Modelling of Measurements for the Evaluation of the Measurement Uncertainty

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Abstract:

The modelling of a measurement is an essential part of the evaluation of the measurement uncertainty in accordance with The *Guide to the Expression of Uncertainty in Measurement (GUM)* [1] requires the modelling of the measurement as a basis for uncertainty propagation calculus. The modelling process establishes mathematically the relationship between the input quantities and the measurand. Input quantities are those which may influence the measurement result and contribute to the combined uncertainty. The modelling is the most difficult part and the first step in uncertainty evaluation.

This paper explains the recommendations of the *GUM* as step-wise procedure. Emphasis is given to a basic step, the mathematical formulation of the so-called model. A straightforward and widely applicable modelling concept has been developed. It is based on the *measuring chain* [2] and the mathematical formulation of the method of measurement, i. e. the logical sequence and relationship of the operations used in the performance of the measurement. It admits a modular formulation of the model equation as well as its completion or partial reformulation. The other steps are fare-less difficult.

The applicability of this modelling concept is demonstrated by examples of the methods of measurement used in various of metrology.

1 Introduction

In the last ten years, the *Guide to the Expression of Uncertainty in Measurement* (*GUM*) has become a world–wide recognized standard for the evaluation of the measurement uncertainty at least at national metrology institutes and in calibration laboratories. For practical use, the *GUM* excels in

- providing a consistent procedure for evaluating and expressing the measurement uncertainty
- offering an easy way to implement computer–aided uncertainty budgeting.

In industry, a realistic evaluation of safety measurement uncertainty based on the *GUM* offers the chance of achieving clear–cut decisions in the fields and increase in efficiency as regards production by

- using measuring and test equipment tailored for specific tasks;
- exhausting product tolerances and process specifications, and
- achieving mutual recognition of calibration and test results (one–stop testing).

The basic steps of the *GUM* procedure for evaluating and expressing uncertainty are

- modelling of the measurement in order to establish mathematically the relationship between the measurand and all relevant input quantities, and
- quantitative description of the existing knowledge of these input quantities by means of probability distributions.

Both steps require first to collect and gain knowledge in order to "convert" it to useful information. The *GUM* sets clear-cut rules for the evaluation both of statistical and of non–statistical information.

But the *GUM* does not provide any guidance for the modelling process. To practitioners, however, modelling appears to be the most difficult problem of uncertainty evaluation in accordance with the *GUM*. Therefore, this part of the procedure has often impeded wider application of the *GUM*.

In order to overcome this problem, this paper describes a straightforward and widely applicable modelling concept. This concept allows a modular formulation of the model equation and, at any time, its completion or partial reformulation. This concept was successfully presented in training courses attended by more than 250 technicians, engineers and physicists.

2 *GUM* procedure for evaluating uncertainty [1]

The *GUM* procedure for evaluating the uncertainty is based on the knowledge of the measuring process and the input quantities which influence the measurement result. Consequently, the subjects of the two initial steps of the procedure are the modelling of the measurement and the evaluation of the relevant input quantities:

- Describing and modelling the measurement:

  Initially, a description the measuring process is required. It must identify both the measurand and the method of measurement used. Based on the useful information taken from this description, the modelling of the measurement establishes mathematically the relationship between the measurand $\boldsymbol{Y}$ and the input quantities $\boldsymbol{X_i}$ which $\boldsymbol{Y}$ depends on:

$$\boldsymbol{Y = f(X_1, X_2, \ldots, X_N)} \tag{2.1}$$

The function $\boldsymbol{f}$ should contain all quantities, including all corrections and correction factors, likely to contribute to the result for the measurand [1]. This relationship is also called *model equation*. The *model equation* provides the information needed for the algorithm of uncertainty propagation.

- Collecting knowledge and describing the input quantities in quantitative terms:

  The aim of the quantitative description of the input quantities is to assign an estimated value x_i and an associated uncertainty $u(x_i)$ to each input quantity X_i. The *GUM* distinguishes between two types of evaluation:

- type-A evaluation based on a statistical analysis of series of observations, and
- type-B evaluation based on other means.

Both types of evaluation use probability distributions to describe the knowledge of the input quantities.

The estimated value x_i of the input quantity X_i is the expectation value

$$x_i = \mathrm{E}[X_i], \tag{2.2}$$

and the associated standard uncertainty is defined by

$$u(x_i) = \sqrt{Var\left[X_i\right]}\,. \tag{2.3}$$

The probability distributions provide the information on the reliability of the values associated with the input quantities. This information is given for all input quantities by their respective the expectation value and the associated standard uncertainty.

The following steps are fare-less difficult:

- Determining the (expectation) value of the measurand and its combined standard uncertainty:

  The model equation (see equation (2.1)) together with the estimated values x_i and their associated standard uncertainties $u(x_i)$ serve as the basis for the determination of the (expectation) value of the measurand,

$$y = f\,(x_1, x_2, \ldots, x_N) \tag{2.4}$$

  and of the associated combined standard uncertainty

$$u_c^2(y) = \sum_{i=1}^{N}\left[\frac{\partial f}{\partial x_i}\right]^2 u^2(x_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j) \tag{2.5}$$

  where

  y - expectation value of Y,
  x_i and x_j - expectation values of X_i and X_j (see equation (2.2)), and
  $u(x_i, x_j)$ - estimated covariance associated with x_i and x_j.

- Determining the expanded uncertainty of measurement:

The expanded uncertainty of measurement U is the half–width of an interval I_Y that may be expected to encompass a large fraction, e.g. 95 % of values that could reasonably be attributed to the measurand Y:

$$I_Y = [y-U;\, y+U], \tag{2.6}$$

The expanded uncertainty of measurement is obtained by multiplying the combined standard uncertainty $u_c(y)$ by a coverage factor k_p:

$$U = k_p \cdot u_c(y) \tag{2.7}$$

The half-width interval I_Y (see equation (2.6)) depends on the coverage probability (usually 95%) and on the probability distribution which is associated with the measurand characterized by its value y and its combined standard uncertainty $u_c(y)$.

- Reporting the result:

  The result of a measurement should be reported as

$$Y = y \pm U \tag{2.8}$$

- Evaluating the uncertainty budget and taking measures.

The above described procedure makes clear that the last four steps –with the exception of taking measures- can be carried out stringently by the mathematical procedure prescribed by the *GUM*. Therefore, a computer–aided calculation is possible.

Only the performance of the two initial steps requires expert knowledge. This paper is stressing the first and most difficult step of the procedure.

3 Concept of the *measuring chain* [2] and its elements

3.1 Basic relationships

Usually, in metrology the *cause-and-effect relationship* of a measuring process is represented by a *measuring chain* [2] [3] that constitutes the path of the measurement signal from the input to the output. The measuring system or –in more general terms– the measuring process is regarded as a series of non-reactive functional elements or a sequence of operational steps to carry out the measurement. Afterwards, both the functional elements and the operational steps are denoted by elements. These elements may be assigned to both targeted functions or operations of the measuring process, e.g. the amplification of a measurement signal, and unwanted effects, e.g. a mismatching connection of two devices [3].

In the steady-state, the intrinsic cause-and-effect relationship of an observed element k may be expressed by the functional relationship of the respective input and output quantities (see also Fig. 1):

$$X_{kOUT} = h_k\,(X_{kIN};\, \underline{Z}_k) \tag{3.1}$$

where

X_{kIN} - (random) quantity acting on the input of the observed element k;
X_{kOUT} - (random) output quantity of the element k;

$\underline{Z}_k$ - *parameter vector* that is composed of the internal and external parameters of the element $\boldsymbol{k}$, e.g. the
time-dependent offset and the operational temperature.

Note: In control engineering, the vector $\underline{Z}_k$ is also denoted by *disturbance vector* that is composed of the disturbing quantities having an impact on the quantity X_{kOUT} .

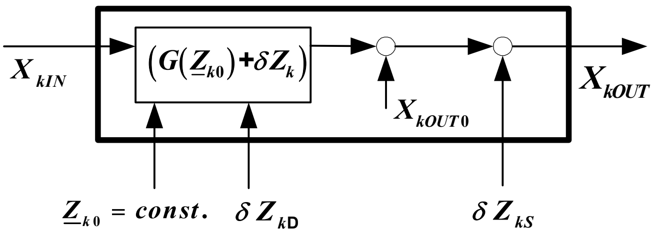

Fig.1 : Cause-and-effect relationship of a single element of a measuring chain. symbols see equation (3.1)

The non-reactive chaining of adjacent elements may be expressed by the following equation:

$$X_{k\,IN} = X_{k-1\,OUT} \tag{3.2}$$

The first element of the measuring chain must be supplied by a measurable parameter X_M. The indication or the record is considered to be the output quantity X_{IND} (Fig. 2). In the case of a linear (non-branched) chain, the parameter X_M is equal to the measurand.

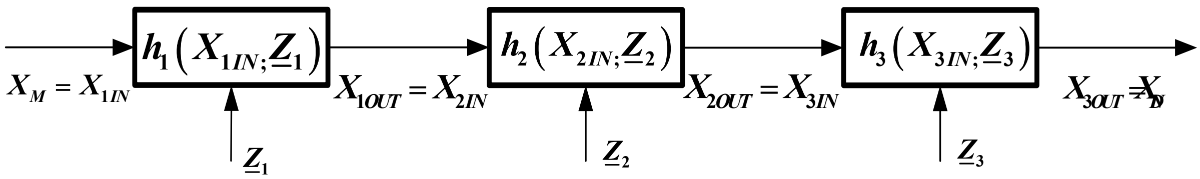

Fig.2 : Linear and non-reactive chaining of three elements

3.2 Concept of the ideal element

Due to its multiple dependencies, the relationship (3.1) is not suitable for practically and comprehensibly expressing steady-state characteristics of elements of the measuring chain.

In metrological practice, however, the above mathematical approach to the description of measuring systems and processes can be simplified on the following assumptions:

- The great majority of measuring systems and devices can be regarded to have linear characteristics or, at least in narrow ranges, a linear characteristic may be assumed.

- In practice, the steady-state characteristic of a measuring system is related to well-adjusted and known operating conditions described by a vector $\underline{Z}_{k0}$. Consequently, on these conditions, the *parameter vector* $\underline{Z}_k$ may be regarded as invariable, i.e. $\underline{Z}_k = \underline{Z}_{k0}$

On the above assumption, equation (3.1) becomes

$$X_{kOUT} = h_k\ (X_{kIN};\ \underline{Z}_{k0}) \tag{3.1a}$$

where

$\underline{Z}_{k0}$ - vector that is composed of the internal and external parameters of the element k under
adjusted operating conditions having well-defined values; $\underline{Z}_{k0}$ = const.

On the above assumption of linearity and well-adjusted operating conditions, the functional relationship can be expressed by

$$X_{kOUT} = X_{kOUT0} + X_{kIN} \cdot G_k(\underline{Z}_{k0}) \tag{3.3}$$

where

X_{kOUT0} - output quantity at the adjusted operating conditions.
$G_k(\underline{Z}_{k0})$ - transmission factor of the element k which demands on the values of the vector $\underline{Z}_k$

Equation (3.3) expresses a linear combination which does not contain any term representing influences, disturbances, instabilities or imperfect knowledge.

When stringently following this line, equation (3.3) mathematically describes a fictitious *ideal element* of the *measuring chain*. It characterizes a perfectly-known *cause-and-effect relationship* with infinite-precisely known parameters and disturbances.

This means that neither the transmission factor $G_k(\underline{Z}_{k0})$ nor the vector $\underline{Z}_{k0}$ would contribute to the uncertainty of measurement. In mathematical terms this reads

- $G_k = \mathrm{E}[G_k]$ and $u(\mathrm{E}[G_k]) = 0$
- $Z_{k0} = \mathrm{E}[Z_{k0}]$ and $u(\mathrm{E}[Z_{k0}]) = 0$

According to this concept, the output quantity X_{kOUT} depends only on the input quantity X_{kIN} and on an infinite-precisely known transmission factor G_k and the vector Z_{k0}. Therefore, the uncertainty of measurement that is associated with the expectation value of the output quantity would be given by the following relationship:

$$u(x_{kOUT}) = u(x_{kIN}) \cdot G_k.$$

Fig. 3 illustrates the *cause-and-effect relationship* of the fictitious *ideal element*.

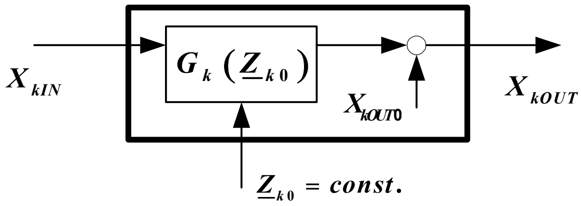

Fig.3 : Illustration of the concept of the ideal element. symbols see equation 3.3

The above concept of the *ideal element* forms the basis of both the mathematical treatment of real elements of the measuring chain (cf. 3.2) and of the modelling concept presented below (cf. 4).

3.3 Disturbed elements of a measuring process

To face facts, the *ideal element* of a measuring system or process is only a theoretical concept that serves as an aid to understand and mathematically simplify the descriptions of measurements. In the real world of measurements, both the internal and the external parameters of a element of the *measuring chain* will be neither well-adjusted and constant nor precisely known. Therefore, one can be sure that almost all components Z_{kl} of the(real) *disturbance vector* $\underline{Z}_k$ will (slightly) deviate from the (ideal) adjusted operating point represented by $\underline{Z}_{k0}$:

$$\underline{Z}_k = \underline{Z}_{k0} - \delta\underline{Z}_k \tag{3.4}$$

where

$\delta\underline{Z}_k$ - deviation vector, $\delta\underline{Z}_k = (\delta Z_{k1}\, ;\, \delta Z_{k2}\, ;\, \dots\, ;\delta Z_{kM})$, that is composed of the deviations of the individual internal and external parameters of the observed element, $\delta Z_{kl} = Z_{k0l} - Z_{kl}$, $l = 1, 2, \dots$.

It should be noted that the above defined deviations are not caused by errors in measurement but are rather a consequence of the concept of the *ideal element* of a measuring system or process. The term deviation is more appropriate than error for what is described and, therefore, being used here.

Consequently, elements of a real measuring system or process may be described not only by means of the functional relationship (3.1) or by similar expressions but also by *disturbed ideal elements*.

Starting from equation (3.3), two types of deviations may be distinguished. Superimposing deviations produce an (additive) offset of the (values of the) output quantity. Deforming deviations result in a change of the transmission factor $G_k(\underline{Z}_{k0})$:

$$X_{kOUT} = X_{kOUT0}(\underline{Z}_{k0}) + \delta\underline{Z}_{kS} + X_{kIN} \cdot [G_k(\underline{Z}_{k0}) + \delta\underline{Z}_{kD}] \tag{3.5}$$

where

$\delta\underline{Z}_{kS}$ - vector that is composed of the superimposing deviations, $\delta\underline{Z}_{kS} = (\delta Z_{kS1};\, \delta Z_{kS2};\, \dots\, ;\, \delta Z_{kSm})$, $m = 1, 2, \dots$

$\delta\underline{Z}_{kD}$ - vector that is composed of the deforming deviations, $\delta\underline{Z}_{kD} = (\delta Z_{kD1}; \delta Z_{kD2}; \ldots ; \delta Z_{kDn})$ $n = 1, 2, \ldots$

Assuming that even real, disturbed measuring systems or processes operate close to their adjusted operating points, both types of deviations may be expected to be predominantly small. Therefore, the impact of the individual deviations may be estimated by a first-order *Taylor series* around the adjusted (ideal) operating point that is described by the vector $\underline{Z}_{k0}$. This reasonable assumption is utilized in both uncertainty propagation (cf. 2) and modelling measurements (cf. 4).

Fig. 4 illustrates the concept of the disturbed ideal element of a measuring system or process.

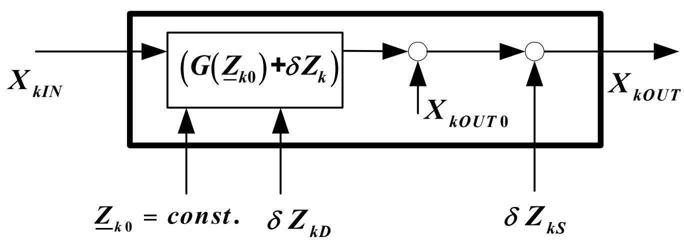

Fig. 4: Illustration of the concept of the disturbed ideal element. symbols see equation (3.5)

In contrast to the mathematical formulation of measurements by analytical expressions with complex functional dependencies according to equations (3.1) and (3.4), the use of disturbed ideal systems makes modular formulations as well as partial reformulations of model equations possible. Therefore, this concept is used here for modelling measurements (cf. 4).

4 Modelling concept

4.1 Starting points

The modelling concept is based on the principles of the *measuring chain* that is composed of the disturbed ideal elements as described in 3.3 and on the method of measurement [2] used. First formulations of this concept have been made by *Bachmaier* [4], *Kessel* [5] and by a common working group of the *Physikalisch–Technische Bundesanstalt (PTB)* and the *Deutsches Institut fuer Normung (DIN)* [6].

The modelling procedure (cf.4.3) uses graphical schemes (signal flow charts) that, in turn, are used to mathematically formulate the *cause-and-effect relationships*.

4.2 Components used

The careful consideration of the concept of the *disturbed ideal element* (cf. 3.3) leads to the conclusion that almost all signal transformations, disturbances, and changes may be described by such an element. This applies both to targeted tasks of the measuring process and unwanted effects, disturbances etc.

The most important exceptions are given by the needed parameter sources and indicating/recording units.

Consequently, for the modelling of *cause-and-effect relationships* of measurements the following components are employed:

- **Parameter source (SRC):**
  The parameter source is designed along the concept of the *disturbed ideal element*: It supplies or reproduces a precisely known and adjusted parameter which is disturbed by real effects, e.g. time-dependent drift, external influences, ranges of ambiguity of parameters etc. The disturbed parameter source supplies always a measurable random quantity on which, prior to other involved quantities, the indication(s) is/are affected.

  Fig. 5 shows the graphical scheme of the parameter source. The following deviations are assigned to this component.

  - deviations due to the imperfect knowledge of generated quantity (range and systematic deviations of the quantity);
  - deviations due to the susceptibility of the source to external conditions.

  The following restriction is made: Disturbances are expressed by superimposing deviations only.

  A parameter source may stand for devices, material measures, substances, processes, human or animal bodies.

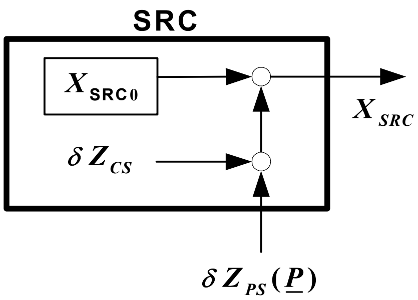

Fig. 5: Graphical scheme of a parameter source.
X_{SRC0} – undisturbed quantity generated by the source; X_{SRC} – disturbed quantity delivered by the source; δZ_{CS} – superimposing deviations due to the imperfect knowledge of X_{SRC0}; $\delta Z_{PS}(\underline{P})$ – superimposing deviation due to the susceptibility of the source to external conditions $\underline{P}$

- **Indicating / recording unit (INDU):**
  The indicating / recording unit serves to indicate or to record its input quantity. Disturbances are given by the limited resolution of the unit, real internal effects, e.g. time-dependent drift, external influences, ranges of ambiguity of parameters etc. Instrumental errors are also assigned to indicating / recording units.

  Fig. 6 shows the graphical scheme of the transforming unit. The following deviations are assigned to this component:

- deviations of the transmission factor due to its instability or an imperfect knowledge of its value;
- deviations due to the susceptibility of the unit to external conditions;
- deviation due to the imperfect coupling of the output to the adjacent unit.

The following restriction is made: Disturbances are expressed by superimposing deviations only.

An indicating / recording unit may stand for a measuring instrument, an indicating device or a recorder.

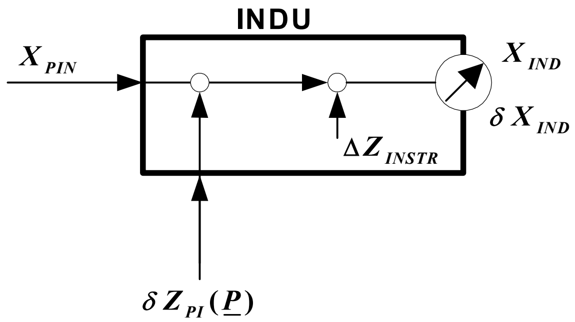

Fig. 6: Graphical scheme of the indicating/recording unit.
ΔZ_{INSTR} – instrumental error of the unit; δX_{IND} – deviation due to the limited resolution; $\delta Z_{PI}(\underline{P})$ – deviation due to the susceptibility of the unit to external conditions $\underline{P}$

- **Transforming unit (TRANS):**

The transforming unit may be recognized to be equal to the *disturbed ideal element* that is described in 3.3.

Fig. 7 shows the graphical scheme of the indicating / recording unit. The following deviations are assigned to this component:

- the instrumental error,
- deviation due to the limited resolution and
- deviations due to the susceptibility of the unit to external conditions.

A transforming unit may stand for a measuring transducer, an amplifier, a transmission or communication path, a measuring bridge as well as for signal matching, interfaces, real coupling / linking effects etc.

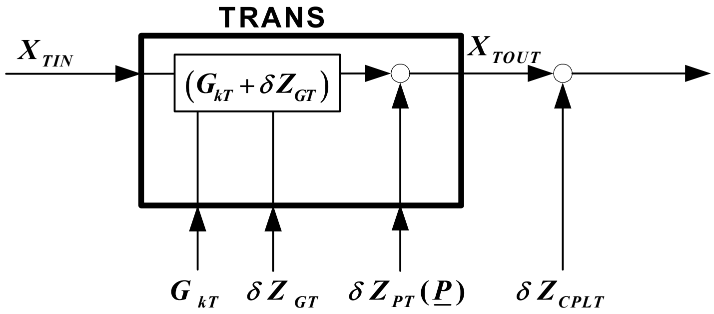

Fig. 7: Graphical scheme of a transforming unit.
$\boldsymbol{G_{kT}}$ – transmission factor; $\boldsymbol{\delta Z_{GT}}$ – deforming deviations (deviations of $\boldsymbol{G_{kT}}$); $\boldsymbol{\delta Z_{PT}(\underline{P})}$ – superimposing deviation due to the susceptibility of the transforming unit to external conditions $\underline{\boldsymbol{P}}$;
$\boldsymbol{\delta Z_{CPLT}}$ – deviation due to the imperfect coupling of the output to the adjacent unit

It is assumed that the components are chained non-reactively (see equation (3.2)).

4.3 Modelling procedure

The modelling procedure consists of four elementary steps (cf.4.3):

- description of the measurement, identification of the measurand and of the method used;
- formulation of the *cause-and-effect relationship* of the *ideal measurement*;
- consideration (mathematical) of all disturbances characterizing the real measurement (cf. 3.3);
- conversion of the *cause-and-effect relationship* of the real measurement into the model equation.

First of all, a clear description of the measurement is necessary, along with the definition of the measurand and the identification of the method of measurement employed.

Example 1: *The aim of the measurement is to determine the electrical resistance* $\boldsymbol{R_{X0}}$ *of a resistor. The resistor is supplied by a constant current. The voltage over the resistor is measured by means of a digital voltmeter (see Fig. 8). The resistance is considered to be the measurand. The method of measurement may be described by direct measurement.*

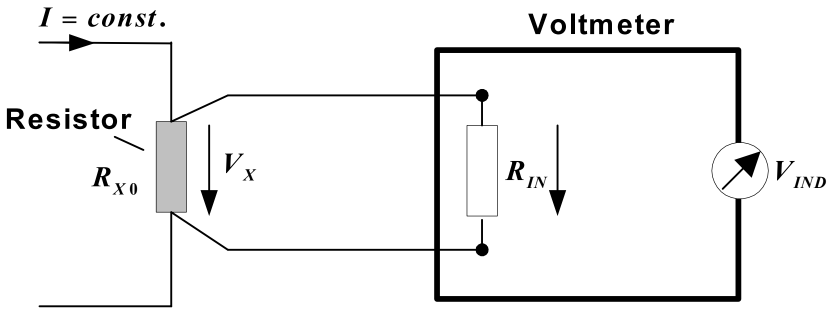

Fig. 8: Example: Measurement of the electrical resistance $\boldsymbol{R_{X0}}$ of a resistor.

The next step is to establish the *cause–and–effect relationship* of the *ideal measurement*. It describes the relationship between the measurable quantity considered to be the main cause of an indication and the effect, i.e. the indication or record. The *ideal measurement* can be assumed as a fictitious measuring process that is consisting of *ideal elements* only (cf. 3.2). This would mean that the parameters of the measuring system or process are completely defined by the operating points of its components which are given by their *parameter vectors* $\{\underline{Z}_k\}$. The measurement is assumed to be undisturbed. Therefore, it may be concluded that the indication depends only on the measurand and on infinite–precisely known parameters of the process, e.g. a amplifying factor.

Firstly, the block diagram of the ideal measurement should be drawn up that employs the above elements.

> ***Example 2:*** *With the above elements being employed, Fig. 9 shows the block diagram of the ideal measurement which is described by Example 1. It is assumed that the current* $\boldsymbol{I}$ *is infinite–precisely known and the voltage over the resistor* $\boldsymbol{V_X}$ *is equal to the indicated voltage* $\boldsymbol{V_{IND}}$.
>
> *From this diagram, the mathematical formulation of the cause–and–effect relationship of the "ideal measurement" can easily be obtained:*
>
> $$V_{IND} = V = V_X = R_X \cdot I$$

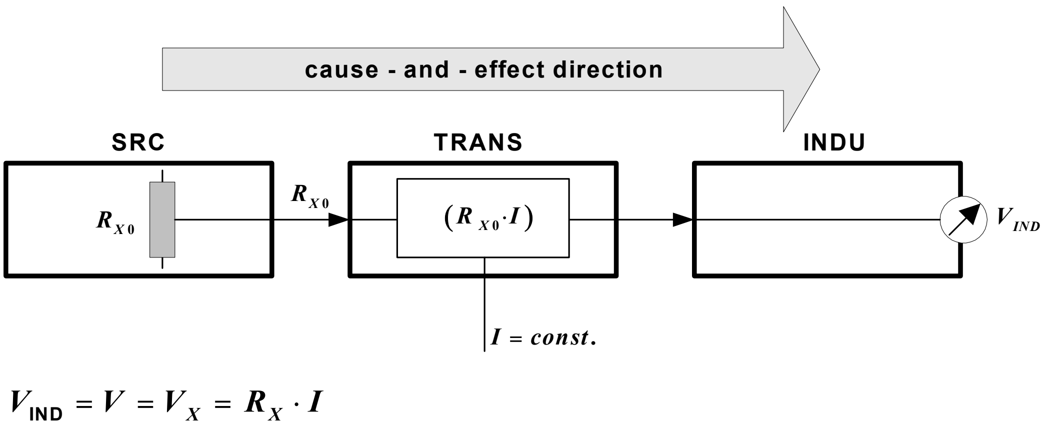

Fig. 9: Cause-and effect relationship of the ideal measurement according to Example 2

The third step is to introduce all relevant disturbing quantities as well as the corrections and relevant imperfections of the real measuring process. In accordance with the concept of the *disturbed ideal element*, they are represented and being introduced in terms of deviations (cf. 3.3). The exclusive use of deviations creates a stringent and comprehensive description of the necessary *cause-and-effect relationship*.

> ***Example 3:*** *The following influences and imperfections are being introduced into the Example 2:*

- *temperature deviations δt which affect the resistor,*
- *thermal voltage δV_{th} due to different temperatures of the connecting points of the resistor,*
- *deviation δI of the current from its nominal value,*
- *ratio $R_X / R_{IN} = r >> 0$, affecting the measured voltage,*
- *instrumental error ΔV_{INSTR} assigned to the voltmeter, and*
- *unknown deviation δV_{IND} due to digital resolution.*

Fig. 10 illustrates this expanded measurement task including all influences and imperfections, and Fig. 11 shows the resulting block diagram of the real measurement.

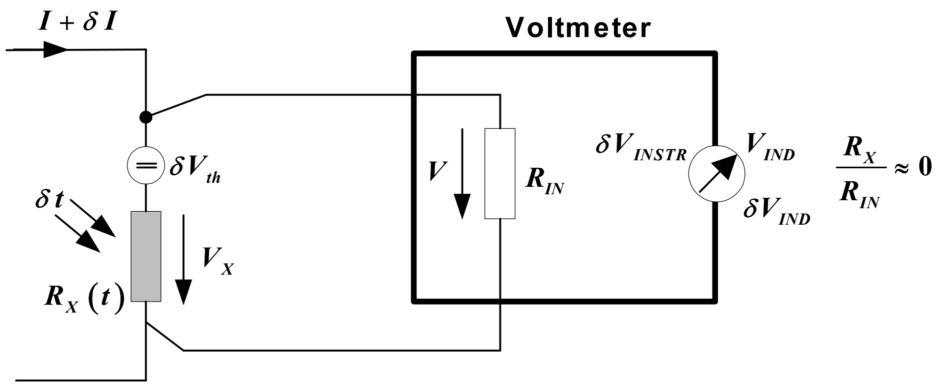

Fig. 10: Example: Measurement of the resistance $R_X(t)$ of a resistor according to *Example 3* (including disturbing influences and imperfections of the measuring system). symbols see *Example 3*

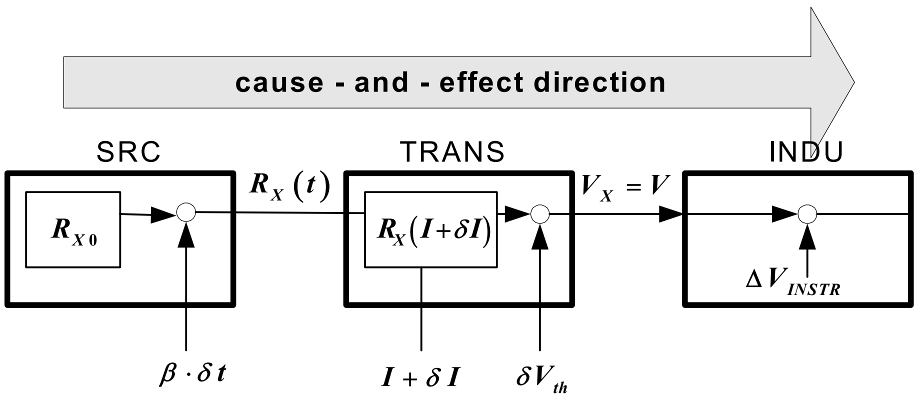

Fig. 11: *Cause-and-effect relationship* of the real measurement according to *Example 3*. symbols see *Example 3*

The expression in mathematical terms of the cause-and-effect relationship can directly be taken from Fig. 11. In the case of more complex systems, it is advisable to carry out this procedure step-by-step and separately for each component.

- $R_X(t) = R_{X0} + \delta t \cdot \beta$ for the parameter source
- $V_X = R_X(t) \cdot (I + \delta I)$ for the transforming unit
- $V = V_X + \delta V_{th}$
- $V_{IND} = V + \Delta V_{INSTR} + \delta V_{IND}$ for the indicating unit

Example 3 illustrates the advantage of expressing all disturbances in terms of deviations. Superimposing deviations can simply be added when going ahead in the cause-and-effect direction. This originates in the definition of the deviation as the *value minus its reference value* [2] or, when applied to the modelling process, *disturbed quantity minus its undisturbed quantity*.

The last step requires the established *cause-and-effect relationship* of the real measuring process being converted into the model equation. The model equation gives an explicit expression for the measurand.

> ***Example 4:*** *From the cause-and-effect relationship that has been established in Example 3, one obtains the following model equation(s) for the resistance* R_{X0}*:*

- $R_{X0} = R_X(t) - \delta t \cdot \beta$
- $R_X(t) = V_X \cdot (I + \delta I)^{-1}$
- $V_X = V - \delta V_{th}$
- $V = V_{IND} - \delta V_{IND} - \Delta V_{INSTR}$

4.3 Role of the method of measurement

The structure and the chaining sequence of the *cause-and-effect relationship* are determined by the method of measurement used.

Direct measurements result in a linear (un-branched) chain of the components used (see Fig. 11). A generic structure of the *cause-and-effect relationship* of a direct measurement is shown in Fig.12. The following relationship is obtained:

$$X_{IND} = h(X_{SRC}; G_T; \Delta Z_{INSTR}; \underline{P}) \tag{4.1}$$

where

X_{SRC} - measurand;
G_T - transmission factor of the transforming unit;
δZ_{INSTR} - instrumental error of the indicating unit;
$\underline{P}$ - vector describing external parameters;
X_{IND} - indicated quantity.

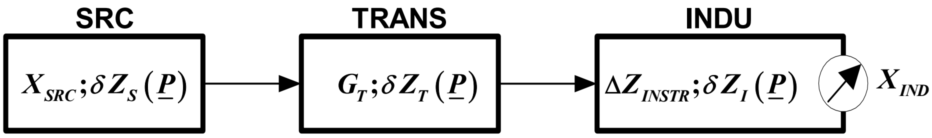

Fig. 12: Generic structure of the cause-and-effect relationship of a direct measurement. symbols: see text

Other methods are used to achieve high accuracies and to ensure traceability of calibration results. These methods often result in branched *cause-and-effect relationships*. Examples are given by the direct comparison of indicating measuring instruments and by the substitution method for the comparison of material measures.

Fig.13 and 14 show the generic structures of the *cause-and-effect relationship* of the two methods. It should be noted that the calibration of a measuring instrument usually does not aim at a measurable quantity but the instrumental error [2] of the instrument under test. Consequently, the uncertainty associated with the result of the calibration is to be determined, i.e. the uncertainty associated with the expectation value of the instrumental error.

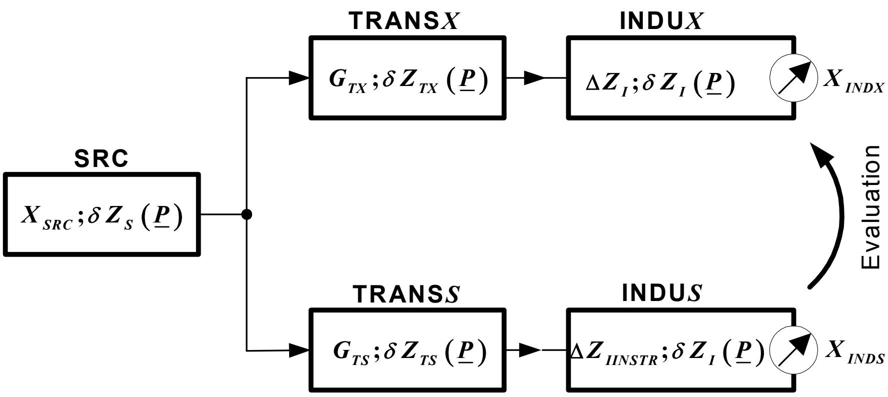

Fig. 13: Generic structure of the cause-and-effect relationship of a direct comparison of indicating measuring instruments. symbols: **TRANS*X*** – transforming unit ***X***-path; **TRANS*S*** – reference transforming unit; **INDU*X*** – instrument under test; **INDU*S*** – indicating standard; other symbols see text

In order to establish the mathematical expression of a *cause-and-effect relationship* having a branched structure, for each branch a separate (partial) relationship must be set up.

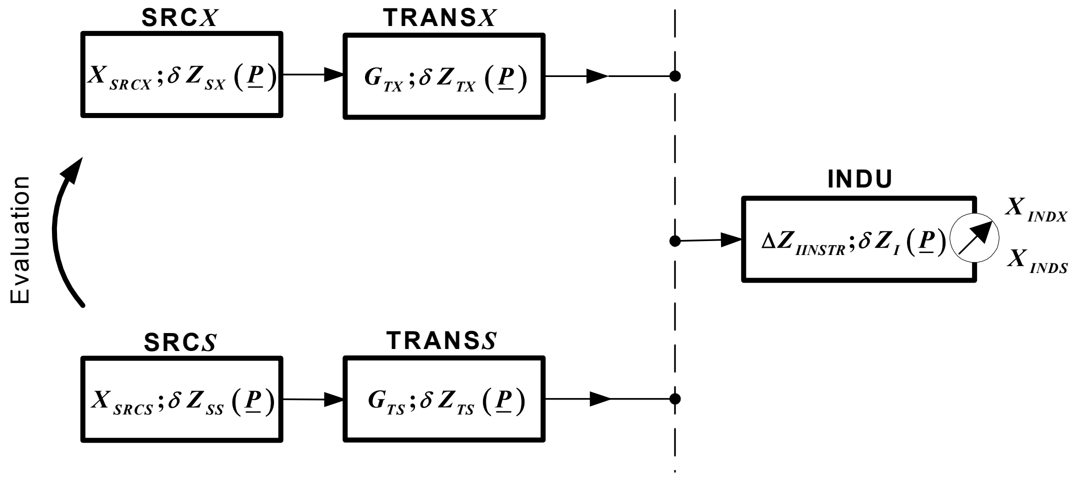

Fig. 14: Generic structure of the cause-and-effect relationship of a measurement using substitution. symbol: **SRC*X*** – material measure under test; **SRC*S*** – standard (material measure); **TRANS*X*** – transforming unit ***X***-path; **TRANS*S*** – transforming unit ***S***-path; other symbols see text

In the case of a direct comparison (see Fig. 13), the following relationships are obtained:

$$X_{INDX} = h_X(X_{SRC};\ G_{TX};\ \delta Z_{INSTRX};\ \underline{P}) \tag{4.2a}$$
$$X_{INDS} = h_S(X_{SRC};\ G_{TS};\ \delta Z_{INSTRS};\ \underline{P}) \tag{4.2b}$$

where

δZ_{INSTRX} - measurand (instrumental error of the instrument under test);
δZ_{INSTRS} - instrumental error of the standard;
X_{INDX} - quantity indicated by the instrument under test;
X_{INDS} - quantity indicated by the standard;
X_{SRC} - measurable quantity;
G_{TX} - transmission factor of the transforming unit of the X-path;
G_{TS} - transmission factor of the transforming unit of the S-path;
$\underline{P}$ - vector describing external parameters and disturbances.

As X_{SRC} appears both in equation (4.2a) and in equation (4.2b), it may be concluded that this quantity does not disturb the measurement whereas the measurand will be influenced by almost all other quantities involved.

Example 5: *Calibration of a thermometer:*

(1) It is the aim of the measurement to determine the instrumental error of a mercury-in-glass thermometer at about 20°C. Together with a standard thermometer which is also a mercury-in-glass thermometer, the instrument to be tested is immersed into a thermostatted and stirred water bath (see Fig. 15). The method of measurement may be considered a direct comparison of two temperatures.

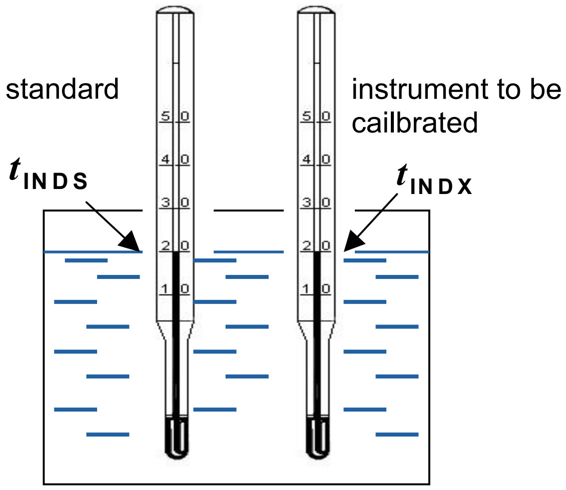

Fig. 15: Example: Calibration of a mercury-in-glass thermometer

(2) Fig. 16 shows the block diagram of the cause-and-effect relationship for the ideal measurement.

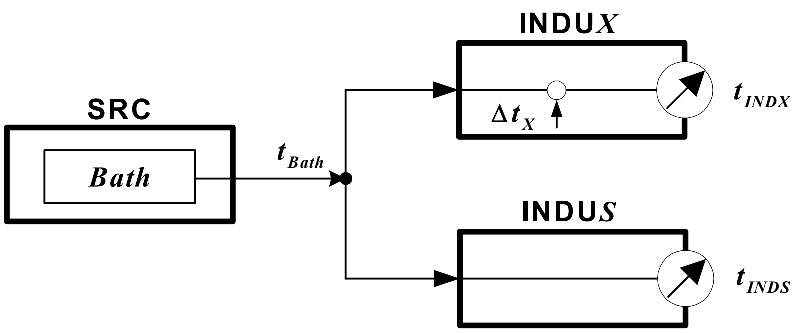

Fig. 16: Cause-and-effect relationship of the *ideal measurement* according to *Example 5*. **SRC** – thermostatted bath; **INDU*X*** – thermometer to be calibrated; **INDU*S*** – standard thermometer: other symbols see *Example 5*

From this block diagram the following relationship may be derived mathematically:

For the X-path:

- $t_{INDX} = t_{Bath} + \Delta t_X$

For the S-path:

- $t_{INDS} = t_{Bath}$

where

t_{Bath} – *temperature of the bath*
t_{INDX} – *temperature indicated by the instrument to be calibrated*
t_{INDS} – *temperature indicated by the standard*
Δt_X – *measurand (instrumental error of the instrument to be calibrated).*

(3) Fig. 17 shows the cause-and-effect relationship of the real measurement. The following imperfections and disturbances have been introduced:

$\boldsymbol{\Delta t_{CS}}$ – *instrumental error of the standard (known and unknown contributions)*
$\boldsymbol{\delta t_{BathX}}$ – *deviation of the temperature of the instrument to be calibrated from the temperature t_{Bath} that is assumed to be equal to the temperature of the standard*
$\boldsymbol{\delta t_{INDX}}$ – *deviation due to imperfect readings of the thermometer*
$\boldsymbol{\delta t_{INDS}}$ – *deviation due to imperfect readings of the standard.*

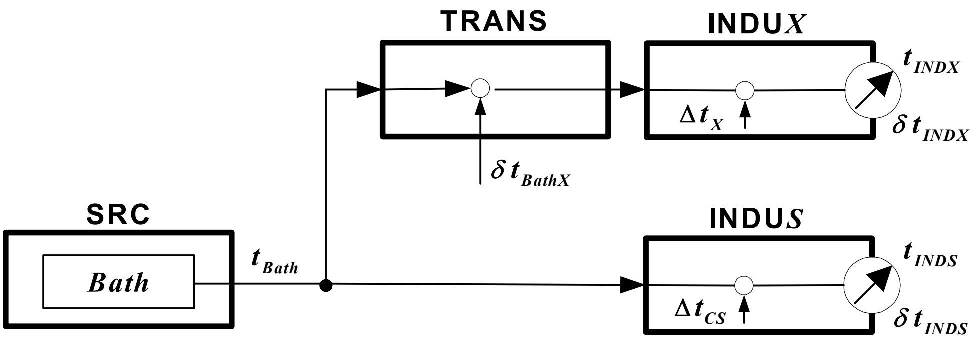

Fig. 17: Cause-and-effect relationship of the *real measurement* according to *Example 5*. **SRC** – thermostatted bath; **TRANS** – temperature gradient in the bath; **INDU*X*** – thermometer to be calibrated; **INDU*S*** – standard thermometer; other symbols see *Example 5*

The following relationships are obtained:

X-path:

- $\boldsymbol{t_{INDX} = t_{Bath} + \delta t_{BathX} + \Delta t_X + \delta t_{INDX}}$

S-path:

- $\boldsymbol{t_{INDS} = t_{Bath} + \Delta t_{CS} + \delta t_{INDS}}$

(3) From the above cause-and-effect relationship, one obtains the following model equation:

- $\boldsymbol{\Delta t_X = t_{INDX} - t_{INDS} + \Delta t_{CS} - \delta t_{BathX} - \delta t_{INDX} + \delta t_{INDS}}$

$\boldsymbol{t_{INDX}}$*,* $\boldsymbol{t_{INDS}}$*,* $\boldsymbol{\delta t_{INSX}}$*, and* $\boldsymbol{\delta t_{INDS}}$ *may be estimated from series of observations. The knowledge of* $\boldsymbol{\Delta t_{CS}}$ *should be taken from the calibration certificate of the standard.* $\boldsymbol{\delta t_{INDS}}$ *may be estimated from the manufacturer's information on the bath.*

In the case of substitution (see Fig. 14), the following generic relationships are obtained:

$$\boldsymbol{X_{INDX} = h(X_{SRCX}; G_{TX}; \Delta Z_{INSTR}; \underline{P})} \quad (4.3a)$$

$$\boldsymbol{X_{INDS} = h(X_{SRCS}; G_{TS}; \Delta Z_{INSTR}; \underline{P})} \quad (4.3b)$$

where

X_{INDX} - indicated quantity for the X-path;
X_{INDS} - indicated quantity for the S-path;

X_{SRCX} - measurand;
X_{SRCS} - quantity represented by a standard;
G_T - transmission factor of the transforming unit;
ΔZ_{INSTR} - instrumental error of the indicating unit (comparator);
$\underline{P}$ - vector describing external parameters and disturbances.

It may be recognized that both equation (4.3a) and equation (4.3b) contain the same relevant quantities ΔZ_{INSTR}, and $\underline{P}$. Therefore, in most cases, their influence can be neglected when the substitution method is used.

Example 6: Calibration of a weight

(1) It is the aim of the measurement to determine the instrumental error of a 10 kg-weight of the class F1 by means of the substitution method a 12 kg-mass comparator. A calibrated 10 kg-weight of the accuracy class E2 serves as a standard using the method of measurement is illustrated by Fig. 18. The weighings are to be carried out in the following sequence: standard-test weight-test weight-standard.

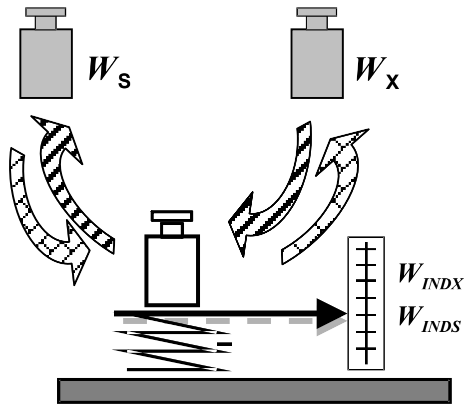

Fig. 18: Example: Calibration of a weight by using the substitution method

(2) Fig. 19 shows the block diagram of the ideal measurement.

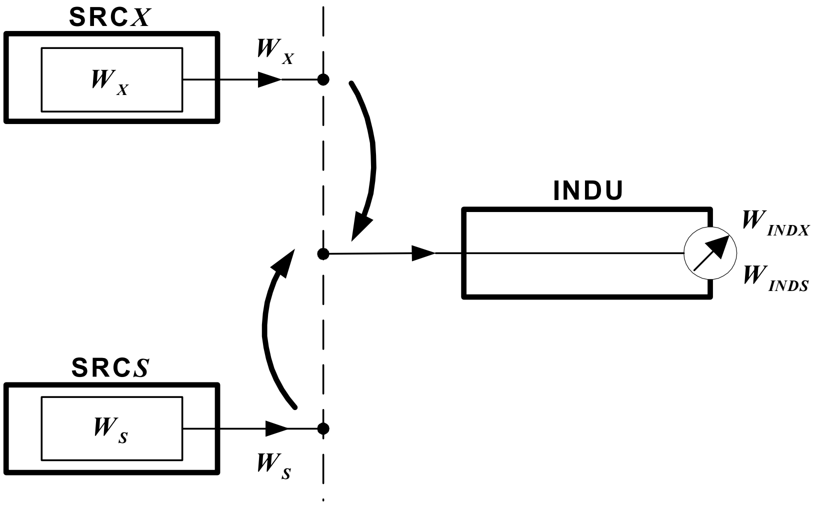

Fig. 19: Graphical illustration of the *cause-and-effect relationship* of the *ideal measurement* according to *Example 6*. **SRC*X*** – weight to be calibrated; **SRC*S*** – standard weight; **INDU** – mass comparator; other symbols see *Example 6*

The following relationships can be derived:

X-path:

- $W_{INDX} = W_X$

S-path:

- $W_{INDS} = W_S$

where

W_X – *actual value of the weight to be calibrated*
W_S – *weighing value of the standard used*
W_{INDX} – *indication for the weight to be calibrated*
W_{INDS} – *indication for the standard*

Note: The measurand ΔW_X is to be calculated from the below relationship by:

$\Delta W_X = W_{Nom} - W_X$

where

ΔW_X – *measurand ("instrumental error" of weight to be calibrated)*
W_{NomX} – *nominal value (10 kg) of the weight to be calibrated*
W_X – *actual value of the weight to be calibrated*

(3) Fig. 20 shows the cause-and-effect relationship of the real measurement. The following imperfections and disturbances have been introduced:

δW_{CPLX} - *deviation due to imperfect "coupling" of the weight to be calibrated with the mass comparator; causes are given by convection, air buoyancy, magnetic susceptibility etc.*
δW_{CPLS} - *deviation due to imperfect "coupling" of the standard weight with the mass comparator; causes: see* δW_{CPLX}

ΔW_{INSTR} - *instrumental error of the masse comparator*

ΔW_C ($\underline{P}$) - *deviation due to the susceptibility of the mass comparator to environmental conditions and incomplete knowledge of the actual operating conditions when standard is measured.*

δW_{INDX}, δW_{INDS} - *deviations due to the imperfect readings of the comparator.*

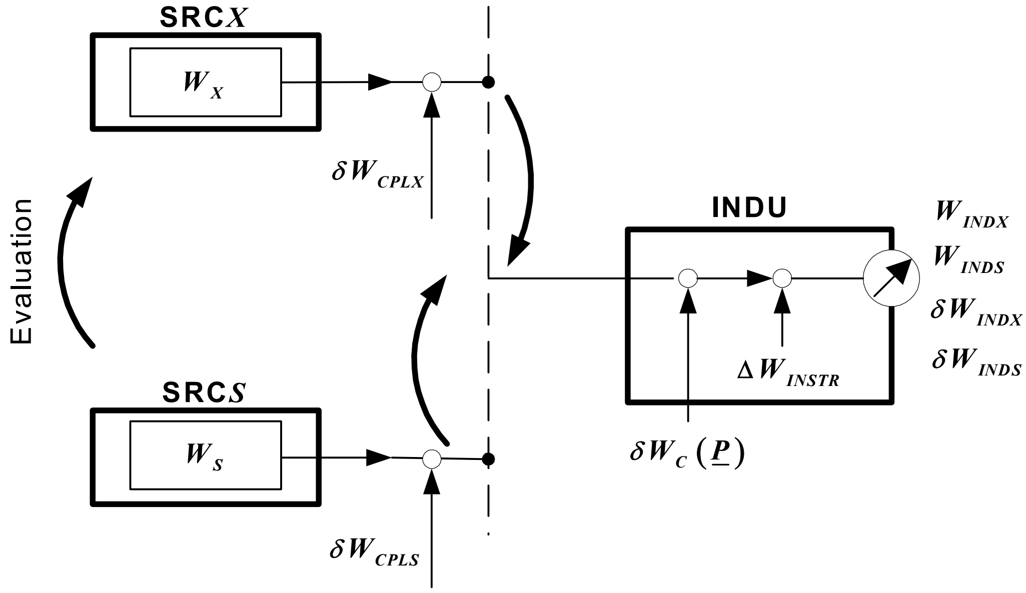

Fig. 20: Graphical illustration of the *cause-and-effect relationship* of the *real measurement* according to *Example 6*. **SRC*X*** – weight to be calibraed; **SRC*S*** – standard weight; **INDU** – mass comparator; other symbols see *Example 6*

The following relationships are obtained:

X-path:

$$W_{INDX} = W_S + \delta W_{CPLX} + \delta W_C\,(\underline{P}) + \Delta W_{INSTR} + \delta W_{INDX}$$

S-path:

$$W_{INDS} = W_S + \delta W_{CPLS} + \delta W_C\,(\underline{P}) + \Delta W_{INSTR} + \delta W_{INDS}$$

With $\delta W_S = W_{NomS} - W_S$ for the "instrumental error" of the standard used, one obtains the following model equation:

$$\Delta W_X = \Delta W_S - (W_{INDX} - W_{INDS}) + \delta W_{CPLX} - \delta W_{CPLS} + \delta W_{INDX} - \delta W_{INDS}$$

The knowledge on ΔW_S can be taken from the calibration certificate of the standard. the differences

(W_{INDX} – W_{INDS}) are delivered by the mass comperator. The remaining deviations can be estimated from the actual measuring process. It should be noted that the performance of the mass comperator does not appear in the above model equation but in the cause-and-effect relationships.

References

[1] Guide to the Expression of Uncertainty in Measurement, BIPM, IEC, IFCC, ISO, IUPAC, IUPAP, OIML, Geneva 1995 (corrected and reprinted)

[2] International Vocabulary of Basic and General Terms in Metrology, BIPM, IEC, FCC, ISO, IUPAC, IUPAP, OIML, 1993.

[3] Kind, D.: Die Kettenschaltung als Modell zur Berechnung von Messunsicherheiten (The measuring chain as a modell for the calculation of measurement uncertainty). PTB-Mitteilungen 111 (2001) 4, pp. 338- 341 Wirtschaftsverlag NW, Bremerhaven 2001 (ISSN 0030-834X)

[4] Bachmair, H.: A simplified method to calculate the uncertainty of measurement for electrical calibration and test equipment. Proceedings of the 9th INTERNATIONAL METROLOGY CONGRESS, Bordeaux, 18-21 October 1999, pp. 342-344, Mouvement Francais pour la Qualité, Nanterre Cedex 1999 (ISBN: 2- 909430-88x)

[5] Kessel, W.: Messunsicherheit – einige Begriffe und Folgerungen für die messtechnische Praxis. PTB- Mitteilungen 111 (2001) 3, pp. 226–244, Wirtschaftsverlag NW, Bremerhaven 2001 (ISSN 0030-834X)

[6] Messunsicherheit nach GUM-praxisgerecht bestimmen (Uncertainty of measurement in accordance with the GUM-practical determinations). Hand-outs of the PTB/DIN training course. Deutsches Institut fuer Normung, Berlin 2001/2002