Analysis of combined subdivision schemes for the interpolation of curves

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An accompanying paper to [7]

Abstract

We analyze the smoothness of several new subdivision schemes that are specially designed for the task of interpolating curves by smooth surfaces. These are instances of a new kind of subdivision schemes called *combined subdivision schemes*, which combine operations on control points with operations on functions.

The analysis is done in the regular setting, where subdivision schemes are defined over the bivariate integers. Due to the locality of those schemes, their smoothness guarantees the smoothness of the surfaces generated by the subdivision algorithm presented in [7], which is defined over control nets of arbitrary topological type.

The general theory of combined subdivision schemes, developed in [5, 6], provides the basis for our analysis.

1 Introduction

In [7] we present a subdivision scheme generating limit surfaces which interpolate given nets of possibly intersecting curves. The basic idea there, is to associate edges of the control net with segments of the given curves, and then to apply different refinement rules near those edges (called *c-edges*), while applying Catmull Clark's scheme everywhere away from them.

Due to the restrictions made on the initial control net in [7], we consider the subdivision process near the c-edges to be uniform, in the sense that it can be described (locally) as a scheme over the bivariate integers. In this setting, the given curves are replaced by functions given over lines in the parametric plane, while the vertices of the control net are replaced by control points associated with the bivariate integers. Notice that the G^2 smoothness of the limit surface away from the given curves is guaranteed by the smoothness of Catmull-Clark's scheme.

The analysis is made in the functional case, i.e. we study the existence and smoothness of scalar limit functions. Since the scheme in [7] applies the same subdivision rules to the three coordinates of each vertex, the existence and smoothness of the limit functions guarantee the existence and smoothness of limit surfaces.

In [7], we restrict the control net by requiring that the control net near each *c-vertex* (i.e. a vertex associated with a point on an interpolated curve) admits to one of six possible configurations. We divide the c-vertices to *regular boundary c-vertices, regular internal c-vertices, internal intersection vertices, inward corner vertices, boundary intersection vertices* and *outward corner vertices*. For the exact definition of those terms, we refer the reader to [7]. The smoothness analysis considers each of the six cases separately.

The paper is organized as follows: Section 2 defines uniform subdivision schemes. In §3 we define combined subdivision schemes. In §4 we discuss Catmull-Clark's scheme, which is the basis of all our schemes, and in §5 we summarize the relevant theorems from [5, 6] that are needed for the smoothness analysis. Sections 6-11 analyze the six cases mentioned above. The relevance of this analysis to the combined subdivision scheme defined in [7] is discussed in §12.

2 Bivariate uniform subdivision schemes

Given $X \subset \mathbb{Z}^2$, let $l(X)$ denote all the functions $P : X \mapsto \mathbb{R}$, let $l_\infty(X)$ denote the Banach space of all the functions $P \in l(X)$ such that $\|P\|_\infty < \infty$, where $\|P\|_\infty$ is the supremum of $|P|$ on X. Let $l_0(X) \subset l_\infty(X)$ denote the space of all the functions $P \in l_\infty(X)$ with finite support.

A bivariate uniform subdivision operator is a linear operator $S : l(\mathbb{Z}^2) \mapsto l(\mathbb{Z}^2)$ which consists of a mask $a \in l_0(\mathbb{Z}^2)$ and is defined by

$$(SP)(\alpha) = \sum_{\beta \in \mathbb{Z}^2} a(\alpha - 2\beta)P(\beta), \quad \forall \alpha \in \mathbb{Z}^2. \tag{1}$$

It is easily seen that for all $P, Q \in l(\mathbb{Z}^2)$

$$P \in l_0(\mathbb{Z}^2) \Rightarrow S(P) \in l_0(\mathbb{Z}^2), \tag{2}$$

and for any $\beta \in \mathbb{Z}^2$

$$P(\alpha + \beta) = Q(\alpha), \quad \forall \alpha \in \mathbb{Z}^2 \Rightarrow (SP)(\alpha + 2\beta) = (SQ)(\alpha), \quad \forall \alpha \in \mathbb{Z}^2. \tag{3}$$

A subdivision scheme S is termed *uniformly convergent*, if for every $P \in l_0(\mathbb{Z}^2)$, there exists a compactly supported function $F \in C(\mathbb{R}^2)$ (called the limit function) such that

$$\lim_{n\to\infty} \left\| S^n P - F(2^{-n}\cdot) \right\|_{\infty,\mathbb{Z}^2} = 0. \tag{4}$$

We denote $S^\infty P = F$.

We say that S is C^m if S is uniformly convergent, and for every P, $S^\infty P \in C^m(\mathbb{R}^2)$. We define $\Phi = S^\infty \delta_0$ where δ_β is defined, for $\beta \in \mathbb{Z}^2$, by

$$\delta_\beta(\alpha) = \begin{cases} 1 & \alpha = \beta \\ 0 & \text{otherwise} \end{cases}.$$

Φ is called the S-refinable function. The limit function $S^\infty P$ can be expressed as a sum of integer translates of Φ

$$S^\infty P = \sum_{\alpha \in \mathbb{Z}^2} P(\alpha)\Phi(\cdot - \alpha) = \sum_{\alpha \in \mathbb{Z}^2} (S^k P)(\alpha)\Phi(2^k \cdot - \alpha), \quad \forall k \geq 0. \tag{5}$$

For convenience, we assume that a is supported on a square centered at the origin, namely that

$$a(\alpha) = 0, \quad \forall \alpha \in \mathbb{Z}^2 \setminus \{-\omega, .., \omega\}^2, \tag{6}$$

for some $\omega \in \mathbb{Z}_+$. It is shown in [2] that the support of Φ is contained in the convex hull of the support of the mask a, therefore

$$\Phi(x) = 0, \quad \forall x \notin \Omega = [-\omega, \omega]^2. \tag{7}$$

The limit functions of subdivision schemes are analyzed in many papers (see e.g [2, 4]).

Throughout the paper we use standard multi-index notations for $\mathbb{Z}^2$: $j = (j_1, j_2) \in \mathbb{Z}^2$, $j \geq 0$ if $j_1, j_2 \geq 0$, $|j| = j_1 + j_2$, $x^j = x_1^{j_1} \cdot x_2^{j_2}$, $j! = j_1! \cdot j_2!$, $D^j = \frac{\partial^{|j|}}{\partial^{j_1} x_1 \partial^{j_2} x_2}$.

3 Combined subdivision schemes

A combined subdivision scheme is a subdivision scheme that operates uniformly away from a closed subset $\mathcal{R} \subset \mathbb{R}^2$. Its limit function is defined over $\mathbb{R}^2 \setminus \mathcal{R}$. The subset $\mathcal{R}$ is called the *exterior* of the combined subdivision scheme, and is invariant to positive scales, namely, $\alpha\mathcal{R} = \mathcal{R}$ for every $\alpha > 0$.

Let S denote a uniform subdivision operator, supported in the cube Ω. Let $\mathcal{R}$ denote a positive-scale-invariant and closed subset of $\mathbb{R}^s$. A combined subdivision operator of order k, which is based on S, with the exterior $\mathcal{R}$, is an operator

$$B : l(\mathbb{Z}^2) \times C^k(\mathbb{R}^2) \mapsto l(\mathbb{Z}^2),$$

such that

$$B(P, f)(\alpha) = SP(\alpha), \quad \forall \alpha \in \mathbb{Z}^2 \setminus (\mathcal{R} + \Omega). \tag{8}$$

We define a *combined subdivision scheme* by the following iterative process:

$$\begin{aligned} P^0 &= P \in l(\mathbb{Z}^2), \\ P^{n+1} &= B(P^n, f(2^{-n}\cdot)), \quad n = 0, 1, \ldots. \end{aligned} \tag{9}$$

We also assume that the operator B uses only values of f and of its derivatives on $\mathcal{R}$. By requiring f to be defined on $\mathbb{R}^s$ instead of $\mathcal{R}$, we make sure that all of its derivatives are defined in $\mathcal{R}$ even when $\mathcal{R}$ is a set of measure zero. By requiring that $\mathcal{R}$ be invariant to positive scales, we get that in every iteration of (9) the only values of f and its derivatives that are used are those in $\mathcal{R}$.

We say that $F \in C(\mathbb{R}^2 \setminus \mathcal{R})$ is the limit function of the combined scheme (9), if for every $x \in \mathbb{R}^2 \setminus \mathcal{R}$ there exists an open domain $D_x \subset \mathbb{R}^2 \setminus \mathcal{R}$, $x \in D_x$, such that

$$\lim_{n \to \infty} \left\| P^n - F(2^{-n}\cdot) \right\|_{\infty, \mathbb{Z}^2 \cap (2^n D_x)} = 0.$$

We denote $B^\infty(P, f) = F$.

Provided that S is C^m, we have that $B^\infty(P, f)$ is always well defined and C^m in $\mathbb{R}^2 \setminus \mathcal{R}$. The goal of the smoothness analysis is to show that $B^\infty(P, f)$ can be extended to a function which is C^m in the closure of its domain $\overline{\mathbb{R}^2 \setminus \mathcal{R}}$.

The combined subdivision schemes that we present are all local and linear in the sense defined in [5, 6], and they are of order $k = 2$, since they involve second derivatives of f. Moreover, they are bounded in the sense that there exists $C_B > 0$ such that

$$|B(0, f)(\alpha)| \leq C_B \sum_{|i|\leq 2} \left\| D^i f \right\|_{\infty, \mathcal{R}\cap\left(\frac{1}{2}\alpha+\frac{1}{2}\Omega\right)}, \quad \forall f \in C^2(\mathbb{R}^2). \tag{10}$$

The above conditions are required by the general smoothness analysis in [5, 6].

4 Catmull Clark's scheme and polynomial precision

The combined subdivision schemes that we construct are based on Catmull Clark's scheme, which is defined over closed nets of arbitrary topology [1, 3, 8]. However, for our smoothness analysis it is sufficient to consider the scheme, as defined on the bivariate integers, where it coincides with the tensor product bicubic B-spline scheme.

We denote the tensor product bicubic B-spline scheme by S. It mask a, is given by

$$a = \frac{1}{64}\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}.$$

The basis function Φ is supported in

$$\Omega = [-2, 2]^2.$$

We denote by $\pi^2 = \pi^2(\mathbb{R}^2)$ the space of the bivariate polynomials of degree ≤ 2. The scheme S is C^2 and the integer translates of Φ are linearly independent. Therefore, by theorem 10.1 in [5, 6], the restriction of S to π^2 is similar to the dilation operator, namely there exists an invertible operator $T : \pi^2(\mathbb{Z}^2) \mapsto \pi^2(\mathbb{Z}^s)$ with no eigenvalues other than the eigenvalue 1, such that

$$ST = T\sigma, \tag{11}$$

where σ is the dilation operator

$$\sigma f = f\left(\frac{\cdot}{2}\right).$$

In the case of the bicubic B-splines, it can be shown that T can be represented as follows:

$$Tf = f - \frac{1}{6}\left(\frac{\partial^2 f}{\partial x_1^2} + \Delta_{x_2}^2 f\right), \tag{12}$$

where $\Delta_{x_i}^2$ is the second difference

$$\Delta_{x_i}^2 f(\cdot) = f(\cdot + e_i) - 2f(\cdot) + f(\cdot - e_i), \quad i \in \{1, 2\},\ e_1 = (1, 0),\ e_2 = (0, 1).$$

An important concept in subdivision schemes is that of *polynomial precision*. In analyzing the smoothness and approximation properties of a subdivision scheme, it is important to find the space of polynomials that can be generated as limit functions of that scheme [2].

The concept of polynomial precision is relevant to the analysis of combined subdivision schemes as well. We say that the combined scheme B belongs to Π^2 (or, that B has quadratic precision), if

$$B\left(Tp|_{\mathbb{Z}^2}, p\right) \equiv T\sigma p|_{\mathbb{Z}^2}, \quad \forall p \in \pi^2(\mathbb{R}^2), \tag{13}$$

where $T : \pi^2(\mathbb{R}^2) \mapsto \pi^2(\mathbb{R}^2)$ satisfies (11) and its only eigenvalue is 1. It is shown in [6] that any scheme B satisfying (13) also satisfies

$$B^\infty\left(Tp|_{\mathbb{Z}^2}, p\right) \equiv p|_{\mathbb{R}^2 \setminus \mathcal{R}}, \quad \forall p \in \pi^2(\mathbb{R}^2).$$

5 The smoothness criteria

For $X \subset \mathbb{R}^2$, we denote by $H^\nu(X)$ the class of functions defined on X, that are Hölder continuous of order $0 < \nu \leq 1$, i.e. functions $F : X \mapsto \mathbb{R}$ such that

$$|F(x) - F(y)| \leq c\|x - y\|^\nu, \quad \forall x, y \in X,$$

for some $c > 0$. We denote by $H^\nu_m(\mathbb{R}^s)$ the class of C^m functions for which all the derivatives of order m are Hölder continuous of order $0 < \nu \leq 1$.

Definition. *We say that $B \in C^m_+$ if there exists $0 < \nu \leq 1$ such that $B^\infty(P, f)$ can be extended to a function in $C^m\left(\overline{\mathbb{R}^s \setminus \mathcal{R}}\right)$ for every $P \in l(\mathbb{Z}^2)$ and $f \in H^\nu_m(\mathcal{R})$.*

Given a combined subdivision scheme B that is based on Catmull-Clark's scheme S, we aim to show that $B \in C^2_+$. Our analysis is based on the analysis tools developed in [5, 6]:

Theorem 5.1 *if $B \in \Pi^2$, and there exists $0 < \nu \leq 1$ such that $B^\infty(P, 0) \in H^\nu_2(\mathbb{R}^2 \setminus \mathcal{R})$ for every $P \in l_0(\mathbb{Z}^s)$ then $B \in C^2_+$.*

The condition $B \in \Pi^2$ is simple to check, using (13). However, the analysis of the limit function of the *homogeneous scheme*, $B^\infty(P, 0)$ (called that way because the function f is set to zero), requires a few additional definitions:

Let $J = \Omega \cap \mathbb{Z}^2 = \{-2, \ldots, 2\}^2$. It is easy to see, for all the combined schemes that we construct, that J is a *refinement set*, namely, that the values of $B(P, 0)$ in J depend only on the values of P in J. Therefore, there exists a matrix M_B such that

$$B(P, 0)|_J = M_B\left(P|_J\right), \quad \forall P \in l(\mathbb{Z}^s).$$

The *refinement matrix* M_B is easy to calculate for any given B. We have the following theorem from [5, 6], which employs the spectral analysis of the matrix M_B:

Theorem 5.2 *Let B denote a combined subdivision scheme such that $B \in \Pi^2$ and*

$$B^\infty(P,0) \in H_2^\nu(\mathbb{R}^2 \setminus (\mathcal{R} \cup \Theta)), \quad \forall P \in l_0(\mathbb{Z}^s), \tag{14}$$

where $\Theta \subset \mathbb{R}^2$ is a bounded set, and $0 < \nu \leq 1$. If the eigenvalues of M_B that are greater than $2^{-(2+\nu)}$ have the same algebraic and geometric multiplicity, and their corresponding eigenvectors q can be written as $q = Q|_J$ where

$$B^\infty(Q,0) \in H_2^\nu(\mathbb{R}^2 \setminus \mathcal{R}), \quad Q \in l_0(\mathbb{Z}^s),$$

then

$$B^\infty(P,0) \in H_2^\nu(\mathbb{R}^2 \setminus \mathcal{R}), \quad \forall P \in l_0(\mathbb{Z}^s).$$

In theorem 5.2 it is assumed that $B^\infty(P,0)$ is well behaved away from the origin. The theorem provides a simple sufficient condition for $B^\infty(P,0)$ to be well behaved near the origin.

6 A combined scheme for the interpolation of a smooth boundary curve

The smoothness analysis of the combined scheme from [7] near boundary edges reduces to the following setting. Let

$$\mathcal{R} = \{x \in \mathbb{R}^2 \mid x_1 \leq 0\}.$$

Let $f \in C^2(\mathbb{R}^2)$. We define the combined subdivision operator B_1 by

$$B_1(P,f)(\alpha) = \left\{ \begin{array}{ll} SP(\alpha), & \alpha_1 \neq 0 \\ T\sigma f(\alpha), & \alpha_1 = 0 \end{array} \right. , \quad \forall \alpha \in \mathbb{Z}^2, \tag{15}$$

where T is defined in (12). It is easy to see that $B_1 \in \Pi^2$, since

$$\begin{aligned} B_1\left(Tp|_{\mathbb{Z}^2}, p\right)(\alpha) &= \left\{ \begin{array}{ll} S\left(Tp|_{\mathbb{Z}^2}\right)(\alpha), & \alpha_1 \neq 0 \\ (T\sigma p)(\alpha), & \alpha_1 = 0 \end{array} \right. \\ &= (T\sigma p)(\alpha), \quad \forall p \in \pi^2\left(\mathbb{R}^2\right). \end{aligned} \tag{16}$$

Let $P \in l_0(\mathbb{Z}^2)$. In order to show that $B_1^\infty(P,0) \in H_2^\nu(\mathbb{R}^s \setminus R)$ for some $0 < \nu \leq 1$, we consider the following scheme. Define $Q \in l_0(\mathbb{Z}^2)$ by

$$Q(\alpha) = \left\{ \begin{array}{cl} B_1(P,0)(\alpha) & \alpha_1 > 0 \\ 0 & \alpha_1 = 0 \\ -B_1(P,0)(-\alpha_1,\alpha_2) & \alpha_1 < 0, \end{array} \right. .$$

Let

$$\begin{aligned} Q^1 &= Q, \\ Q^{n+1} &= SQ^n, \quad n = 1, 2, \ldots. \end{aligned} \tag{17}$$

Consider the combined scheme

$$\begin{aligned} P^0 &= P, \\ P^{n+1} &= B_1(P^n, 0), \quad n = 0, 1, \ldots \end{aligned} \tag{18}$$

It is easy to see from the symmetries of S that

$$Q^n(\alpha) = P^n(\alpha), \quad \forall n \geq 1,\ \forall \alpha \in \mathbb{Z}^2,\ \alpha_1 \geq 0.$$

Therefore

$$S^\infty Q^1\big|_{\mathbb{R}^s \setminus \mathcal{R}} = B_1^\infty\left(P^1\big|_{\mathbb{R}^s \setminus \mathcal{R}}, 0\right),$$

hence the limit function $B_1^\infty(P, 0)$ has Lipschiez continuous second derivatives, and it follows from theorem 5.1 that $B_1 \in C_+^2$.

7 A combined scheme for the interpolation of a smooth curve

The smoothness analysis the combined scheme from [7] near internal c-edges reduces to the following combined scheme defined over the bivariate integers. Let

$$\mathcal{R} = \{x \in \mathbb{R}^2 \mid x_1 = 0\}.$$

We define the combined subdivision operator B_2 as a composition of B_1 with a *correction operator* C that operates near R:

$$B_2(P, f) = C(B_1(P, f), \sigma f), \tag{19}$$

where

$$\begin{aligned} C(P, f)(\alpha) &= \frac{1}{2}\left(P(\alpha_1, \alpha_2) - P(-\alpha_1, \alpha_2)\right) \\ &+ P(0, \alpha_2) + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(0, \alpha_2), \end{aligned}$$

if $\alpha_1 \in \{-1, 1\}$, and $C(P, f)(\alpha) = P(\alpha)$ otherwise. First, we show that $B_2 \in \Pi^2$. It is easy to see that

$$C\left(p|_{\mathbb{Z}^2}, p\right) = p|_{\mathbb{Z}^2}, \quad \forall p \in \pi^2, \tag{20}$$

hence

$$B_2\left(Tp|_{\mathbb{Z}^2}, p\right) = C\left(T\sigma p|_{\mathbb{Z}^2}, \sigma p\right) = C\left(T\sigma p|_{\mathbb{Z}^2}, T\sigma p\right) = T\sigma p|_{\mathbb{Z}^2}. \tag{21}$$

Here we used the fact that

$$\frac{\partial^2}{\partial x_1^2} Tp \equiv \frac{\partial^2}{\partial x_1^2} p, \quad \forall p \in \pi^2,$$

In order to show that $B_2^\infty(P,0) \in H_2^\nu(\mathbb{R}^s \setminus R)$, we consider the following scheme:

$$\begin{aligned} Q^1 &= B_2(P,0), \\ Q^{n+1} &= SQ^n, \quad n = 1,2,\ldots \end{aligned} \tag{22}$$

and

$$\begin{aligned} P^0 &= P, \\ P^{n+1} &= B_2(P^n,0), \quad n = 0,1,\ldots \end{aligned} \tag{23}$$

It is easy to see that

$$Q^n \equiv P^n, \quad \forall n \geq 1,$$

which implies that $B_2^\infty\left(P^1,0\right) = S^\infty Q^1$, hence the limit function $B_2^\infty(P,0)$ has Lipschiez continuous second derivatives, and it follows from theorem 5.1 that $B_2 \in C_+^2$.

8 A combined scheme near inward corners

In this section we analyze the smoothness of the combined scheme from [7] near *inward corners*, after reducing it to the following setting. Let

$$\mathcal{R} = \{x \in \mathbb{R}^2 \mid x_1 \leq 0 \text{ or } x_2 \leq 0\}.$$

We define the transpose operator $(\cdot)^t$ by

$$F^t(x_1,x_2) = F(x_2,x_1), \quad \forall x_1,x_2. \tag{24}$$

For $f \in C(\mathbb{R}^2)$ and $P \in l(\mathbb{Z}^2)$, we define the combined subdivision scheme B_3 by

$$B_3(P,f)(\alpha) = \begin{cases} SP(\alpha) & \alpha_1 \cdot \alpha_2 \neq 0 \\ (T\sigma f)(\alpha) & \alpha_1 = 0 \text{ and } \alpha_2 \neq 0 \\ \left(T\sigma f^t\right)^t(\alpha) & \alpha_2 = 0 \text{ and } \alpha_1 \neq 0 \\ f(0) - \frac{2}{3}\big(\Delta_{x_1}^2\sigma^2 f(1,0) & \\ \qquad +\Delta_{x_2}^2\sigma^2 f(0,1)\big) & \alpha = 0 \end{cases}. \tag{25}$$

From the fact that T commutes with the transpose operator $(\cdot)^t$, and from the following observation:

$$B_3(P,p)(0) = T\sigma p(0), \quad \forall p \in \pi^2(\mathbb{R}^2),\ \forall P \in l(\mathbb{Z}^2),$$

it follows that $B_3 \in \Pi^2$. Moreover, from the fact that $B_1 \in C_+^2$, and from the fact that B_3 coincides with B_1 (or its transpose) away from the origin, it is easy to see that the homogeneous scheme of B_3 has Lipschiez continuous second derivatives away from the origin, namely, there exists a bounded $\Theta \subset \mathbb{R}^2$ such that

$$B_3^\infty(P,0) \in H_2^\nu(\mathbb{R}^2 \setminus (\mathcal{R} \cup \Theta)), \quad \forall P \in l_0(\mathbb{Z}^s), \tag{26}$$

with $\nu = 1$. According to theorem 5.2 it is only left to examine the spectral properties of M_{B_3}. We find that M_{B_3} has exactly four eigenvectors corresponding to the eigenvalue $\frac{1}{4}$, which is the dominant eigenvector. The four eigenvectors are

$$q_1 = \begin{bmatrix} 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 4 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$q_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 & 0 \end{bmatrix}, \quad q_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}.$$

Clearly, q_1 can be expressed as

$$q_1 = Q|_J ,$$

where

$$Q(\alpha) = \begin{cases} \alpha_1 \cdot \alpha_2 & \alpha \in \mathbb{Z}^2 \cap \mathcal{R} \\ 0 & \alpha \in \mathbb{Z}^2 \setminus \mathcal{R} \end{cases}$$

It is easy to see that $B^\infty(Q, 0)$ is the polynomial $x_1 \cdot x_2$ restricted to $\mathbb{R}^2 \setminus \mathcal{R}$, therefore $B^\infty(Q, 0) \in H_2^\nu(\mathbb{R}^2 \setminus \mathcal{R})$. Similarly each of q_2, q_3 and q_4 can be extended to $Q \in l(\mathbb{Z}^s)$ which is zero in $\mathbb{Z}^2 \cap \mathcal{R}$, which yields $B^\infty(Q, 0) \equiv 0$. Finally, using theorem 5.2, we get that $B_3 \in C_+^2$.

9 A combined scheme for the interpolation of two intersecting curves

In this section we analyze the smoothness of the combined scheme from [7] near *internal intersection vertices*. Let

$$\mathcal{R} = \{x \in \mathbb{R}^2 \mid x_1 \cdot x_2 = 0\}.$$

We define the combined subdivision scheme B_4 by

$$B_4(P, f)(\alpha) = \begin{cases} B_2(P, f)(\alpha) & \alpha_2 \notin \{-1, 0, 1\} \\ B_2(P^t, f^t)^t(\alpha) & \alpha_1 \notin \{-1, 0, 1\} \\ C\left(C(SP, \sigma f)^t, \sigma f^t\right)^t(\alpha) & \alpha \in \{-1, 1\}^2 \\ T\sigma f(\alpha) & \alpha \in \{(0, -1), (0, 1)\} \\ \left(T\sigma f^t\right)^t(\alpha) & \alpha \in \{(-1, 0), (1, 0)\} \\ f(0) - \frac{1}{6}\left(\Delta_{x_1}^2 \sigma f(0) + \Delta_{x_2}^2 \sigma f(0)\right) & \alpha = 0 \end{cases}. \tag{27}$$

For $f \in \pi^2(\mathbb{R}^2)$ we get

$$B_4(P, f)(0) = T\sigma f(0).$$

It is easy to see that $B_4 \in \Pi^2$ since $B_2 \in \Pi^2$ and from (20). Also, from the fact that $B_2 \in C^2_+$, there exists a bounded $\Theta \subset \mathbb{R}^2$ such that

$$B_4^\infty(P,0) \in H_2^\nu\left(\mathbb{R}^s \setminus (\mathcal{R} \cup \Theta)\right), \quad \forall P \in l_0(\mathbb{Z}^2).$$

The dominant eigenvalue of M_{B_4} is $\frac{1}{4}$ and it is simple, with the eigenvector

$$q = \begin{bmatrix} -4 & -2 & 0 & 2 & 4 \\ -2 & -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & -2 \\ 4 & 2 & 0 & -2 & -4 \end{bmatrix}.$$

Clearly, q is a restriction of the polynomial $x_1 \cdot x_2$ to J. It follows from theorem 5.2 that $B_4 \in C^2_+$.

10 A combined scheme near an intersection between a boundary curve and an interpolated curve

In this section we analyze the smoothness of the combined scheme from [7] near *boundary intersection vertices.* Let

$$\mathcal{R} = \{x \in \mathbb{R}^2 \mid x_1 \leq 0 \text{ or } x_2 = 0\}.$$

We define the combined subdivision scheme B_4 by

$$B_5(P,f)(\alpha) = \begin{cases} B_2(P^t,f^t)^t(\alpha) & \alpha_1 \neq 0 \\ T\sigma f(\alpha) & \alpha_1 = 0,\ \alpha_2 \neq 0 \\ f(0) - \frac{1}{6}\left(4\Delta^2_{x_1}\sigma^2 f(1,0) + \Delta^2_{x_2}\sigma f(0)\right) & \alpha = 0 \end{cases}. \tag{28}$$

For $f \in \pi^2(\mathbb{R}^2)$ we get

$$B_5(P,f)(0) = T\sigma f(0).$$

It is easy to see that $B_5 \in \Pi^2$ since $B_2 \in \Pi^2$. Also, from the fact that $B_2 \in C^2_+$, there exists a bounded $\Theta \subset \mathbb{R}^2$ such that

$$B_5^\infty(P,0) \in H_2^\nu\left(\mathbb{R}^s \setminus (\mathcal{R} \cup \Theta)\right), \quad \forall P \in l_0(\mathbb{Z}^2).$$

The dominant eigenvalue of M_{B_4} is $\frac{1}{4}$ and it has multiplicity 2. The corresponding eigenvectors are

$$q_1 = \begin{bmatrix} 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -2 & -4 \end{bmatrix}, \quad q_2 = \begin{bmatrix} -4 & -2 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that q_1 can be extended to $Q_1 \in l(\mathbb{Z}^s)$ such that $B^\infty(Q_1,0)$ is the restriction of the polynomial $x_1 \cdot x_2$ to $\mathbb{R}^2 \setminus \mathcal{R}$. Similarly, q_2 can be extended to $Q_1 \in l(\mathbb{Z}^s)$ such that $B^\infty(Q_2,0) \equiv 0$. It follows from theorem 5.2 that $B_5 \in C^2_+$.

11 A combined scheme near outward corners

In this section we analyze the smoothness of the combined scheme from [7] near *outward corners*. The analysis reduces to the following setting: Let

$$\mathcal{R} = \{x \in \mathbb{R}^2 \mid x_1 \le 0 \text{ and } x_2 \le 0\}.$$

Let $f \in C^2(\mathbb{R}^2)$. We define the combined subdivision operator B_6 by a composition of two operators $B_{6,1}$ and $B_{6,2}$:

$$B_6(P,f) = B_{6,2}\left(B_{6,1}(P,f), \sigma f\right) \tag{29}$$

where

$$B_{6,1}(P,f) = \begin{cases} SP(\alpha), & \alpha \in \mathbb{Z}^2 \setminus \partial\mathcal{R} \\ T\sigma f(\alpha), & \alpha_1 = 0 \text{ and } \alpha_2 < 0 \\ \left(T\sigma f^t\right)^t(\alpha), & \alpha_2 = 0 \text{ and } \alpha_1 < 0 \\ f(0) - \frac{2}{3}\big(\Delta^2_{x_1}\sigma^2 f(-1,0) & \\ \qquad +\Delta^2_{x_2}\sigma^2 f(0,-1)\big) & \alpha = 0 \end{cases} . \tag{30}$$

Using the same arguments as we used when analyzing B_3, it follows that $B_{6,1} \in \Pi^2$.

The correction operator $B_{6,2}$ is defined as follows:

$$B_{6,2}\left(P,f\right)(\alpha) = P(\alpha), \quad \forall \alpha \notin L,$$

where

$$L = \{(-1,1),(0,1),(1,1),(1,0),(1,-1)\}.$$

For every $P \in l(\mathbb{Z}^2)$, let

$$\begin{aligned} t(P) \;&=\; \frac{1}{4}\cdot\big(P(-1,0) + P(0,-1) - P(1,-1) - P(1,0) \\ &+\; 2P(1,1) - P(0,1) - P(-1,1)\big) \end{aligned}$$

We denote, for convenience $Q := B_{6,2}(P,f)$, and we define Q in L by

$$\begin{aligned} Q(1,0) &= \frac{1}{3}P(1,0) + \frac{2}{3}\left(2P(0,0) - P(-1,0) + \Delta^2_{x_1}\sigma f(-1,0)\right), \\ Q(0,1) &= \frac{1}{3}P(0,1) + \frac{2}{3}\left(2P(0,0) - P(0,-1) + \Delta^2_{x_2}\sigma f(0,-1)\right), \\ Q(1,-1) &= \frac{1}{3}P(1,-1) + \frac{2}{3}\left(Q(1,0) + P(0,-1) - P(0) - t(P)\right). \\ Q(-1,1) &= \frac{1}{3}P(-1,1) + \frac{2}{3}\left(Q(0,1) + P(-1,0) - P(0) - t(P)\right). \\ Q(1,1) &= \frac{1}{3}P(-1,1) + \frac{2}{3}\left(Q(0,1) + Q(1,0) - P(0) + t(P)\right). \end{aligned} \tag{31}$$

We have defined $B_{6,2}$ such that

$$B_{6,2}\left(Tp|_{\mathbb{Z}^2}, p\right) = Tp|_{\mathbb{Z}^2}, \quad \forall p \in \pi^2, \tag{32}$$

therefore we get $B_6 \in \Pi^2$. Finally, the dominant eigenvalue of M_{B_6} is $\frac{1}{4}$. It is simple, and the corresponding eigenvector is

$$q = \begin{bmatrix} -4 & -2 & 0 & 2 & 4 \\ -2 & -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & -2 \\ 4 & 2 & 0 & -2 & -4 \end{bmatrix},$$

which is the restriction of the polynomial $x_1 x_2$ to J. Hence we get $B_6 \in C_+^2$.

12 Discussion

In this section we explain the relevance of the combined schemes defined in sections 6-11 to the scheme described in [7]. We refer the reader to [7] for specific terms from there that we do not redefine.

As mentioned in §6, the smoothness of the scheme B_1 is supposed to guarantee the smoothness of the scheme in [7] near *regular boundary c-vertices*. Notice that $B_1(P, f)$ uses the second derivatives of f in the cross boundary direction $\partial^2 f / \partial x_1^2$, while the scheme in [7] uses only the values of the given curves, which correspond to values of the given function f.

What replaces the cross boundary second derivatives are the *cross boundary second differences* $d(v)$ defined in [7]. From the propagation rules for $d(v)$ it is easy to see that the values of $d(v)$ multiplied by 4^k, where k is the number of iterations made so far, are values of a piecewise linear function. This exactly corresponds to $d(v)$ being the values of $4^{-k}\partial^2 f / \partial x_1^2$ which is piece-wise linear with knots at the integers. The factor 4^{-k} comes naturally from the dilation applied to f in each iteration of the combined subdivision scheme (9).

In the same way we explain the relevance of $B_2, \ldots, B_6$ to the smoothness of the limit surfaces in [7] near the other five types of c-vertices. However, the association between cross boundary second derivatives $d(v)$ and $\partial^2 f / \partial x_1^2$ suggests that $\partial^2 f / \partial x_1^2$ is not piece-wise linear with knots at the integers, in the cases of $B_3, \ldots, B_6$.

For smoothness, we do not necessarily require that $\partial^2 f / \partial x_1^2$ is piece-wise linear. We only need it to be Hölder continuous. In the following, we define the function $\partial^2 f / \partial x_1^2$ in a way that corresponds to the propagation rules of $d(v)$ near intersection vertices, and we prove that we get a Hölder continuous function:

We demonstrate this analysis in the case of *inward corners*, which corresponds to the scheme B_3, with

$$\mathcal{R} = \{x \in \mathbb{R}^2 \mid x_1 \leq 0 \text{ or } \quad x_2 \leq 0\}.$$

We denote

$$\Delta^{(k)} = 4^{k+1} \Delta_{x_1}^2 \sigma^{k+1} f(1, 0).$$

Let $\{d_n\}_{n=1,2,\ldots}$ denote a sequence of real numbers, that correspond to the values $d(v)$ on the original control net. We define $d_{2^{-k}}$ recursively as follows:

$$d_{2^{-k}} = \frac{d_{2^{1-k}} + \Delta^{(k)}}{2}, \quad k = 1, 2, \ldots. \tag{33}$$

We then proceed by defining the function $\frac{\partial^2 f}{\partial x_1^2}(0, x_2)$ when $x_2 > 0$ to be the piecewise linear function, with knots at $\ldots, 2^{-2}, 2^{-1}, 1, 2, 3, \ldots$, that satisfies

$$\frac{\partial^2 f}{\partial x_1^2}(0, n) = d_n, \quad n = 1, 2, \ldots \quad \text{or } n = 2^{-k},\ k = 1, 2, \ldots.$$

In case $\frac{\partial^2 f}{\partial x_1^2}$ is Hölder continuous in $\{x \in \mathbb{R}^2 \mid x_1 > 0, \text{ and } x_2 = 0\}$. we have that

$$\left| \Delta^{(k)} - \frac{\partial^2 f}{\partial x_1^2}(0) \right| = O\left(2^{-\nu k}\right), \tag{34}$$

for some $0 < \nu \leq 1$. Then it follows from (33) and (34) that

$$|d_{2^{-k}} - d_{2^{1-k}}| = O\left(2^{-\nu k}\right),$$

for some $0 < \nu \leq 1$. From this, it can be shown that $\frac{\partial^2 f}{\partial x_1^2}$ as we defined it, is Hölder continuous in $\{x \in \mathbb{R}^2 \mid x_2 > 0, \text{ and } x_1 = 0\}$.

13 Conclusions

The analysis in this paper guarantees that the surfaces generated by the scheme defined in [7] are C^2 near the interpolated curves (therefore G^2 everywhere except near internal extraordinary vertices of the original control net, where the surfaces are only almost G^2), in case the given curves have Hölder continuous second derivatives, and under all the restriction given in [7].

Furthermore, for the analysis we assume that not only are the curves given, but also the cross-curve second derivatives are given as a continuos function (More precisely, they have to be Hölder continuous). Therefore, the piecewise linear propagation rules for $d(v)$ that are given in [7] can be replaced by other propagation rules. In fact, the designer can prescribe the cross-curve second derivatives everywhere across the given curves, and the scheme will still generate surfaces that are G^2 almost everywhere.

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