorname{cov}(L_i, w_i)$$

$$\sum b_i^2 = \sum (w_i - \mu_w)^2 = N\sigma_w^2$$

This allows for a statistical formulation for v, as

$$v = \left(\frac{-\operatorname{cov}(L_i, w_i)}{\sigma_w^2} \right). \tag{21}$$

Here, σ_w is known *a priori* from the wavelengths that will be used (σ_w is the standard deviation of the population of w_i). The covariance is calculated after the data has been taken from the standard and detector signals. Next T_{test}, is calculated from (20). Once we have the T_{test}, we can calculate the average test emissivity, ε_{test} from (6) and (8) as

$$\varepsilon_{test} = \tfrac{1}{N} \sum_{i=1}^{N} \varepsilon_i = \frac{\varepsilon_{std}}{N} \sum_{i=1}^{N} R(\lambda_i) \exp(w_i v).$$

The whole process can be enumerated as follows:
Algorithm 1

1. Set the blackbodies to the appropriate setpoints and allow them to stabilize within 0.5 K.
2. Predetermine at what wavelengths (λ_i, $i=1, 2, \ldots, N$) the signals will be recorded.
3. Perform the following calculations, (recall σ^2_w is the standard deviation of a population, not the standard deviation of a sample, hence the denominator is N rather than $N - 1$.)

$$w_i = c_2/\lambda_i, \quad \sigma_w^2 = \frac{\sum (w_i - \mu_w)^2}{N}$$

4. Record detector signals for test and standard blackbody (S_{test} and S_{std}) at each wavelength.
5. Perform the following calculations,

$$R(\lambda_i) = S_{test}/S_{std}, L_i = \ln[R(\lambda_i)], v = \frac{-\operatorname{cov}(L_i, w_i)}{\sigma_w^2}$$

$$T_{test} = \frac{T_{std}}{1 + vT_{std}}, \varepsilon_{test} = \frac{\varepsilon_{std}}{N} \sum_{i=1}^{N} R(\lambda_i) \exp(w_i v)$$

4. Examples

Let's do some examples using this methodology. These examples were selected because they are close to the 0.5 K criterion. Example 1 has $T_{test} - T_{std}$ close to 0.5 K and Example 2 has $T_{test} - T_{std}$ close to – 0.5 K. All calculations were performed using more precision, but the extra digits are suppressed here for aesthetics.

4.1. Example 1

Here is the data:

$$T_{std} = 508.5170 \text{ K}, \ \varepsilon_{std} = 0.999840$$
$$\lambda_i = \{2.0,\ 2.3,\ 3.0\} \ \mu\text{m}$$
$$R(\lambda_i) = \{0.993954,\ 0.989498,\ 0.988997\}.$$

The calculations proceed as follows:

$$w_i = c_2/\lambda_i = \{7193.88,\ 6255.54,\ 4795.92\} \text{ K}, \ \sigma^2_w = 973465 \text{ K}^2$$
$$L_i = \ln(R(\lambda_i)) = \{-6.064\text{x}10^{-3},\ -1.056\text{x}10^{-2},\ -1.106\text{x}10^{-2}\}$$
$$v = -1.93401\text{x}10^{-6} \text{ K}^{-1}$$
$$T_{test} = 509.0176 \text{ K}, \ \varepsilon_{test} = .979072$$

The exact solution is

$$T_{test} = 509.0201 \text{ K}, \ \varepsilon_{test} = .979014$$

The error in test emissivity is $5.8\text{x}10^{-5}$ and the error in test temperature is $-2.5\text{x}10^{-3}$ K.

4.2. Example 2

Here is the data:

$$T_{std} = 693.278 \text{ K}, \ \varepsilon_{std} = 1.0002400$$
$$\lambda_i = \{1.7,\ 2.0,\ 2.3,\ 3.0\} \ \mu\text{m}$$
$$R(\lambda_i) = \{0.982405,\ 0.984300,\ 0.984858,\ 0.986245\}.$$

The calculations proceed as follows:

$$w_i = c_2/\lambda_i = \{8463.38,\ 7193.88,\ 6255.54,\ 4795.92\} \text{ K}$$
$$\sigma^2_w = 1793606 \text{ K}^2$$
$$L_i = \ln(R(\lambda_i)) = \{-1.775\text{x}10^{-2},\ -1.583\text{x}10^{-2},\ -1.526\text{x}10^{-2},\ -1.385\text{x}10^{-2}\}$$
$$v = 1.03073\text{x}10^{-6} \text{ K}^{-1}$$
$$T_{test} = 692.78295 \text{ K}, \ \varepsilon_{test} = .9914877$$

The exact solution is

$$T_{test} = 692.78240 \text{ K}, \ \varepsilon_{test} = .9914967$$

The error in test emissivity is $-9.0\text{x}10^{-6}$ and the error in test temperature is $5.5\text{x}10^{-4}$ K.

We can see that even though three approximations were made in getting to this result, the final errors are very small. In fact, these errors are likely to be much less than other uncertainty contributions, providing for a measurement methodology that does not require extensive calculations.

5. Summary and Conclusions

One particular measurement setup was described for calibrating a blackbody for temperature and emissivity. Approximations for simplifying the calculations to minimize the standard deviation of the test item emissivity were derived. A procedure that uses these approximations to analyze the data from a test item measurement was given. We worked through two examples using this procedure. These two examples show that this new approximation method gives results that are very close to the results of an exact solution.

The exact solution requires some method of iteration to solve the nonlinear equation. The number of steps depends on the method of solving the nonlinear equation that is employed, and on how close the initial guess is to the solution. The new method requires no iteration, yet gives results very close to the exact results. Also, the functions required to compute the approximation method are readily available in spreadsheet software. The fact that the new method requires no iteration makes it a more suitable alternative when analyzing a large set of data taken from many different measurements. The error that results from the approximation method may be much smaller than the overall uncertainty of the calibration. If more precision is required, an exact method can be employed using the result of the approximation method as the initial guess. Because the initial guess is very close to the final value, the number of iterations required to reach a prescribed tolerance is reduced.

6. References

1. Air Force Technical Order 33K4-5-182-1
2. The Photonics Handbook 2004 50th Edition, Book 3, Laurin Publishing, Pittsfield, MA, p. H-29, 2004.
3. http://physics.nist.gov/cuu/Constants/index.html, April 6, 2006.
4. Richard C. Dorf and Ronald J. Tallarida, Pocket Book of Electrical Engineering Formulas, CRC Press, Inc., Boca Raton, FL, 1993, pp. 57-58.