Standard Maps 3.1

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What you need

Standard Maps has been compiled for Macintoshes with an FPU, and assumes you are using System 6.0 or later. If you have color, then it will run in full color (I have tested it only for 8 bit); however, it will also run on an old Mac with black and white only. Otherwise, as far as I know, this program will run on any Mac. If you would like a version that does not require the FPU, you can find one on newton.colorado.edu, in /pub/dynamics/programs for anonymous ftp.

For help

There are two "help" facilities in the program. The first is <u>Balloon help</u>. Turn this on if you have System 7 to see what each menu item does. Additionally, if you turn on "Verbose Text" by checking the menu item, I attempt to give you more information about what is happening.

What it does

Standard Maps displays the dynamics of several "area preserving" mappings. It will also find periodic orbits, cantori, and stable and unstable manifolds of any periodic orbit. A basic reference on all this is (Meiss, 1992). You can also consult a number of textbooks, such as (Lichtenberg and Lieberman, 1982), (Arrowsmith and Place, 1990), and (MacKay and Meiss, 1987).

The standard map itself is

$$T: \begin{array}{l} y' = y - \dfrac{k}{2\pi}\sin(2\pi x) \\ x' = x + y' \end{array}$$

Since the map is periodic in x with period one, we wrap x into the interval (-.5,.5). Thus the standard map is really a map on the cylinder. However, for many purposes we do need to keep track of x on the real line. Though you probably don't care, in the program "x" is a structure where x.f is the fractional part and x.i is the integer part.

To iterate a map, simply click on the initial point. Click again to change initial conditions.

To find a periodic orbit, you must choose a "frequency" (p,q), where p and q are integers. This means that the orbit rotates "p" times around in "q" iterations. "Around" for the standard map means that x has increased by one, so a (p,q) orbit satisfies

$$T^q(x,y) = (x+p,y)$$

The frequency could also be written p/q.

For the Hénòn map, we measure "around" by counting how many times the point rotates around the fixed point. This leads to a problem that we haven't quite resolved satisfactorily. See the bugs section.

You can find orbits with irrational frequency ω by looking at periodic orbits that limit on that frequency, $p/q \rightarrow \omega$.

Cantori and all that

A cantorus is the remnant of an invariant circle. For an integrable, or slightly perturbed integrable mapping (e.g. k small for the standard map), the quasiperiodic orbits with "sufficiently" irrational frequencies cover a circle, forming invariant circles (i.e. the KAM theorem). When you increase k the invariant circles are destroyed, but there are still quasiperiodic obits—they just cover a Cantor set instead of a circle.

To find such a cantorus, set k = 1.0 for the standard map, and select the "Farey Path" menu. What then happens is that you are asked to select a pair of orbits. The default is (0,1) and (1,1). These orbits form the head of a Farey tree. Then you are asked to select a Farey path, the default being "LRLRLR...". This path leads to the frequency, $\omega = 1/\gamma^2$, where $\gamma = (1 + \sqrt{5})/2$, the golden mean. This happens to be the "most irrational" frequency for the standard map, that is, its invariant circle lasts the longest, up to k = 0.97163540631.

Any Farey path that never terminates, and doesn't eventually become all L's or all R's, corresponds to an irrational, and thus either an invariant circle or cantorus. Farey paths that are eventually "LLLLL..." or "RRRR...." correspond to homoclinic orbits to periodic points.

Reversibility

Each of the maps in the program is reversible. This means that there is an involution S that inverts the map: $STS = T^{-1}$, where S is an involution ($S^2 = I$), that reverses orientation. A good example is simply $S(x,y) = (-x,y)$. The implications of this for our purposes are that periodic orbits can be found by a one-dimensional, instead of a two-dimensional search. In general S will have a fixed set which is a curve (we call it a symmetry line, though it might not be straight). Periodic orbits that have points on the fixed set are called "symmetric". The program only finds symmetric periodic orbits.

Actually, associated with any symmetry $S = S_1$, there is a family of symmetries The first member is $S_2 = TS$. This is also reversor for T. Additionally, since maps on the cylinder have a translation symmetry $R(x,y) = (x-1,y)$, there are other symmetries $S_3 = SR$, and $S_4 = TSR$. The fixed lines of these four symmetries allow us to find periodic orbits of different kinds.

There are two primary kinds of periodic orbits, stable (elliptic) or minimax orbits and unstable (hyperbolic) minimizing orbits. For more info on this see (Meiss, 1992). These orbits are found by searching for an orbit that starts on different symmetry sets. The program, can automatically choose which symmetry line is appropriate for minimizing and minimax orbits, though this doesn't work completely yet (for example we demand that k be positive for the standard map).

For the standard map, we set

$$
S : \begin{cases} y' = y - \dfrac{k}{2\pi}\sigma\iota\nu(2\pi\xi) \\ \xi' = -\xi \end{cases}
$$

The fixed set of S is Fix(S) = {x = 0}. For the other symmetries we have

$$
\begin{aligned}
\mathrm{Fix}(S) &= \{x=0\} \\
\mathrm{Fix}(SR) &= \left\{x = \tfrac{1}{2}\right\} \\
\mathrm{Fix}(TS) &= \{y = 2x\} \\
\mathrm{Fix}(TSR) &= \{y = 2x + 1\}
\end{aligned}
$$

It turns out that when $k > 0$ the minimax orbits (they are elliptic for small k) have points on Fix(S), and the minimizing orbits have points on Fix(TS) for q even and Fix(SR) for q odd.

Another simplification is that symmetric orbits have points on two of the symmetry lines, and so one can find the orbits by iterating for half the period (q). This is what we do.

For more information on symmetry see (Devaney, 1976; Kook and Meiss, 1989; Quispel and Roberts, 1988; Sevryuk, 1986).

Known Problems (might be called bugs)

1) Overly sensitive to initial guesses for the Hénòn map. Thus we sometimes are unable to find a p.o. even though it is not too unstable. One problem is that the fixed points are solutions of our zero finder.

2) If you search for a p.o. for Hénòn that ends on Fix(S), the program fails. This is because we increment x.i upon crossing Fix(S) = {x=y}, giving a discontinuity that the secant method doesn't like. You can always start the orbit on Fix(S) though.

3) The choice of symmetry line for the "minimizing" and "minimax" orbits is not always correct. E.g. the standard map for $k < 0$, and the two frequency map when $k_2 \geq k_1$.

4) Choice of palletes for the off screen pixmap are wrong sometimes. This is true if you are running another application that mucks with the palette. I think I'll change the offscreen routines to GWorlds. Probably I just have to worry about setting the Device properly.

5) Text window doesn't scroll when user drags mouse outside of window. Delete key doesn't delete text. Text window is not always redrawn properly (when a modal dialog is cancelled).

Versions

3.0.1 Couldn't choose the two-harmonic map. Now fixed.
3.0.2 Minor cleaning up
3.0.3 Cleaned up SetValues code—new dialogs, made common libraries
3.0.4 Added "Standard NonTwist Map".
Moved labels on printed graphs.
3.0.5 Added support for printing and redrawing stable manifolds.
3.0.6 Redid background drawing routines.
Fixed Print error on cancel. Renamed Print menus.

3.0.7 Added McMillan Map
Reorganized resouces for help and balloons
Fixed spacing for points on stable manifolds
3.0.8 Added "Choose Initial Condtion" command and reorganized logic of "Iterate"
3.0.9 Fixed bug in Paste Bitmap when faster graphics on
Added Position Window.
Choose Niterates based on 10 tick response time instead of arbitrarily
Made TextEdit copy & paste to/from clipboard
3.1 Added Default-outline to Stop Button

Latest Version

The latest version of this program can always be found on "newton.colorado.edu". Use anonymous ftp, in the directory /pub/dynamics/programs. There is also a NOFPU version of this program in that directory.

References

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Devaney, R. (1976), "Reversible Diffeomorphisms and Flows," Trans. Am. Math. Soc. **218**: 89-113.

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