TECHNOTE:
Understanding Conic Splines

Conic splines are a useful graphic primitive. They exactly represent any conic section: line, circle, ellipse, parabola, or hyperbola. Lines and circles are of obvious importance, and parabolic splines are a primitive building block for shapes in QuickDraw GX. To maintain closure under the full set of perspective transformations allowed by QuickDraw GX, the full set of conic sections must be used.

This Technote gives a derivation of some of the mathematical formulas associated with conic splines. It defines a quadratic rational spline as a weighted mean of three control points whose weights vary quadratically in the parameter t. A canonical form is derived for the most general form of the weighted mean. Then the effect of perspective transforms on the weights and control points is explained. Finally, a method is derived for determining which conic section contains a given conic spline.

This Technote is directed primarily at developers working with the paths and perspective transforms defined in QuickDraw GX. A firm grasp of those concepts is necessary to understanding this Technote.

Technote 1052 - QuickDraw GX ConicLibrary.c in Detail: Description, and Derivations also addresses the concept of conic splines, and approaches it from a different perspective. See *Inside Macintosh: QuickDraw GX Graphics* and *Inside Macintosh: QuickDraw GX Objects* for further documentation.

Introduction to Quadratic Rational Splines

Definition 1.1: The quadratic rational spline (QRS) with vertices (a,b,c) and weights

(A,B,Γ) is defined as the image of $[0,1]$ under the function

$$\eta(t) = \frac{Aa(1-t)^2 + 2Bbt(1-t) + \Gamma ct^2}{A(1-t)^2 + 2Bt(1-t) + \Gamma t^2} \qquad (1.1)$$

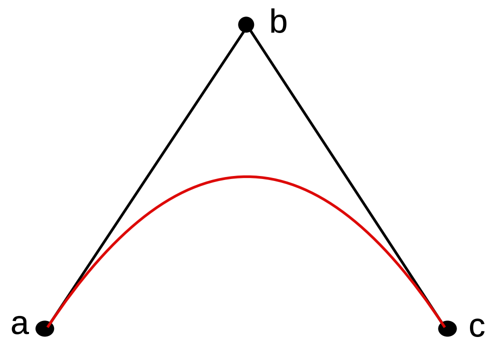
Definition 1.2: A proper QRS (PQRS) is a QRS with positive weights.

Notice that $\eta(t)$ is a linear combination of a, b, and c whose coefficients sum to 1. Thus, applying an affine map to a, b, and c is the same as applying that map to $\eta(t)$.
From (1.1), the following are easily derived:

$$\eta(0) = a \qquad \frac{d}{dt}\eta(0) = \frac{2B}{A}(b-a)$$

$$\eta(1) = c \qquad \frac{d}{dt}\eta(1) = \frac{2B}{\Gamma}(c-b)$$

Figure 1.1 A Quadratic Rational Spline with Vertices (a,b,c)



Lemma 1.1: The PQRS with vertices (a,b,c) and weights (A,B,Γ) is the same as the PQRS with vertices (a,b,c) and weights $(Ar,Br,\Gamma r)$ for $r>0$.
Proof: The PQRS with vertices (a,b,c) and weights (A,B,Γ) is

$$\eta(t) = \frac{Aa(1-t)^2 + 2Bbt(1-t) + \Gamma ct^2}{A(1-t)^2 + 2Bt(1-t) + \Gamma t^2}$$
$$= \frac{Ara(1-t)^2 + 2Brbt(1-t) + \Gamma rct^2}{Ar(1-t)^2 + 2Brt(1-t) + \Gamma rt^2}$$

This is the PQRS with vertices (a,b,c) and weights $(Ar,Br,\Gamma r)$. **QED**

Lemma 1.2: The PQRS with vertices (a,b,c) and weights (A,B,Γ) is the same as the PQRS with vertices (a,b,c) and weights $(A,Br,\Gamma r^2)$ for $r>0$.

Proof: Consider the relation between s and t defined by $\frac{t}{1-t} = r\frac{s}{1-s}$. As t ranges from 0 to 1, s also ranges from 0 to 1.
The PQRS with vertices (a,b,c) and weights (A,B,Γ) is

$$\begin{aligned}
\eta(t) &= \frac{Aa(1-t)^2 + 2Bbt(1-t) + \Gamma ct^2}{A(1-t)^2 + 2Bt(1-t) + \Gamma t^2} \\
&= \frac{Aa + 2Bb\frac{t}{1-t} + \Gamma c\left(\frac{t}{1-t}\right)^2}{A + 2B\frac{t}{1-t} + \Gamma\left(\frac{t}{1-t}\right)^2} \\
&= \frac{Aa + 2Brb\frac{s}{1-s} + \Gamma r^2c\left(\frac{s}{1-s}\right)^2}{A + 2Br\frac{s}{1-s} + \Gamma r^2\left(\frac{s}{1-s}\right)^2} \\
&= \frac{Aa(1-s)^2 + 2Brbs(1-s) + \Gamma r^2cs^2}{A(1-s)^2 + 2Brs(1-s) + \Gamma r^2s^2}
\end{aligned}$$

This is the PQRS with vertices (a,b,c) and weights $(A,Br,\Gamma r^2)$. **QED**

Definition 1.3: The normal QRS (NQRS) with vertices (a,b,c) and weight Λ is the QRS with vertices (a,b,c) and weights $(1,\Lambda,1)$ where $\Lambda>0$.

Lemma 1.3: The PQRS with vertices (a,b,c) and weights (A,B,Γ) is the same as the NQRS with vertices (a,b,c) and weight Λ = **Error!**.

Proof: Given the weights (A,B,Γ), first apply Lemma 1.1 with $r = \frac{1}{A}$, then apply Lemma 1.2 with $r^2 = \frac{A}{\Gamma}$ to get the equivalent weights $(1,\Lambda,1)$ where Λ = **Error!**. **QED**

Notice that the NQRS with vertices (a,b,c) and weight Λ is the image of [0,1] under the function

$$\eta(t) = \frac{a(1-t)^2 + 2\Lambda bt(1-t) + ct^2}{1 + 2(\Lambda-1)t(1-t)} \qquad (1.2)$$

Operations on Normal Quadratic Rational Splines

Definition 2.1: The perspective map $P = (A,h,u,w)$ consists of a 2x2 matrix A, two points h and u, and a scalar w. P acts on the point a by $Pa = \frac{Aa + h}{u\cdot a + w}$.

Theorem 2.1: If $u\cdot a+w>0$, $u\cdot b+w>0$, and $u\cdot c+w>0$, then $P = (A,h,u,w)$ acting on the NQRS with vertices (a,b,c) and weight Λ, produces the NQRS with vertices (Pa,Pb,Pc) and weight **Error!**$\cdot\Lambda$.

Proof: Because $A\eta+h$ and $u\cdot\eta+w$ are affine maps of η,

$$A\eta(t)+h = \frac{(Aa+h)(1-t)^2 + 2\Lambda(Ab+h)t(1-t) + (Ac+h)t^2}{1 + 2(\Lambda-1)t(1-t)}$$

$$= \frac{(u\cdot a+w)Pa(1-t)^2 + 2\Lambda(u\cdot b+w)Pbt(1-t) + (u\cdot c+w)Pct^2}{1 + 2(\Lambda-1)t(1-t)}$$

$$u\cdot\eta(t)+w = \frac{(u\cdot a+w)(1-t)^2 + 2\Lambda(u\cdot b+w)t(1-t) + (u\cdot c+w)t^2}{1 + 2(\Lambda-1)t(1-t)}$$

Taking the quotient of these last two equations, we arrive at

$$P\eta(t) = \frac{A\eta(t)+h}{u\cdot\eta(t)+w}$$

$$= \frac{(u\cdot a+w)Pa(1-t)^2 + 2\Lambda(u\cdot b+w)Pbt(1-t) + (u\cdot c+w)Pct^2}{(u\cdot a+w)(1-t)^2 + 2\Lambda(u\cdot b+w)t(1-t) + (u\cdot c+w)t^2}$$

This is the proper quadratic rational spline with vertices (Pa,Pb,Pc) and weights $(u\cdot a+w, \Lambda(u\cdot b+w), u\cdot c+w)$. According to Lemma 1.3, this is the normal quadratic rational spline with vertices (Pa,Pb,Pc) and weight **Error!**$\cdot\Lambda$. **QED**

Definition 2.2: The $[t_0,t_1]$ section of $\eta([0,1])$ is $\eta([t_0,t_1])$ where $0 \le t_0 < t_1 \le 1$.

Theorem 2.2: The $[t_0,t_1]$ section of the NQRS with vertices (a,b,c) and weight Λ is the NQRS with vertices (a^*,b^*,c^*) and weight Λ^* where

$$a^* = \frac{a(1-t_0)^2 + 2\Lambda bt_0(1-t_0) + ct_0^2}{1 + 2(\Lambda-1)t_0(1-t_0)} = \eta(t_0)$$

$$b^* = \frac{a(1-t_0)(1-t_1) + \Lambda b(t_0+t_1-2t_0t_1) + ct_0t_1}{1 + (\Lambda-1)(t_0+t_1-2t_0t_1)}$$

$$c^* = \frac{a(1-t_1)^2 + 2\Lambda bt_1(1-t_1) + ct_1^2}{1 + 2(\Lambda-1)t_1(1-t_1)} = \eta(t_1)$$

Λ^* = **Error!**

Proof: Let $t = (1-s)t_0+st_1$, then as s ranges from 0 to 1, t ranges from t_0 to t_1. It follows that $1-t = (1-s)(1-t_0)+s(1-t_1)$. Furthermore,

$$\eta(t) = \frac{a(1-t)^2 + 2\Lambda bt(1-t) + ct^2}{1 + 2(\Lambda-1)t(1-t)}$$

$$= a\frac{[(1-s)(1-t_0)+s(1-t_1)]^2}{1 + 2(\Lambda-1)t(1-t)} + 2\Lambda b\frac{[(1-s)t_0+st_1][(1-s)(1-t_0)+s(1-t_1)]}{1 + 2(\Lambda-1)t(1-t)} + c\frac{[(1-s)t_0+st_1]^2}{1 + 2(\Lambda-1)t(1-t)}$$

$$= \frac{[a(1-t_0)^2 + 2\Lambda bt_0(1-t_0) + ct_0^2]}{1 + 2(\Lambda-1)t(1-t)}(1-s)^2 + 2\frac{[a(1-t_0)(1-t_1) + \Lambda b(t_0+t_1-2t_0t_1) + ct_0t_1]}{1 + 2(\Lambda-1)t(1-t)}s(1-s) + \frac{[a(1-t_1)^2 + 2\Lambda bt_1(1-t_1) + ct_1^2]}{1 + 2(\Lambda-1)t(1-t)}s^2 \quad (2.1)$$

Let us stop to consider where we are headed. We wish to make $\eta(t)$ into a PQRS in terms of s. That is, we wish to be able to write

$$\eta(t) = \frac{\mathbf{A}a^*(1-s)^2 + 2\mathbf{B}b^*s(1-s) + \Gamma c^*s^2}{\mathbf{A}(1-s)^2 + 2\mathbf{B}s(1-s) + \Gamma s^2} \quad (2.2)$$

(2.2) says that the coefficients of a*, b*, and c* in $\eta(t)$ should sum to 1. Because the coefficients of a, b, and c in $\eta(t)$ sum to 1, (2.2) is true if we make the coefficients of a, b, and c in each of a*, b*, and c* sum to 1. Therefore, assume the following

$$\mathbf{A} = 1 + 2(\Lambda-1)t_0(1-t_0)$$

$$a^* = \frac{a(1-t_0)^2 + 2\Lambda bt_0(1-t_0) + ct_0^2}{1 + 2(\Lambda-1)t_0(1-t_0)}$$

$$\mathbf{B} = 1 + (\Lambda-1)(t_0+t_1-2t_0t_1)$$

$$b^* = \frac{a(1-t_0)(1-t_1) + \Lambda b(t_0+t_1-2t_0t_1) + ct_0t_1}{1 + (\Lambda-1)(t_0+t_1-2t_0t_1)}$$

$$\Gamma = 1 + 2(\Lambda-1)t_1(1-t_1)$$

$$c^* = \frac{a(1-t_1)^2 + 2\Lambda bt_1(1-t_1) + ct_1^2}{1 + 2(\Lambda-1)t_1(1-t_1)}$$

These were chosen so that the coefficients of a, b, and c in each of a*, b*, and c* sum to 1. Substituting them into (2.1) we have the following

$$\eta(t) = \frac{\mathbf{A}a^*(1-s)^2 + 2\mathrm{B}b^*s(1-s) + \Gamma c^*s^2}{1 + 2(\Lambda-1)t(1-t)}$$

$$= \frac{\mathbf{A}a^*(1-s)^2 + 2\mathrm{B}b^*s(1-s) + \Gamma c^*s^2}{\mathbf{A}(1-s)^2 + 2\mathrm{B}s(1-s) + \Gamma s^2}$$

Again, the last equation is true because the coefficients of a^*, b^*, and c^* in $\eta(t)$ sum to 1. Thus, the $[t_0,t_1]$ section of the NQRS with vertices (a,b,c) and weight Λ is the PQRS with vertices (a^*,b^*,c^*) and weights $(\mathbf{A},\mathrm{B},\Gamma)$, and according to Lemma 1.3 this is the NQRS with vertices (a^*,b^*,c^*) and weight Λ^* = **Error!**.
QED

The Conic of a Normal Quadratic Rational Spline

A normal quadratic rational spline is an arc of a conic section. In this section, we will investigate that conic.

First, let us recall a few properties of affine maps. Given three noncollinear points, there is a unique invertible affine map that sends those points to any other three noncollinear points. Affine maps preserve parallel lines and linear combinations whose coefficients sum to 1. In particular, affine maps preserve midpoints.

Recall also that any circle or ellipse is similar to an ellipse $x^2+cy^2=1$ for some $c>0$, any parabola is similar to the parabola $y=x^2$, and any hyperbola is similar to a hyperbola $x^2+cy^2=1$ for some $c<0$.

The Conic Center

Definition 3.1: The center of a conic is the midpoint of any two points on the conic at which the tangents are parallel.

Lemma 3.1: All conics, except parabolas, have one center; parabolas have none.

Proof: Any conic except a parabola is similar to a conic $x^2+cy^2=1$ for some $c\neq 0$. Differentiation yields the relation $x+cmy=0$, where m is the slope of the tangent. For a given m, $x+cmy=0$ intersects $x^2+cy^2=1$ in at most two points; if (x,y) is one of these points, then (-x,-y) is the other, and their midpoint is the origin.

All parabolas are similar to $y=x^2$. Differentiation yields the relation $m=2x$, where m is the slope of the tangent. For a given m, $m=2x$ intersects $y=x^2$ in only one point. Therefore, there can not be two points with parallel tangents. **QED**

Theorem 3.2: The center of the conic containing the NQRS with vertices (a,b,c) and weight Λ is $\dfrac{a-2\Lambda^2 b+c}{2(1-\Lambda^2)}$.

Proof: There is an affine map that sends a, b, and c to (-1,0), (0,1), and (1,0) respectively. Because affine maps preserve linear combinations whose coefficients sum to 1, this map sends the NQRS with vertices (a,b,c) and weight Λ to the NQRS with vertices ((-1,0),(0,1),(1,0)) and weight Λ.

Let us find the extrema in y for the NQRS with vertices ((-1,0),(0,1),(1,0)):

$$\eta_y(t) = \frac{a_y(1-t)^2 + 2\Lambda b_y t(1-t) + c_y t^2}{1 + 2(\Lambda-1)t(1-t)}$$

$$= \frac{2\Lambda t(1-t)}{1 + 2(\Lambda-1)t(1-t)}$$

$$\frac{d}{dt}\eta_y(t) = \frac{2\Lambda(1-2t)}{[1 + 2(\Lambda-1)t(1-t)]^2}$$

The extrema in y occur at $t = \frac{1}{2}$ and $t = \infty$. The tangents at these points are parallel to the line connecting (-1,0) and (1,0). Therefore, in the NQRS with vertices (a,b,c), the tangents at $t = \frac{1}{2}$ and $t = \infty$ are parallel to the line connecting a and c.

The corresponding points on the conic are $\eta(\frac{1}{2}) = \frac{a+2\Lambda b+c}{2(1+\Lambda)}$ and $\eta(\infty) = \frac{a-2\Lambda b+c}{2(1-\Lambda)}$.

Their midpoint is the conic center: $\frac{1}{2}\left(\frac{a+2\Lambda b+c}{2(1+\Lambda)} + \frac{a-2\Lambda b+c}{2(1-\Lambda)}\right) = \frac{a-2\Lambda^2 b+c}{2(1-\Lambda^2)}$.

QED

Corollary 3.2.1: The conic center lies on the parametric bisector.

Proof: $\frac{a+c}{2} = (1-\Lambda^2)\frac{a-2\Lambda^2 b+c}{2(1-\Lambda^2)} + \Lambda^2 b$. **QED**

Lemma 3.3: The parametric bisector of a parabolic arc is parallel to the axis of the parabola.

Proof: All parabolas are similar to $y=x^2$. Suppose (s,s^2) and (t,t^2) are the endpoints of an arc of $y=x^2$. The tangents at these points are $y=2sx-s^2$ and $y=2tx-t^2$. The tangential intercept is then $(\frac{s+t}{2}, st)$, and the terminal midpoint is $(\frac{s+t}{2}, \frac{s^2+t^2}{2})$. Thus, the parametric bisector is the line $x = \frac{s+t}{2}$, which is parallel to the y axis. **QED**

The Implicit Form

The implicit form of a conic is $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$. This is equivalent to

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^T Q \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0 \qquad (3.1)$$

where Q is a symmetric 3x3 matrix.

Theorem 3.4: The NQRS with vertices (a,b,c) and weight Λ is an arc of the conic

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^T Q \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

where $Q = 2\Lambda^2(uw^T+wu^T)-vv^T$ and u, v, and w are columns of the matrix

$$[\,u\ v\ w\,] = \operatorname{cof}\begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} b_y - c_y & c_y - a_y & a_y - b_y \\ c_x - b_x & a_x - c_x & b_x - a_x \\ b_x c_y - b_y c_x & c_x a_y - c_y a_x & a_x b_y - a_y b_x \end{bmatrix}.$$

Proof: First recall that according to (1.2)

$$\eta_x(t) = \frac{a_x(1-t)^2 + 2\Lambda b_x t(1-t) + c_x t^2}{1 + 2(\Lambda-1)t(1-t)} = \frac{p(t)}{r(t)}$$

$$\eta_y(t) = \frac{a_y(1-t)^2 + 2\Lambda b_y t(1-t) + c_y t^2}{1 + 2(\Lambda-1)t(1-t)} = \frac{q(t)}{r(t)}$$

where

$$\begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} (1-t)^2 \\ 2\Lambda t(1-t) \\ t^2 \end{bmatrix}.$$

Next notice that

$$\begin{bmatrix} (1-t)^2 \\ 2\Lambda t(1-t) \\ t^2 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 2\Lambda^2 \\ 0 & -1 & 0 \\ 2\Lambda^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} (1-t)^2 \\ 2\Lambda t(1-t) \\ t^2 \end{bmatrix} = 0.$$

Combining the last two equations, we have that

$$\begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix}^T \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{bmatrix}^{-1T} \begin{bmatrix} 0 & 0 & 2\Lambda^2 \\ 0 & -1 & 0 \\ 2\Lambda^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} = 0.$$

Because $\operatorname{cof}A = \det A\,(A^{-1})^T$, if we set

$$Q = \operatorname{cof}\begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2\Lambda^2 \\ 0 & -1 & 0 \\ 2\Lambda^2 & 0 & 0 \end{bmatrix} \operatorname{cof}\begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{bmatrix}^T$$

then

$$\begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix}^T Q \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} = 0.$$

Dividing this by $r(t)^2$ gives (3.1) for $\eta(t)$.

The cofactor matrix is

$$\operatorname{cof}\begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} b_y - c_y & c_y - a_y & a_y - b_y \\ c_x - b_x & a_x - c_x & b_x - a_x \\ b_x c_y - b_y c_x & c_x a_y - c_y a_x & a_x b_y - a_y b_x \end{bmatrix} = [\,u\ v\ w\,].$$

Therefore,

$$Q = [\,u\ v\ w\,]\begin{bmatrix} 0 & 0 & 2\Lambda^2 \\ 0 & -1 & 0 \\ 2\Lambda^2 & 0 & 0 \end{bmatrix}\begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix} = 2\Lambda^2(uw^T + wu^T) - vv^T.$$ **QED**

Example: Find the implicit form of the NQRS with vertices $((1,1),(2,3),(4,5))$ and weight $\frac{1}{2}$. First, $2\Lambda^2 = \frac{1}{2}$.

$$\text{cof}\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 & -2 \\ 2 & -3 & 1 \\ -2 & -1 & 1 \end{bmatrix} \Rightarrow u = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} \text{ and } v = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix} \text{ and } w = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$uw^T + wu^T = \begin{bmatrix} 4 & -2 & -2 \\ -4 & 2 & 2 \\ 4 & -2 & -2 \end{bmatrix} + \begin{bmatrix} 4 & -4 & 4 \\ -2 & 2 & -2 \\ -2 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 4 & 0 \\ 2 & 0 & -4 \end{bmatrix}$$

$$vv^T = \begin{bmatrix} 16 & -12 & -4 \\ -12 & 9 & 3 \\ -4 & 3 & 1 \end{bmatrix}$$

$$Q = \frac{1}{2}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 4 & 0 \\ 2 & 0 & -4 \end{bmatrix} - \begin{bmatrix} 16 & -12 & -4 \\ -12 & 9 & 3 \\ -4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -12 & 9 & 5 \\ 9 & -7 & -3 \\ 5 & -3 & -3 \end{bmatrix}$$

Therefore, the implicit form is $-12x^2+18xy-7y^2+10x-6y-3=0$.

Summary

As with most mathematical concepts, there are many ways to view conic splines. This Technote presents conic splines both as a quadratically-varying weighted mean of control points, and as a segment of a conic section. Using the proper representation for a given problem will help in finding a solution.

Further References

- Technote 1052 - QuickDraw GX ConicLibrary.c in Detail: Description, and Derivations
- Inside Macintosh: QuickDraw GX Graphics
- Inside Macintosh: QuickDraw GX Objects

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