Chapter 5

Matrices and linear algebra

The HP 49G has extensive capabilities for entering and manipulating arrays. An array object can be a vector or matrix.

Many of the matrix operations described in this chapter also apply to vectors. Wherever this is the case, the more general term *array* is used instead of *matrix*.

By default, Matrix Writer will interpret a one-row array as a vector rather than as a matrix. If you want a one-row array interpreted as an array, press VEC first.

For instructions on how to create an array—and how to open an array in Matrix Writer—see chapter 8 of the *User's Guide*.

Matrix Writer operations

The following summarizes the operations that are available in Matrix Writer.

To move around an array

Method 1: using arrow keys

Press the arrow keys to move the cursor from cell to cell.

You can press ↱ before an arrow key to move the cursor directly to an outermost row or column:

- ↱ ▶| moves the cursor to the last column
- ↱ ▲ moves the cursor to the first row
- ↱ |◀ moves the cursor to the first column
- ↱ ▼ moves the cursor to the last row.

Method 2: the GOTO command

1. Press GOTO.

   The Matrix Writer input form is displayed.

   Note that the GOTO command is on the second page of the Matrix Writer menu, so you may need to press (NXT) to display it.
2. Enter the row number of the cell you want to go to.
3. Press (ENTER).
4. Enter the column number of the cell you want to go to.
5. Press (ENTER).
6. Press OK or (ENTER).

   The matrix is redisplayed and the cursor is now in the cell whose row–column coordinates you specified.

To edit an array

1. Move the cursor to a cell to be edited.

   See the previous section for instructions on cursor movement.
2. Press EDIT.

   The contents of the cell are copied to the command line.

If you want to completely change the contents of the cell, you can do so without first pressing EDIT. Just start typing the new contents. What you type appears on the command line.

3. Make your changes.
4. Press (ENTER) to move your changes from the command line to the cell.
5. Repeat from step 1 if you want to change other cells.
6. Press (ENTER) to close Matrix Writer and place the changed array on the command line (or press (CANCEL) to close Matrix Writer and discard your changes).

   If you discarded your changes, ignore the next step.
7. Press (ENTER) again to save your changes.

To make the cells narrower or wider

- Press ←WID to make the cells narrower.
  More columns are displayed.
- Press WID→ to make the cells wider.
  Fewer columns are displayed.

Note that these commands modify the width of *all* columns, not just the column with the highlighted cell.

To control how the cursor moves after an entry

By default, the cursor moves to the adjacent cell in the next *column* after you place an object in a cell. You can change this setting in one of two ways:

- To make the cursor move to the adjacent cell in the next *row* after entry, press GO↓.
  A small box (■) appears beside the command on the menu.
- To prevent the cursor from advancing at all after entry, press GO→ and GO↓ until neither command shows a box (■) beside it on the menu.

The setting you choose remains the mode of cursor movement until you change the setting again.

To return to the default setting, press GO→ until a box (■) appears beside it on the menu.

If you want to change the way the cursor moves through an array, make your change *before* entering objects into the array. Once you start entering objects, the way the cursor moves cannot be changed.

Manipulating columns and rows

To insert a column

1. Move the cursor to the column where you want the new column to appear.
2. Press +COL. A column of zeros is inserted.

   Note that the +COL command is on the second page of the menu, so you may need to press NXT to display it.

You can also add a column to an array without using Matrix Writer. See "To insert one or more new columns into an array" on page 5-11.

To add a column to the right of the last column of data

1. Move the cursor to the right of the last column of data.

   Pressing ↱ ▶ will move the cursor directly to the last column of data. You then press ▶ to move to the next column.
2. Enter an object.
3. Press ENTER to move your object to the highlighted cell.

   The rest of the column fills with zeros and your array now includes this new column.

To delete a column

1. Move the cursor to the column you want to delete.
2. Press –COL.

   Note that the –COL command is on the second page of the menu, so you may need to press NXT to display it.

To insert a row

1. Move the cursor to the row where you want the new row to appear.
2. Press +ROW. A row of zeros is inserted.

   Note that the +ROW command is on the second page of the menu, so you may need to press NXT to display it.

You can also add a row to an array without using Matrix Writer. See "To insert one or more new rows into a matrix" on page 5-11.

To add a row below the bottom row of data

1. Move the cursor to the row below the last row of data.

   Pressing ↱ ⏬ will move the cursor directly to the last row of data. You then press ▼ to move to the next row.
2. Enter an object.
3. Press ENTER to move your object to the highlighted cell.

   The rest of the row fills with zeros and your array now includes this new row.

To delete a row

1. Move the cursor to the row you want to delete.
2. Press –ROW.

To delete the contents of a selection of cells

1. Move the cursor to the first cell in the group of cells you want to delete.
2. Press ↱ BEGIN to select BEGIN.
3. Move the cursor to the last cell in the group of cells you want to delete.
4. Press ↱ END to select END.

   The cells between your beginning and end cell are now highlighted.
5. Press DEL to delete the contents of the highlighted cells.

   The DEL command does *not* leave the selected cells empty. Instead, the contents of a cell are replaced with a zero.

Summary of Matrix Writer operations

Key	Description
EDIT	Places the contents of the current cell on the command line for editing.
VEC	For one-row arrays, toggles between vector entry and matrix entry. If this command is selected, one-row arrays are entered onto the command line as vectors (example: [1 2 3]); if it's not selected, one-row arrays are entered as matrices (example: [[1 2 3]]).
←WID	Decreases the width of all cells.
WID→	Increases the width of all cells.
GO→	Sets left-to-right entry mode. The cursor moves to the next *column* after data entry.
GO↓	Sets top-to-bottom entry mode. The cursor moves to the next *row* after data entry.
+ROW	Inserts a row of zeros at the current cursor position.
−ROW	Deletes the current row.
+COL	Inserts a column of zeros at the current cursor position.
−COL	Deletes the current column.
→STK	Copies the current cell to history (level 1 of the stack).
GOTO	Move the cursor to a specified cell.
DEL	Replace the contents of selected cells with zeros.

Advanced matrix operations

Creating special matrices

To create an array filled with a given constant

1. Select the Constant Array command.
   ↰ (MATRICES) CREATE CON
2. For the first argument of the command, enter either:
   - a list containing the dimensions of the desired constant array: { *rows*, *columns* } or
   - an existing array.
3. For the second argument, enter the constant that you want in the array.
4. Press (ENTER).

   The result is an array of the dimensions you entered (or of the dimensions of the specified array) filled with the specified constant.

To create an identity matrix

1. Select the Identity Matrix command.
   ↰ (MATRICES) CREATE IDN
2. Enter either:
   - a number representing the number of rows and columns you want in the identity matrix or
   - an existing array.
3. Press (ENTER).

   The result is an identity matrix of the specified dimensions (that is, a square matrix with zero for all elements except the diagonal elements, which are all 1).

To create an array filled with random integers

1. Select the Random Matrix command.
   ↰ (MATRICES) CREATE RANM
2. Enter either:
   - a list containing the dimensions of the desired random matrix: { *rows*, *columns* } or
   - an existing array.
3. Press (ENTER).
   The result is a random array of the specified dimensions (or of the dimensions of the specified array). The elements are integers within the range –9 to 9.

Assembling matrices

To assemble a matrix by rows from a series of vectors

1. Select the Rows-to-Matrix command.
   ↰ (MATRICES) CREATE ROW ROW→
2. Enter each vector in the order you want them to appear in the matrix. Enter the row 1 vector first, then the row 2 vector, and so on, separating each vector with a comma.
3. Enter the number of rows in the desired matrix.
4. Press (ENTER).
   The result is a matrix composed of the vectors you entered.

To assemble a matrix by columns from a series of vectors

1. Select the Columns-to-Matrix command.
   ↰ (MATRICES) CREATE COLUMN COL→
2. Enter each vector in the order you want them to appear in the matrix. Enter the column 1 vector first, then the column 2 vector, and so on, separating each vector with a comma.
3. Enter the number of columns in the desired matrix.
4. Press (ENTER).
   The result is a matrix composed of the vectors you entered.

To assemble a matrix with a particular diagonal from a vector

1. Select the Vector-to-Matrix Diagonal command.
   ↰ (MATRICES) CREATE DIAG→
2. Enter the vector containing the diagonal elements.
3. Enter either:
   - a list containing the dimensions of the desired matrix: {*rows columns*} or
   - a real number representing the number of rows and columns in the desired square matrix.
4. Press (ENTER).

   The result is a matrix of the desired dimensions using the elements of the vector as the diagonal elements of the matrix. If the vector contains more diagonal elements than are needed to create the matrix, the extra elements are discarded. If the vector does not contain enough elements to complete the matrix, the undefined diagonal elements are set to zero.

To assemble a matrix from a sequence of elements

1. Select the Stack-to-Array command.
   ↰ (PRG) TYPE →ARRY
2. Enter the elements in *row-major order*.

   Row-major order begins with the first element (the element in row 1 and column 1). The next element is the next one in the *row*. If there are no more elements in the row, the next element is the first element in the next row, and so on.
3. Enter a list containing the dimensions of the desired matrix: {*rows*, *columns*}.
4. Press (ENTER) to assemble the matrix.

For example, →ARRY(1, 2, 3, 4, {2, 2}) results in $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Disassembling matrices

To disassemble a matrix into its elements

1. Select the Object-to-Stack command.
   ↰ (PRG) TYPE OBJ→
2. Enter or select the matrix you want to disassemble.
3. Press (ENTER).
   The matrix is disassembled in row-major order. A list indicating the dimensions of the matrix is also returned.

To disassemble a matrix into row vectors

1. Select the Matrix-to-Rows command.
   ↰ (MATRICES) CREATE ROW →ROW
2. Enter or select the matrix you want to disassemble.
3. Press (ENTER).
   The result is a list of row vectors (first row to last), and the number of rows in the matrix.

To disassemble a matrix into column vectors

1. Select the Matrix-to-Columns command.
   ↰ (MATRICES) CREATE COLUMN →COL
2. Enter or select the matrix you want to disassemble.
3. Press (ENTER).
   The result is a list of column vectors (first column to last), and the number of columns in the matrix.

To extract the vector of diagonals from a matrix

1. Select the Matrix-Diagonal-to-Array command.
   ↰ (MATRICES) CREATE →DIAG
2. Enter or select the matrix.
3. Press (ENTER).
   The result is a vector whose elements were the diagonal elements of the matrix.

Inserting rows and columns

To insert one or more new rows into a matrix

1. Select the Insert Row command.
   ↰ (MATRICES) CREATE ROW ROW+
2. Enter or select the array you want to modify.
3. Enter the vector or matrix that you want to insert.
   An inserted array must have the same number of columns as the array into which it is being inserted.
4. Enter the row number you want the first (or only) inserted row to be.
5. Press (ENTER).
   The rows beneath, and including, the row specified at step 4 are pushed down to make room for the inserted rows.

You can also add rows to an array using Matrix Writer. See "To insert a row" on page 5-4.

To insert one or more new columns into an array

1. Select the Insert Column command.
   ↰ (MATRICES) CREATE COLUMN COL+
2. Enter or select the array you want to modify.
3. Enter the vector or matrix that you want to insert.
   An inserted array must have the same number of rows as the array into which it is being inserted.
4. Enter the column number you want the first (or only) inserted column to be.
5. Press (ENTER).
6. The columns to the right of, and including, the column specified at step 4 are pushed to the right to make room for the inserted columns.

You can also add columns to an array using Matrix Writer. See "To insert a column" on page 5-4.

Extracting rows and columns

To extract a particular row from an array

1. Select the Delete Row command.
   ↰ (MATRICES) CREATE ROW ROW–
2. Enter or select the array with the row you want to extract.
3. Enter the number of the row you want to extract.
4. Press (ENTER).
   The result is the array without the extracted row, and the extracted row as a vector.

To extract a particular column from an array

1. Select the Delete Column command.
   ↰ (MATRICES) CREATE COLUMN COL–
2. Enter or select the matrix with the column you want to extract.
3. Enter the number of the column you want to extract.
4. Press (ENTER).
   The result is the matrix without the extracted column, and the extracted column as a vector.

Swapping rows and columns

To swap two rows in an array

1. Select the Row Swap command.
   ↰ (MATRICES) CREATE ROW RSWP
2. Enter or select the array with the rows you want to swap.
3. Enter the number of the one of the rows you want to swap.
4. Enter the number of the other row you want to swap.
5. Press (ENTER).
   The result is the array with the two specified rows swapped.

To swap two columns in an array

1. Select the Column Swap command.
   ↰ (MATRICES) CREATE COLUMN CSWP
2. Enter or select the matrix with the columns you want to swap.
3. Enter the number of the one of the columns you want to swap.
4. Enter the number of the other column you want to swap.
5. Press (ENTER).
   The result is the array with the two specified columns swapped.

Extracting and replacing elements of matrices

To extract the element at a specified position

1. Select the Get Element command.
   ↰ (MATRICES) CREATE GET
2. Enter or select the array with the element you want to extract.
3. Enter either:
   - a list containing the row number and column number of the element you want to extract, or
   - the position number—that is, row-major number—of the element you want to extract.
4. Press (ENTER).
   The result is the extracted element.

To replace an element at a specified position

1. Select the Put Element command.
   ↰ (MATRICES) CREATE PUT
2. Enter or select the array with the element you want to replace.
3. Enter either:
   - a list containing the row number and column number of the element you want to replace, or
   - the position number—that is, row-major number—of the element you want to replace.
4. Press (ENTER).
   The result is a modified array.

Characterizing matrices

Matrix calculations are often sensitive to special characteristics of the matrices used. The HP 49G has a number of commands that return characteristics of matrices. Note that some commands are only defined for *square* matrices, some for any rectangular matrix.

Commands for characterizing matrices

Keys	Description
↰ MATRICES OPERATIONS SIZE	Returns the dimensions of the array (that is, the number of rows and number of columns).
↰ MATRICES OPERATIONS ABS	Returns the Frobenius norm of a matrix and the Euclidean length of a vector: the square root of the sums of the squares of the absolute values of the elements.
↰ MATRICES OPERATIONS SNRM	Returns the spectral norm of a matrix. The spectral norm of a matrix is equal to the largest singular value of the matrix. Same as ABS for a vector.
↰ MATRICES OPERATIONS RNRM	Returns the row norm of a matrix. The row norm of a matrix is the maximum value (over all rows) of the sums of the absolute values of all elements in a row. The row norm of a vector is the maximum absolute value of its elements.
↰ MATRICES OPERATIONS CNRM	Returns the column norm of a matrix. The column norm of a matrix is the maximum value (over all columns) of the sums of the absolute values of all elements in a column. The column norm of a vector is the sum of the absolute values of its elements.

Keys (Continued)	Description
↰ MATRICES OPERATIONS SRAD	Returns the spectral radius of a square matrix. The spectral radius is the absolute value of the largest eigenvalue of the matrix.
↰ MATRICES OPERATIONS COND	Returns the column-norm condition number of a square matrix. The condition number is defined to be the product of the column norm of a square matrix and the column norm of its inverse.
↰ MATRICES OPERATIONS RANK	Returns an estimate of the rank of a matrix. The rank of a matrix is equal to the number of non-zero singular values of the matrix. If flag –54 is clear (default), RANK treats any computed singular value less than 10^{-14} times the size of the largest computed singular value as zero. If flag –54 is set, RANK counts all non-zero singular values no matter what their size.
↰ MATRICES OPERATIONS DET	Returns the determinant of a square matrix. DET checks flag –54, and will refine its computed value only if –54 is clear (default).
↰ MATRICES OPERATIONS TRACE	Returns the trace of a square matrix. The trace of a matrix is equal to the sum of the diagonal elements and also equal to the sum of the eigenvalues of the matrix.

Transforming matrices

To transpose a matrix

1. Select the appropriate Transpose Matrix command:
   - ↰ (MTH) MATRIX MAKE TRN (if you want the conjugate transpose of a complex matrix), or
   - ↰ (MATRICES) OPERATIONS TRAN (if you want transposition without conjugation).
2. Enter or select the array you want to transpose.
3. Press (ENTER) to transpose the matrix.

   The first row of the original matrix is now the first column, the original second row is now the second column and so on.

To invert a square matrix

1. Press (1/x).
2. Enter or select the array you want to invert.
3. Press (ENTER) to transpose the matrix.

To change the dimensions of an array

1. Select the Redimension Array command.
   ↰ (MATRICES) CREATE RDM
2. Enter or select the array that you want to redimension.
3. Enter a list containing the new dimensions of the array:
   { *row*, *column* }.
4. Press (ENTER).

   Elements from the original array are placed *in row-major order* into the newly-dimensioned array. If there are fewer elements in the new array than in the original one, the excess elements are dropped. If there are more elements in the new array than in the original one, the missing elements are filled with zeros (or (0,0) if the array is complex).

For example, RDM($\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, {3,4}) results in: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

More matrix arithmetic

Simple matrix arithmetic is covered in chapter 8 of the HP 49G *User's Guide*. This section covers some of the other arithmetic options.

To change the sign of each element in a matrix

1. Press +/- ( ).
2. With the cursor between the parentheses, enter or select the matrix.
3. Press ENTER.

To multiply a matrix and vector

1. Enter or select the matrix.
2. Press ×.
3. Enter or select the vector.

   The number of elements in the vector must equal the number of columns in the matrix.
4. Press ENTER.

To divide an array by a square matrix

1. Enter or select the array.
2. Press ÷.
3. Enter the square matrix.

   The number of rows in the matrix must equal the number of rows in the array.
4. Press ENTER.

Transforming complex matrices

To combine two real matrices into a complex matrix

1. Select the Real-to-Complex command.
   ◁ (MTH) COMPLEX R→C
2. Enter or select the real matrix that will become the real part of the complex matrix.
3. Enter or select the real matrix that will become the imaginary part of the complex matrix.
   This matrix must have the same dimensions as the matrix entered at step 2.
4. Press (ENTER).
   The two real matrices are combined to form a complex matrix.

To split a complex matrix into two real matrices

1. Select the Complex-to-Real command.
   ◁ (MTH) COMPLEX C→R
2. Enter or select the complex matrix that you want to split.
3. Press (ENTER).
   The result is two real matrices formed from the complex matrix.

To conjugate each element of a complex matrix

1. Select the Conjugate function.
   ▷ COMPLEX CONJ
2. Enter or select the complex matrix that you want to conjugate.
3. Press (ENTER).

To extract the matrix of real parts from a complex matrix

1. Select the Real Part function.
   ▷ COMPLEX RE
2. Enter or select the complex matrix whose real components you want to extract.
3. Press (ENTER).
   The result is a matrix comprising just the real components of the complex matrix.

To extract the matrix of imaginary parts from a complex matrix

1. Select the Imaginary Part function.
   ↱ COMPLEX IM
2. Enter or select the complex matrix whose imaginary components you want to extract.
3. Press ENTER.
   The result is a matrix comprising just the imaginary components of the complex matrix.

Linear algebra topics

The use of matrix functions to solve systems of linear equations is covered in chapter 8 of the HP 49G *User's Guide*. This section covers other important linear algebra commands.

Eigenvalues and eigenvectors

A square $(n \times n)$ matrix $\mathbf{A}$ is said to have an *eigenvalue* λ and a corresponding *eigenvector* $\mathbf{x}$ if $\mathbf{Ax} = \lambda\mathbf{x}$.

Eigenvalues are the roots of the *characteristic equation*—$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$—which is a polynomial of degree n. Thus, $\mathbf{A}$ has n eigenvalues, although they are not always distinct. Each eigenvalue has a corresponding eigenvector.

The HP 49G allows you to compute either the eigenvalues only (a faster computation) or both the eigenvalues and their corresponding eigenvectors.

To compute the eigenvalues for a square matrix

1. Select the Eigenvalues command.
   ↰ MATRICES EIGENVECTOR EGVL
2. Enter or select the square $(n \times n)$ matrix whose eigenvalues you want to calculate.
3. Press ENTER.
   The result is a vector of n eigenvalues.

To compute the eigenvalues and eigenvectors for a square matrix

1. Select the Eigenvalues and Eigenvectors command.
   ↰ (MATRICES) EIGENVECTOR EGV
2. Enter or select the square ($n \times n$) matrix whose eigenvalues and eigenvectors you want to calculate.
3. Press (ENTER).
   The result is an $n \times n$ matrix of eigenvectors and an n-element vector of eigenvalues.

To compute the singular values of a matrix

1. Select the Singular Values command.
   ↰ (MATRICES) FACTORIZATION SVL
2. Enter or select the matrix.
3. Press (ENTER).
   The result is a vector—of length MIN(m,n)—of the singular values of the matrix. The values are returned in non-increasing order.

To decompose or factor a matrix

The HP 49G offers a set of matrix decomposition and factorization tools that you can use either alone or in program routines to solve specialized problems. These tools are explained in the following table:

Keys	Description
↰ (MATRICES) FACTORIZATION LU	**Crout LU Decomposition.** This procedure is used in the process of solving an exactly-determined system of linear equations, inverting a matrix, and computing the determinant of a square matrix. It factors the square matrix (**A**) into a lower-triangular matrix **L**, an upper-triangular matrix **U** containing ones on its diagonal, and a permutation matrix **P** such that **PA** = **LU**.
↰ (MATRICES) FACTORIZATION LQ	**LQ Factorization.** This command factors an $m \times n$ matrix **A** into an $m \times n$ lower trapezoidal matrix **L**, an $n \times n$ orthogonal matrix **Q**, and an $m \times m$ permutation matrix **P** such that **PA** = **LQ**.
↰ (MATRICES) FACTORIZATION QR	**QR Factorization.** This command factors an $m \times n$ matrix **A** into an $m \times m$ orthogonal matrix **Q**, an $m \times n$ upper trapezoidal matrix **R**, and an $n \times n$ permutation matrix **P** such that **AP** = **QR**.
↰ (MATRICES) FACTORIZATION SCHUR	**Schur Decomposition.** This command factors a square matrix **A** into an orthogonal matrix **Q** and an upper-triangular matrix (or, if **A** is real-valued, the upper quasi-triangular matrix) **U** such that $\mathbf{A} = \mathbf{QUQ}^{\mathbf{T}}$ (where $\mathbf{Q}^{\mathbf{T}}$ is the transpose of matrix **Q**).

Keys (Continued)	Description
↰ (MATRICES) FACTORIZATION SVD	**Singular Value Decomposition.** This command factors an $m \times n$ matrix **A** into an $m \times m$ orthogonal matrix **U**, an $n \times n$ orthogonal matrix **V**, and a vector **S** of the singular values of **A** such that **A** = **US‘V** (where **S‘** is the $m \times n$ matrix formed by using the elements of **S** as its diagonal elements).