Coordinates2.3a Read Me

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A UTILITY APPLICATION FOR THOSE MEASURING THE EARTH OR OTHER PLANETS

FREEWARE

OUTLINE

INTRODUCTION

Coordinates2.3a is a Macintosh application which allows the user to convert coordinates from geodetic to Cartesian coordinate systems, and vice-versa. It will also compute distances along the ellipsoidal surface of the Earth and baseline distances as well as chord distances. All of these concepts are defined and described in this Read-Me file.

The program was written in FutureBasic™ (from Zedcor, Inc. of Tucson, AZ) and was benchmarked on a Mac 660AV and an older Mac II (w/ Daystar 50 MHz PowerCache w/FPU). It has not been tested on PowerPC platforms, but presumably will work in 68k emulation mode.

What's New in Version 2.3a?

About a week after the first public release of Coordinates2.3a, a user (thanks, Traci!) suggested that a useful capability would be to compute the coordinates of a distant point given the geodesic distance and azimuth (bearing). Never declining a good challenge, one could squeeze the task buttons a little closer together and write a new local function that would make the calculation. While making the modifications, it became apparent that handling routine events in the interface could be done a little better (i.e., the author is still climbing the learning curve!). A better way of handling window refreshes was made by moving all the drawing of the text and graphics into their own local function. The keyboard entry was beefed-up to allow the tab key to control movement between edit fields and the enter key to activate the outlined buttons, in the same way that one can in "mainstream" applications. Also, colors were added to the user-addressable edit fields (but refreshing them brings them back in B/W (The author will figure that one out someday!). A new chapter in this read me file explains how the coordinates are calculated given the location of a starting point and a distance and azimuth to a second point.

The remainder of this file describes the definitions, theory and source material from which I worked to create the application.

BACKGROUND AND HISTORY

As we are all well aware, we live on a (basically) round planet (although, believe it or not, chapters of the *Flat Earth Society* still exist!). For some scientists, engineers and surveyors, the exact shape of the planet is significant in their calculations. The study of the shape of planets falls under the discipline of *Geodesy*, which itself falls under the realm of Geophysics–a branch of Earth Science. The history of Geodesy stretches all the way back to the early Egyptians and Aztecs, where it mixed intimately with astronomical observations. Of course, the classic tale that we all learned in primary school was the story of Eratosthenes, who in about 220 BC, noticed that on the longest day of the year, the Sun shone straight down a water well in the city of Syene, along the Nile (near present day Aswan). A year or two later, he noticed, again, on the longest day of the year, in the city of Alexandria (which is north of Syene) that a tall pole in the center of city still cast a shadow at mid-day. Using the Sun as a point source, he deduced that the only explanation for these occurrences was that the surface of the Earth must be curved. After riding repeatedly between Alexandria and Syene, he measured the distance between the two cities and then was able to calculate the diameter of the Earth. He came up with a value that was about 16% too large, but the fact remains that he deduced the round shape of the Earth 1800 years before Magellan sailed around the world, thereby proving the roundness of Earth!

Much can be found in the literature about the early history of the knowledge of the Earth's shape, which, in the earliest musings of scientific thinkers, was intertwined with Man's understanding of the cosmos and his developing cosmological model. For those seriously interested, I recommend *Lindberg* [1992] and *Grant* [1994] among others. Of the same ilk, but more on the popular side are: *Koestler* [1959], *Ferris* [1988] and *Harrison* [1986]. Later, through the Renaissance, Geodesy got intermixed with cartography and mapping, see, e.g., *Wilford* [1981] and *Hale* [1994]. In the 20th century, Geodesy entered the space age with the launching of Sputnik. Its little transmitter, beeping its way around the world, gave scientists a chance to measure its Doppler frequency shift and hence derive certain basic parameters regarding the path it followed and to deduce the flattening of the Earth. Of course, the flattening had been theorized and detected long before the space age.

DEFINITIONS AND GENERAL THEORY

Much of the material given here can be found in any good volume dealing with Geodesy, such as *Bomford* [1980] or *Vanicek & Krakiwsky* [1986].

The Ellipsoidal Shape of the Earth

The simplest geometric description for the shape of the Earth is that of a sphere. However, a better approximation is that described by an oblate ellipsoid of revolution (which is sometimes called the spheroid). As shown in cross section in Figure 1, its equatorial radius (the *semi-major axis*, denoted by the letter "*a*") is larger than its polar axis (*semi-minor axis*, *b*). For the Earth, the semi-major axis value adopted by the International Earth Rotation Service in 1992 (IERS-92) is *a* = 6378136.3 m. The ratio of the difference between the two axes taken with respect to the semi-major axis is called the flattening, which is given by:

$$f = \frac{(\alpha - \beta)}{\alpha} = 1 - \frac{\beta}{\alpha} \qquad (1)$$

Typically, the flattening is expressed as its reciprocal, which for IERS-92 is

$$1/f = 298.257 \qquad (2)$$

This means that the Earth's semi-minor axis is about three tenths of one percent smaller than its semi-major axis. Its not all that different, but for those who need to know the relative distance between points on the Earth's surface to an accuracy of a hundred meters or better, the flattening must be accounted for. My personal need was to distinguish changes in relative distances between points separated by thousands of kilometers in order to detect and monitor tectonic motion (see *Smith et al.* [1990]).

Another parameter commonly associated with ellipses and ellipsoids is called the *first eccentricity*. It is the distance between the center of the ellipse and one of it's foci divided by the semi-major axis:

$$e = \frac{(\alpha^2 - \beta^2)^{1/2}}{\alpha} \qquad (3a)$$

It is more common to use the square of the first eccentricity:

$$e^2 = \frac{(\alpha^2 - \beta^2)}{\alpha^2} = 1 - \frac{\beta^2}{\alpha^2} = 2\phi - \phi^2 \qquad (3b)$$

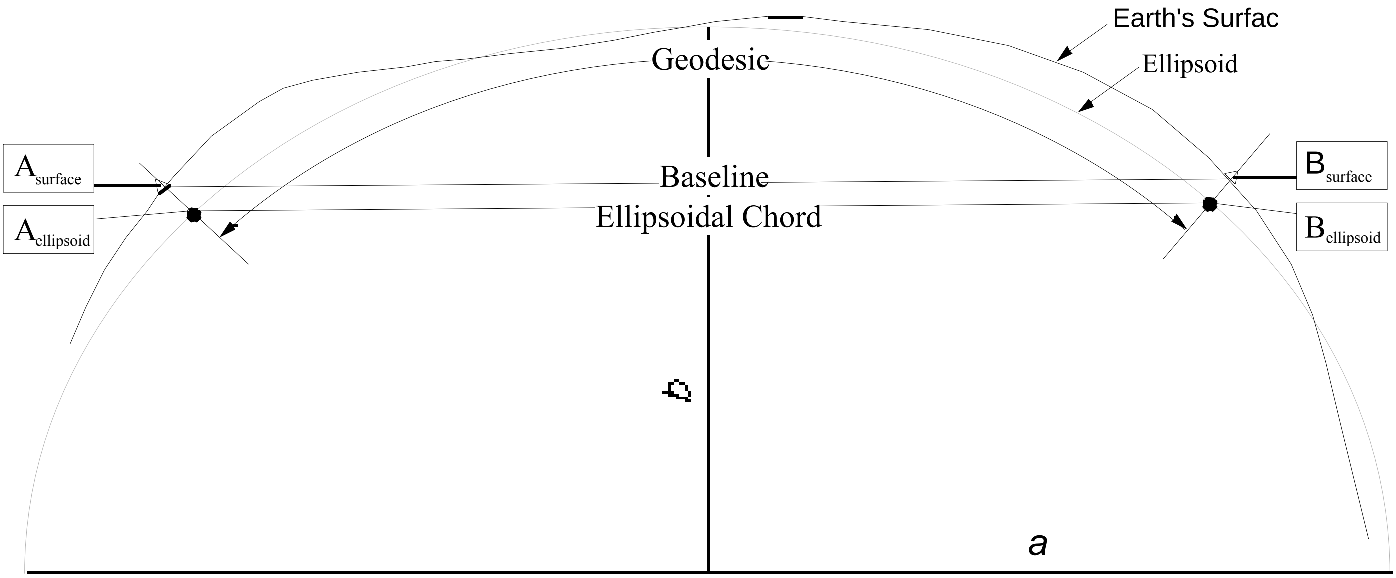

Figure 1. Cross-sectional view of the Earth ellipsoid demonstrating the terms geodesic, baseline and ellipsoidal chord. The case of the geodesic passing through one of the poles only occurs when points A and B lie on complementary longitudes. The Earth's flattening has been greatly exaggerated in this figure.

As seen in (3b), the eccentricity can be calculated on the basis of the flattening, thus only the semi-major axis and the flattening are the fundamental quantities from which most further calculations can be made.

Coordinate Systems

The simplest coordinate system is the Cartesian coordinate system defined by a triad of numbers (usually denoted as X, Y, and Z or X_1, X_2, and X_3). A location in three-dimensional space can be assigned coordinates and relative distances and directions to other points in the same space can be easily determined. However, since we are creatures that exist on the outer surface of an ellipsoid, our minds are more receptive to the notion of distances along the surface rather than through a chunk of the Earth. For really short distances it doesn't matter, that's why regional surveyors rely on principles of plane trigonometry. Many short triangulation chains and level lines (say, < 20 km) do not require any special treatment to account for the curved surface of the Earth, a planar approximation is usually sufficient. Longer triangulation chains and level lines, sometimes spanning continents, require a "mapping", one way or another, onto a curved surface.

The natural coordinate system for a spherical planet would be to use spherical latitude and longitude, with angles being measured at the center of the Earth, as its origin. This could be used for the flattened Earth as well, but the system has difficulties in that the line connecting the surface point to the center of the Earth does not intersect the ellipsoidal surface at right angles, which causes problems if one is also measuring height. Optimally, surveyors and geodesists like to keep the local horizontal and vertical orthogonal to one another–it just makes for a cleaner way to treat the local coordinate frame. So, on an ellipsoidal Earth, *geodetic latitude* and *longitude* have been defined (see Figure 2). Of course, geodetic longitude is equivalent to spherical longitude since the Earth's Equator is a circle (as are lines of equal

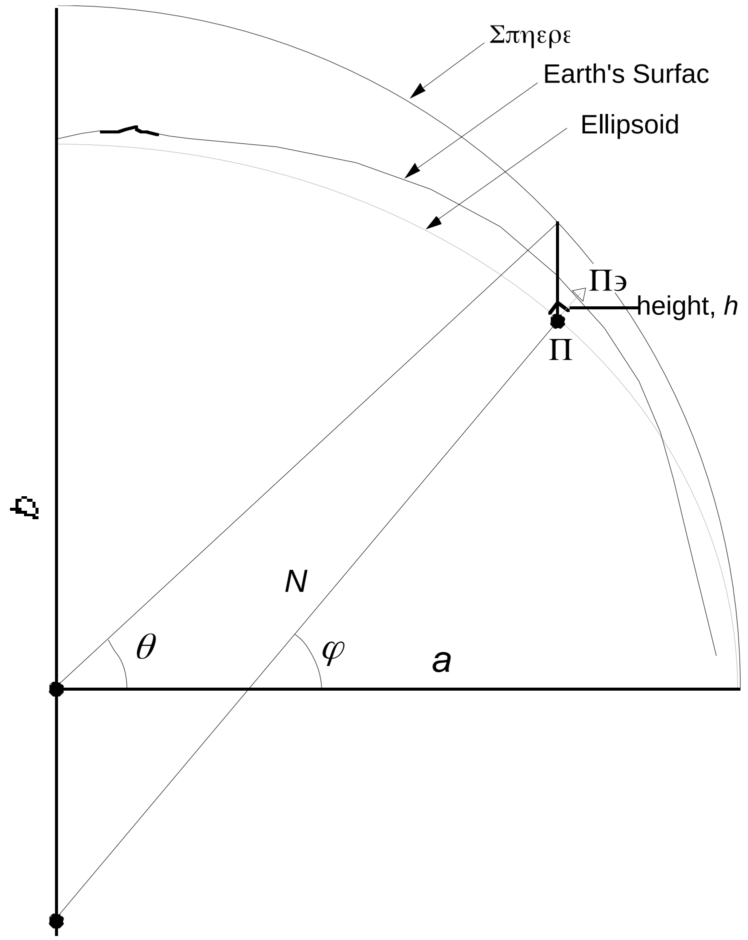

Figure 2. Definition of latitudes. The geodetic latitude is denoted φ, and is the angle between the equatorial plane and the line orthogonal to the ellipsoid. The reduced latitude is denoted θ. *N* denotes the radius of curvature in the prime meridian and is measured from the Z-axis to the point P on the ellipsoid. The height *h*, gives the distance from P to P', on the Earth's surface.

latitude). The geodetic latitude is related to the spherical latitude, sometimes called *reduced* latitude by the relation

$$\tan\varphi = \frac{\tau\alpha\nu\theta}{(1-\varepsilon^2)^{1/2}} = \frac{\alpha}{\beta}\tau\alpha\nu\theta \qquad (4)$$

Thus, by using geodetic latitude, longitude and the height above the ellipsoid, we gain a convenient orthogonal triad of coordinates by which each point in and around the Earth can be located in an intuitive way. One of the purposes of Coordinates2.3a is to make the transformation between this and the Cartesian coordinate system.

Lines upon the Ellipsoid: The Geodesic

The shortest line which lies on a sphere connecting two points on its surface is defined to be a *great circle*. For example, on a sphere, the shortest path between New York and Istanbul (which are almost at the same latitude) is <u>not</u> via the latitude line connecting the two cities, but <u>is</u> along a line which passes just south of Iceland. One can easily prove this with a piece of string and a globe. We can easily visualize these great circles and understand that they define a plane which passes through the center of the sphere.

On an ellipsoidal Earth, the shortest line between two points is no longer a great circle. It is a strange geometric concoction (from differential geometry), called a *geodesic*. A geodesic line between two points does not lie in the plane defined by the two end points and the Earth's center. Weird, huh? That is why computing geodesic distances and azimuths is such a chore. We no longer can depend on the simple relationships from spherical trigonometry. The details of all the calculations are given below.

THE ALGORITHMS

Geodetic Coordinates to Cartesian Coordinates

The transformation from geodetic coordinates to Cartesian coordinates is straight forward. We are given, for a point on or near the Earth's surface the following:

φ	geodetic latitude
λ	longitude
h	height above the ellipsoid

we desire to find the corresponding *X*, *Y*, and *Z*. These are found via the closed formulas:

$$X = (N + h)\cos\varphi\cos\lambda$$

$$Y = (N + h)\cos\varphi\sin\lambda \qquad (5)$$

$$Z = \left[N(1 - e^2) + h\right]\sin\varphi$$

where *N* is the radius of curvature in the prime vertical (Figure 2), given by

$$N = \frac{a}{\sqrt{1 - e^2\sin^2\varphi}} \qquad (6)$$

Cartesian Coordinates to Geodetic Coordinates

The computation from Cartesian to geodetic coordinates is much more complicated since equations (5) can not be easily inverted. Through the years, several iterative algorithms have been used and more recently, some very good closed ones have been derived. The one I've used for many years is the one by *Bowring* [1976]. The computation goes as follows.

First, an approximation for the reduced latitude is made via

$$\tan\theta = \left(\frac{Z}{\sqrt{X^2 + Y^2}}\right)\left(\frac{a}{b}\right) = \left(\frac{Z}{\sqrt{X^2 + Y^2}}\right)\left(\frac{1}{1 - f}\right) \qquad (7a)$$

from this, using some trig identities we can compute

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} \tag{7b}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} \tag{7c}$$

where the sign of the $\sin\theta$ is the same as the sign of $\tan\theta$. These equations are used to calculate the geodetic latitude from

$$\tan\varphi = \frac{Z + \varepsilon'^2\, b \sin^3\theta}{\sqrt{X^2+Y^2} - \varepsilon^2\, a \cos^3\theta} = \frac{Z + \left[\left(a^2-b^2\right)/\,b\right]\sin^3\theta}{\sqrt{X^2+Y^2} - \left[\left(a^2-b^2\right)/\,a\right]\cos^3\theta} \tag{7d}$$

The longitude is calculated simply by

$$\lambda = \tan^{-1}\left(\frac{Y}{X}\right) \tag{7e}$$

and the height is computed from

$$h = \frac{\sqrt{X^2+Y^2}}{\cos\varphi} - N \tag{7f}$$

where N is given from equation (6) and the computation of λ are made using a function similar to a **ATAN2** function from FORTRAN. Bowring notes that this algorithm should provide accurate latitudes to within 0.00000003" (< 1 μm) for points near the Earth's surface (i.e. between -5000m and +10000m).

Geodesic Length Computations

As mentioned previously, the computation of the geodesic length and its associated azimuths is no trivial exercise. I have used the iterative algorithm devised by *Vincenty* [1975]. *Bowring* [1986] has devised a closed formulation, but I've not yet explored this algorithm, but hope to do so in the near future. A description of the Vincenty algorithm follows.

In this problem (known as the inverse problem, the direct problem is treated in the next chapter), we are given the latitudes (φ_1, φ_2) and longitudes (λ_1, λ_2) of two points on the Earth's surface. Height is irrelevant in the computation for geodesic lines on the ellipsoid. We assume that the point on the Earth's surface is reduced to the ellipsoidal surface by simply setting the height to zero.

First, we the difference the longitudes as a first approximation for the quantity $\delta\lambda$:

$$\delta\lambda = \lambda_2 - \lambda_1 \tag{8a}$$

The iteration begins with the computation of intermediate values σ, α and σ_m by calculating the sin and cos terms, then using the **ATAN2** function mentioned above. Also used, are the reduced latitudes, θ_1, θ_2, which can be found by inverting equation (4).

$$\sin^2\sigma = (\cos\theta_2 \sin\delta\lambda)^2 + (\cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2 \cos\delta\lambda)^2 \tag{8b}$$

$$\cos\sigma = \sin\theta_1 \sin\theta_2 - \cos\theta_1 \cos\theta_2 \cos\delta\lambda \tag{8c}$$

$$\tan\sigma = \frac{\sin\sigma}{\cos\sigma} \tag{8d}$$

and

$$\sin\alpha = \frac{\cos\theta_1 \cos\theta_2 \sin\delta\lambda}{\sin\sigma} \tag{8e}$$

$$\cos 2\sigma_\mu = \cos\sigma - \frac{2\sin\theta_1 \sin\theta_2}{\cos^2\alpha} \tag{8f}$$

These quantities are used in the following two equations to determine a new value of $\delta\lambda$, and make another iteration starting with equation (8b) until the value $\delta\lambda$ reaches some convergence level (I used 10^{-20}, or 20 iterations, whichever comes first):

$$C = \frac{\phi}{16}\cos^2\alpha\left[4 + \phi(4 - 3\cos^2\alpha)\right] \tag{8g}$$

$$L = \delta\lambda - (1 - X)\phi\sin\alpha\left\{\sigma + X\sin\sigma\left[\cos 2\sigma_\mu + X\cos\sigma(-1 + 2\cos^2 2\sigma_\mu)\right]\right\} \tag{8h}$$

then $\delta\lambda = L$ and, as mentioned, the iteration begins again at equation (8b). Upon convergence, then the following expressions are evaluated to obtain the geodesic distance, *s*, and the azimuth from site 1 to site 2, α_1, and the azimuth from site 2 to site 1, α_2:

$$s = \beta A(\sigma - \Delta\sigma) \tag{9a}$$

$$\tan\alpha_1 = \frac{\cos\theta_2 \sin\delta\lambda}{\cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2 \cos\delta\lambda} \tag{9b}$$

$$\tan\alpha_2 = \frac{\cos\theta_1 \sin\delta\lambda}{-\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2 \cos\delta\lambda} \tag{9c}$$

where

$$A = 1 + \frac{\upsilon^2}{16384}\left\{4096 + \upsilon^2\left[-768 + \upsilon^2(320 - 175\upsilon^2)\right]\right\} \tag{10a}$$

$$B = \frac{\upsilon^2}{1024}\left\{256 + \upsilon^2\left[-128 + \upsilon^2(74 - 47\upsilon^2)\right]\right\} \tag{10b}$$

$$u^2 = \frac{a^2 - b^2}{b^2}\cos^2\alpha \qquad (10c)$$

$$\Delta\sigma = B\sin\sigma\left\{\cos 2\sigma_m + \frac{1}{4}B\left[\cos\sigma(-1+2\cos^2 2\sigma_m) - \frac{1}{6}B\cos 2\sigma_m(-3+4\sin^2\sigma)(-3+4\cos^2 2\sigma_m)\right]\right\} \qquad (10d)$$

So, now I'm sure you can appreciate why this isn't so easy!

Coordinates of Second Point, Given Distance and Bearing

We now will treat the direct problem of long lines on the ellipsoid. In this case we are given the location (φ_1,λ_1) of a point and the geodesic length (s) and bearing (or azimuth, α_1) to a second point. We want to compute the latitude and longitude (φ_2,λ_2) for the second point. Again, we follow *Vincenty* [1975] (borrowing equations from above, when needed) and use an iterative approach, which should yield coordinates with an accuracy of 0.00005" of arc.

The computations begin with a computation for the angular distance on the sphere from the equator to the first point (σ_1). Using the reduced latitude, θ_1, from equation (4) we write

$$\tan\sigma_1 = \tan\theta_1/\cos\alpha_1 \qquad (11)$$

and the azimuth of the geodesic at the equator

$$\sin\alpha = \cos\theta_1\sin\alpha_1 \qquad (12)$$

Next, some auxiliary quantities are computed

$$u^2 = \frac{a^2 - b^2}{b^2}\cos^2\alpha \qquad (13a)$$

$$A = 1 + \frac{u^2}{16384}\{4096 + u^2[-768 + u^2(320 - 175u^2)]\} \qquad (13b)$$

$$B = \frac{u^2}{1024}\{256 + u^2[-128 + u^2(74 - 47u^2)]\} \qquad (13c)$$

Now we begin an iterative loop on σ, the angular distance between the two points on the sphere, until the change between iterations becomes negligible (a criteria of 1.E-10 or 20 iterations was used in the program). To start the iterations, a value of $\sigma = s/(bA)$ is used. Then

$$2\sigma_m = 2\sigma_1 + \sigma \qquad (14)$$

$$\Delta\sigma = B\sin\sigma\left\{\cos 2\sigma_m + \frac{1}{4}B\left[\cos\sigma(-1+2\cos^2 2\sigma_m)\right.\right.$$

$$-\frac{1}{6} B \cos 2\sigma_{\mu} (-3 + 4\sin^2 \sigma)(-3 + 4\cos^2 2\sigma_{\mu})\Big]\Big\} \qquad (15)$$

$$\sigma = \frac{s}{bA} + \Delta\sigma \qquad (16)$$

and the sequence goes back to equation(14) until it reaches convergence. Now, the latitude for the second point can be computed from

$$\tan \varphi_2 = \frac{\sin\theta_1 \cos\sigma + \cos\theta_1 \sin\sigma \cos\alpha_1}{(1-f)\left[\sin^2\alpha + (\sin\theta_1 \sin\sigma - \sin\theta_1 \sin\sigma \cos\alpha)^2\right]^{1/2}} \qquad (17)$$

The difference in longitude on the auxiliary sphere, $\delta\lambda_S$ is given by

$$\tan \delta\lambda_s = \frac{\sin\sigma \sin\alpha_1}{\cos\theta_1 \cos\sigma - \sin\theta_1 \sin\sigma \cos\alpha_1} \qquad (18)$$

Now using *C* from equation (8g) and other quantities computed here, we insert all these into equation (8h) to get the difference in longitude on the ellipsoid. The azimuth from the second point to the first (sometimes called back-azimuth) is

$$\tan \alpha_2 = \frac{\sin\alpha}{-\sin\theta_1 \sin\sigma + \cos\theta_1 \cos\sigma \cos\alpha_1} \qquad (19)$$

Again, for the computation of the inverse tangent function in equations (18) and (19), an **ATAN2**-type function has been used to insure proper quadrant placement. Vincenty notes that an approximation can be made by trimming equation (13b) back a bit, but this was not adopted here. The full expression given in equation (13b) was used.

Baselines and Chords

The rest is a piece of cake! The computation for the baseline distance, *b*, is made in Cartesian space simply by extending the Pythagorean theorem to three-dimensions:

$$b = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \qquad (11)$$

The chord distance uses the same equation, but where the *X*, *Y*, and *Z* terms are computed for the reduced point on the ellipsoid. This is the *ellipsoidal chord*, not the spherical chord.

HOW TO USE COORDINATES2.3a

Once you've grasped the concepts discussed in the section on General Theory, then using the application is pretty much self evident. Upon launching the application, the user is faced with two sets of buttons. The six on the left side of the window define the task to be performed and the five on the right side define the ellipsoid to be used (or specified). If you forget to choose an ellipsoid, and one is needed for the task you've selected, then a reminder window will ask you to select an ellipsoid. The steps are quite simple.

Step 1: Select which task to perform from the set of six buttons on the left.

Step 2: Choose which ellipsoid to use (not needed for baseline calculations if you are working in Cartesian coordinates).

Step 3: Enter in coordinates into appropriate fields (the ones with a color outline). Be sure to select the appropriate units (e.g., decimal degrees, degrees-minutes-seconds or X, Y, Z)

Step 4: Click the "Calculate" button and read answers in appropriate fields.

Step 5: To quit the program, either click on the "Quit" button or select "Quit" from the File menu.

That's all there is to it!

REACHING THE AUTHOR OF THIS APPLICATION

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Please notify him if you discover bugs or suspect incorrect answers.

Legal Junk:

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