Algorithms for Combinatorial Coalition Formation and Payoff Division in an Electronic Marketplace[1]

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Abstract

In an electronic marketplace coalition formation allows buyers to enjoy a price discount for each item while combinatorial auction enables buyers to place bids for a bundle of items that are complementary. Coalition formation and combinatorial auction both help to improve the efficiency of a market and have received much attention from economists and computer scientists. But neither in laboratories nor in practice has there been literature on the situations where both coalition formation and combinatorial auctions exist. In this paper we consider an e-market where each buyer places a bid on a combination of items with a reservation cost, and sellers offer price discounts for each item based on volumes. We call coalition formation under this condition a Combinatorial Coalition Formation (CCF) problem since coalition formation is motivated by price discounts on single items while multiple items are complementary for buyers. By artificially dividing the reservation cost of each buyer appropriately among the items we can construct optimal coalitions with respect to each item. We then try to make these coalitions satisfy the complementarity of the items, and thus induce the optimal solution. Based on this idea we present polynomial-time algorithms to find a semi-optimal solution of CCF and a payoff division scheme that is in the core of the coalition when linear price functions are applied, and in the pseudo-core when general price functions are applied. Simulation results show that the algorithms obtain solutions in a satisfactory ratio to the optimal value.

1 Introduction

Coalition formation and combinatorial auctions have received much attention from both economists and computer scientists. By offering price discounts suppliers drive customers to buy in wholesale lots while by forming coalitions customers take advantage of the price discount without purchasing more than their real demand [18]. Auction is an efficient mechanism to allocate resources when a real market in which each resource is equipped with a price does not exist. We say goods g and h are complementary to buyer b if $u_b(\{g,h\}) > u_b(\{g\}) + u_b(\{h\})$, where $u_b(G)$ is the utility of the set of goods G to b [31]. For example, a customer may want to buy both a cellular phone and a battery but does not need only a cellular phone without a battery or a battery without a cellular phone. In combinatorial auctions bidders can express the complementarity of items explicitly by placing bids on combinations of items [25]. In the example above, the customer can place a bid including both the cellular phone and the battery with a reservation cost, which is equal to the utility of both of them to the customer.

It is common in real markets that price discounts and complementarity among items exist simultaneously. In this condition coalition formation and combinatorial auctions are both needed to improve the efficiency of the markets. Suppose in the example above there are two buyers b_1 and b_2. They both need a cellular phone g_1 and a battery g_2. A customer needs to pay \$500 to buy one unit of g_1 or \$405 for each to buy two units. Also one needs to pay \$50 to buy one unit of g_2 or \$40 for each to buy two units. The utility of both of g_1 and g_2 for each buyer is \$450, while either g_1 or g_2 values zero for both b_1 and b_2. Suppose only the mechanism of coalition formation is considered. b_1 and b_2 need to split the value of $\{g_1, g_2\}$ and bid for g_1 and g_2 separately. Assume $p_{b_1}(\{g_1\}) = \$405$, $p_{b_2}(\{g_1\}) = \$400$, $p_{b_1}(\{g_2\}) = \$45$, $p_{b_2}(\{g_2\}) = \$50$, where $p_b(G)$ is the bidding price of buyer b for a set of goods G. The result of the bidding is that b_1 and b_2 both only get g_2, which has no use for them, although they enjoy the price discount of g_2 by coalition. On the other side suppose only the mechanism of combinatorial auctions is considered. Let $p_{b_1}(\{g_1, g_2\}) = p_{b_2}(\{g_1, g_2\}) = \$450$, which are the highest bidding prices they can endure. In this case neither b_1 nor b_2 can win their bids without coalition. But when both of the mechanisms are applied, then both b_1 and b_2 can win their bids and furthermore have \$10 profit together.

Although economists have provided much insight into the stability analysis of coalitions [2, 4] and mechanism design of combinatorial auctions [40, 41, 25], both the determination of optimal coalition structure and stable payoff division in coalition formation problems, and winner determination in combinatorial auction problems are computationally intractable. There is some research discussing the computational problems by computer scientists in each of these two fields (for example, [12, 13, 15, 19, 14] in coalition formation, and [31, 33, 26, 28, 29, 30, 32, 34] in winner determination in combinatorial auctions), but there is limited work to date on considering both behaviors simultaneously.

In this paper we consider an electronic market in which both coalition formation and combinatorial auction exist. There are many sellers and buyers with some items traded from sellers to buyers in the market. Each buyer may want a bundle of items that complement each other and have a reservation cost, the highest cost he can pay for his request. Suppose a one-round sealed auction mechanism (the buyers submit sealed bids and the allocation and payment are determined at one round of bidding) is applied in the market and each buyer unstrategically places a bid including all the items he desires with his reservation cost. Assume a fractional part of the bid has value zero for the buyer and will not be accepted[1]. Each seller has a price schedule for each item he provides and offers price discounts in each transaction based on the quantity of the item that is sold. The larger the quantity, the lower the price. By forming coalitions buyers can enlarge the quantity in each transaction and take advantage of price discounts. We call such a coalition formation problem Combinatorial Coalition Formation (CCF) since the coalition formation is motivated by price discounts of single items while multiple items are complementary for buyers. Since buyers are self-interested a stable payoff division mechanism for each coalition is needed so that no coalition members have incentive to leave the coalitions. In this paper we consider the problem of combinatorial coalition formation and payoff division in such an electronic market. The objective is to find a subset of the buyers who have the maximum coalition value among all the possible coalitions and distribute the payoff among the coalition members such that the division is in the core (we say a payoff division of a coalition is in the core if the payoff assigned to any subset of the coalition is no less than the profit that the subset can make by themselves. A coalition with the payoff division in the core is stable since no members have incentives to leave the coalition.).

In [1] an efficient algorithm of coalition formation and payoff division has been given for the case when each buyer wants an XOR listing of multiple items within a category (for instance,

[1]Otherwise the buyer needs to place multiple bids, one for each set of items that has positive value for him. This more general case is discussed at the end of the paper.

a buyer wants to buy either a camera A for 300 or lower, or a camera B for 400 or lower). It gives an algorithm to form an optimal coalition for each item and a mechanism to distribute the payoff for one coalition in the core. A suboptimal coalition configuration is constructed by choosing in each round a coalition with the maximal value among all the optimal coalitions, which are formed one for each item by the buyers who have not been picked into the coalition configuration. Based on the approach with respect to one item in that paper, our solution to CCF is as follows. Suppose each buyer has a virtual reservation cost for each item he desires. The sum of the virtual reservation costs of a buyer is equal to his real reservation cost for his whole request. Based on the virtual reservation costs the optimal coalition of each item (called subcoalition) can be constructed and the surplus is shared in a way stable in the core. We prove that when the optimal subcoalitions are compatible (satisfying the complementarity of the items for each buyer), the coalition induced from the subcoalitions is optimal and the payoff division by summing up the sharing in the subcoalitions for each buyer is in the core of the optimal coalition. A virtual reservation cost transfer scheme is proposed to reach compatible optimal subcoalitions when they exist. Evolution of the optimal subcoalitions in the transfer procedure is analyzed. Based on the analysis we give two polynomial-time approximation algorithms for coalition formation in the two situations where linear price functions and general price functions are applied. The payoff division obtained is in the core with linear price functions and in the pseudo-core with general price functions. Simulation results show that the approximation algorithms reach a solution with good ratio to the optimal value.

The paper is organized as follows. Section 2 formulates the problem mathematically. In Section 3 the approach is introduced and analyzed. The algorithm for optimal subcoalition formation, the basic transfer scheme, the approximation algorithms of coalition formation in the conditions of linear and general price functions are presented correspondingly in Section 4, 5, 6 and 7. Section 8 gives the experimental results and Section 9 presents some conclusion.

2 Problem Formulation

An electronic market is composed of buyers, sellers and items.

Let $G = \{g_1, g_2, \ldots, g_K\}$ indexed by k, $B = \{b_1, b_2, \ldots, b_N\}$ indexed by n, and $E = \{e_1, e_2, \ldots, e_L\}$ indexed by l denote the collection of items, buyers and sellers respectively. Each buyer b_n places a bid, $bid_n = \{Q_n, r_n\}$, where $Q_n = \{q_n^1, \ldots, q_n^K\}$ is the quantity of each item that b_n requests, and r_n is the reservation cost, the highest cost that b_n can pay for his request Q_n. Denote by $G_n = \{g_k \in G | q_n^k > 0\}$ the set of items that b_n requests, and by $B_k = \{b_n \in B | q_n^k > 0\}$ the set of buyers that request g_k. Each seller has a price schedule $p_{l,k}(\cdot) : Z^+ \to R^+$ for each item g_k he provides. The function $p_{l,k}(m)$ is a decreasing step function of m, which represents the unit price of g_k when m units of g_k are sold together. If we assume the sellers have no capacity constraint then we can obtain an integrated price schedule $p_k(m) : Z^+ \to R^+$ for each item which is the minimum unit price of g_k when m units are sold together among all those price functions $p_{l,k}(m)$ offered by the sellers.

A **coalition** C is a subset of the buyers with a **coalition value** which is the difference between the sum of the reservation cost of the coalition members and the minimum cost needed to satisfy the requests of all the members:

$$v(C) = \sum_{b_n \in C} (r_n - \sum_{k=1}^{K} q_n^k \times p_k(q_C^k))$$

where

$$q_C^k = \sum_{b_n \in C} q_n^k$$

A **payoff division** X_C of a coalition C is a vector $\{x_C(b) : b \in C\}$ with the sum of the elements equal to the value of C:

$$\sum_{b \in C} x_C(b) = v(C)$$

The **core** of a coalition C is the collection of all payoff divisions of C such that each element X_C satisfies, for any $C' \subset C$,

$$v_{C'} \leq x_C(C')$$

where $x_C(C') = \sum_{b \in C'} x_C(b)$. With a payoff division in the core any subset of the coalition members can get at least as much by joining the coalition as the value of the coalition formed by themselves.

If the payoff division X_C is in the core of the coalition C, we say C is stable in the core.

The objective of the problem is to find a set of exclusive coalitions such that the sum of the coalition values is maximized, and to distribute the payoff for each coalition such that they are stable in the core. Under the assumption that the price functions are decreasing, we have the conclusion that the values of disjoint coalitions are superadditive:

Claim 1 *If* $C^1, C^2 \subset B$, $C^1 \cap C^2 = \emptyset$, *then*

$$v(C^1 \cup C^2) \geq v(C^1) + v(C^2).$$

Proof: Since the price functions are decreasing for any $g_k \in I$, the purchasing price of a coalition for an item is no greater than the price in any subset of the coalition, and thus the purchasing cost of a coalition is no greater than the sum of the purchasing costs of all subsets which form a partition of the coalition:

$$p_k(q^k_{C^1 \cup C^2}) \leq min\{p_k(q^k_{C^1}), p_k(q^k_{C^2}))\}$$

$$p_k(q^k_{C^1 \cup C^2}) \cdot q^k_{C^1 \cup C^2} \leq p_k(q^k_{C^1}) \cdot q^k_{C^1} + p_k(q^k_{C^2}) \cdot q^k_{C^2}.$$

It follows that the value of the coalition $C^1 \cup C^2$ is no less than the sum of the values of C^1 and C^2.

$$\sum_{b \in C^1} r_b + \sum_{b \in C^2} r_b - \sum_{k \in G} p_k(q^k_{C^1 \cup C^2}) \cdot q^k_{C^1 \cup C^2} \geq \sum_{b \in C^1} r_b + \sum_{b \in C^2} r_b - \sum_{k \in G} p_k(q^k_{C^1}) \cdot q^k_{C^1} - \sum_{k \in G} p_k(q^k_{C^2}) \cdot q^k_{C^2}.$$

The left hand side of the inequality above is $v(C^1 \cup C^2)$ and the two items of the right hand side are $v(C^1)$ and $v(C^2)$ respectively. Therefore

$$v(C^1 \cup C^2) \geq v(C^1) + v(C^2).$$

□

From Claim 1 the optimal coalition configuration always consists of no more than one coalition which is the optimal coalition with the largest non-negative value among all possible coalitions. The problem is to find an optimal coalition C^* and the payoff division in the core of C^*.

Example 1 gives a simple example of the problem:

Example 1 *The set of buyers is* $B = \{b_1, b_2, b_3\}$, *the set of items is* $G = \{g_1, g_2\}$. *Buyer* b_1 *asks for one unit of* g_2 *with the reservation cost 1, b2 and b3 ask for one unit of both* g_1 *and* g_2 *with the reservation cost 5 and 6 respectively.*

b_n	q_n^1	q_n^2	r_n
b_1	0	1	1
b_2	1	1	5
b_3	1	1	6

The price functions for the two items are decreasing step functions:

$$p_1(m) = \begin{cases} 3 & \text{if} \quad 0 < m < 3 \\ 2 & \text{if} \quad m \geq 3 \end{cases} \qquad p_2(m) = \begin{cases} 3 & \text{if} \quad 0 < m < 2 \\ 2 & \text{if} \quad m \geq 2 \end{cases}$$

where m *is the quantity of each item sold together.*

We can see that an optimal coalition C^* is $\{b2, b3\}$. The value of C^* is $v(C^*) = 1$. The payoff division $X_{C^*} = \{0, 1\}$ is in the core of C^*.

3 Approach

3.1 Reservation cost transfer scheme

Considering that coalition formation is motivated by price discounts of each item, and multiple items requested by a buyer complement each other, we can artificially divide the reservation cost r_n of each buyer to r_n^k for each item such that $r_n = \sum_{g_k \in G} r_n^k$, find the optimal coalitions for each item with the reservation cost division and then balance the coalitions to satisfy the complementarity of the items required by a buyer. We call a coalition with respect to item g_k a subcoalition denoted by C_k.

Definition 1 (Reservation cost division) *A reservation cost division* $RD \in R^{K^N}$ *is a set of* K*-dimensional real numbers* $\{\{r_n^k\}_{k=1}^K\}_{n=1}^N$ *satisfying*

$$\sum_{k=1}^{K} r_n^k = r_n$$

and $r_n^k = 0$ *if* $q_n^k = 0$.

Call r_n^k *the virtual reservation cost and* $p_n^k = r_n^k / q_n^k$ *the virtual reservation price of buyer* b_n *for item* g_k.

Definition 2 (Subcoalition) *A subcoalition* $C_k \subset B_k$ *with respect to item* g_k *is a subset of buyers requesting* g_k *with the coalition value* $v_k(C_k)$ *equal to*

$$v_k(C_k) = \sum_{b_n \in C_k} (r_n^k - q_n^k \cdot p_k(\sum_{b_n \in C_k} q_n^k)) \tag{1}$$

Denote by $C_k^*(RD)$ *the optimal subcoalition of the item* g_k *with the reservation cost division* RD.

With a reservation cost division a set of optimal subcoalitions can be constructed one for each item. If a buyer b_n is involved in all or none of the subcoalitions of the items he requests, we say that the subcoalitions are compatible with respect to the buyer b_n.

Definition 3 (Compatible) *A set of subcoalitions* $C_1, \ldots, C_K$ *is compatible with respect to the buyer* b_n *if* $b_n \in \bigcap_{g_k \in G_n} C_k$ *or* $b_n \notin \bigcup_{g_k \in G_n} C_k$ *where* $G_n = \{g_k \in G | q_n^k > 0\}$.

A set of subcoalitions are compatible if they are compatible with respect to all the buyers. If the set of subcoalitions $C_1, \ldots, C_K$ *are compatible, we can induce a coalition* $C = C_1 \cup C_2 \ldots \cup C_K$ *which is composed of all the members in the subcoalitions. From a coalition* C *we can induce a subcoalition* $(C)_k = C \cap B_k$ *for the item* g_k *which is composed of all the members in* C *that request* g_k. *The subcoalitions induced from a coalition are compatible. When the coalition* C *is induced from a set of compatible optimal subcoalitions* $C_1, \ldots, C_K$, *we have* $(C)_k = C_k$.

We now prove some properties of optimal subcoalitions: The coalition induced by compatible optimal subcoalitions is optimal. Furthermore, if the payoff division for each subcoalition is in the core of the subcoalitions, then if we let the payoff of a member in the induced coalition be the sum of his payoffs in the subcoalitions, we get a payoff division which is in the core of the optimal coalition.

Claim 2 *Let* C_k^* *be an optimal subcoalition with the reservation cost division* RD *with respect to the item* g_k, *with the payoff division* $X_{C_k^*}$ *in the core of* C_k^*. *If the optimal subcoalitions* *of all the items, C_k^*, $k = 1, \ldots, K$, are compatible then $C^* = \bigcup_{k=1}^K C_k^*$ is an optimal coalition and $X_{C^*} = \{x_{C^*}(b_n) | b_n \in C^*, x_{C^*}(b_n) = \sum_{k: b_n \in C_k^*(RD)} x_{C_k^*}(b_n)\}$ is in the core of C^*.*

Proof: For any coalition $C \subseteq B$

$$v(C) = \sum_{b_n \in C} [r_n - \sum_{k=1}^K q_n^k \times p_k(q_C^k)].$$

Since $\sum_{k=1}^K r_n^k = r_n$,

$$v(C) = \sum_{k=1}^K \sum_{b_n \in C} [r_n^k - q_n^k \times p_k(q_C^k)].$$

The items to be summed up at the outer round in the equation above are the value of the subcoalitions induced from C,

$$v_k((C)_k) = \sum_{b_n \in C} [r_n^k - q_n^k \times p_k(q_C^k)].$$

It follows that

$$v(C) = \sum_{k=1}^K v_k((C)_k) \tag{2}$$

(i) Prove C^* is an optimal coalition:

Suppose $C \neq C^*$ and $v(C) > v(C^*)$, from Equation 2 we have

$$\sum_{k=1}^K v_k((C)_k) > \sum_{k=1}^K v_k((C^*)_k).$$

There exists some k such that $v_k((C)_k) > v_k((C^*)_k)$. But $(C^*)_k = C_k^*$. This contradicts that C_k^* is an optimal subcoalition with respect to g_k.

(ii) Prove X_{C^*} is in the core of C^*:

For any $C \subset C^*$, the payoff of C according to X_{C^*} is equal to the sum of the payoff of the subcoalitions $(C)_k$ induced from C according to the payoff division of the optimal subcoalitions $X_{C_k^*}$,

$$x_{C^*}(C) = \sum_{k=1}^{K} x_{C_k^*}((C)_k).$$

Since the subcoalitions C_k^*, $k = 1, \ldots, K$, are stable in the core,

$$v_k((C)_k) \leq x_{C_k^*}((C)_k).$$

It follows from Equation 2 that

$$v(C) \leq x_{C^*}(C).$$

□

Balancing the subcoalitions to make them compatible can be realized by transferring payoffs between the subcoalitions. For one buyer who is involved in the optimal subcoalitions of some of the items he desires but not in the others, if his contribution to the former subcoalitions covers the loss of the later ones caused by his joining, the buyer can be accepted by all the subcoalitions by having the former subcoalitions transfer some payoff to the later ones to compensate for their loss, else he is rejected by all the subcoalitions since he causes more loss to some subcoalitions than benefit to the others. The contribution of a buyer to a subcoalition, however, is decided by his virtual reservation cost for the item.(We can see from Equation 1 that the value of a subcoalition C_k is an increasing function of the virtual reservation cost of the members for the item g_k). The more the virtual reservation cost, the more the contribution and the higher the coalition payoff. Therefore this leads to the transfer of virtual reservation costs of buyers among items. For one buyer that is involved in some of the optimal subcoalitions he desires but not the others, we can make the optimal subcoalitions compatible with respect to him by transferring some virtual reservation cost of the buyer from the items with optimal subcoalitions involving him, to those he desires but with the optimal subcoalitions not involving him. If the transfers end up with a reservation cost division such that all the optimal subcoalitions are compatible, then the coalition induced by the subcoalitions is optimal.

Example 2 *Take the electronic market described in Example 1 for an example. Suppose the initial reservation cost division* RD^0 *is:*

b_n	r_n	r_n^1	r_n^2
b_1	1	0	1
b_2	5	2	3
b_3	6	3	3

The optimal subcoalitions with RD^0 of each item are: $C_1^*(RD^0) = \{b_3\}$, $C_2^*(RD^0) = \{b_2, b_3\}$. The two subcoalitions are compatible with respect to b_1 and b_3 but not b_2. If b_2 has 1 virtual reservation cost transferred from item g_2 to g_1, then with the new reservation cost division RD^1

b_n	r_n	r_n^1	r_n^2
b_1	1	0	1
b_2	5	3	2
b_3	6	3	3

we have compatible optimal subcoalitions $C_1^*(RD^1) = \{b_2, b_3\}$, $C_2^*(RD^1) = \{b_2, b_3\}$. The coalition $C^* = \{b_2, b_3\}$ induced from $C_1^*(RD^1)$ and $C_2^*(RD^1)$ is the optimal coalition. The payoff divisions $X_1(C_1^*(RD^1)) = \{0, 0\}$, $X_2(C_2^*(RD^1)) = \{0, 1\}$ are in the core of $C_1^*(RD^1)$ and $C_2^*(RD^1)$ respectively. Then $X(C^*) = \{0, 1\}$ constructed by summing up $X_1(C_1^*(RD^1))$ and $X_2(C_2^*(RD^1))$ is in the core of C^*.

3.2 Existence of compatible optimal subcoalitions

Although we can find an optimal coalition by inducing one from compatible optimal subcoalitions, compatible optimal subcoalitions do not exist for all problem instances. Example 3 is such a counter example.

Example 3 *The set of buyers is $B = \{b_1, b_2, b_3\}$. The set of items is $G = \{g_1, g_2\}$. The quantities requested for each item and the reservation cost of the buyers are listed in the following table:*

b_n	q_n^1	q_n^2	r_n
b_1	1	1	3
b_2	1	1	3.5
b_3	1	1	6

The price schedules for the two items are:

$$p_1(m) = \begin{cases} 3 & if \quad 0 < m < 3 \\ 2 & if \quad \quad m \geq 3 \end{cases} \quad p_2(m) = \begin{cases} 3 & if \quad 0 < m < 2 \\ 2 & if \quad \quad m \geq 2 \end{cases}$$

where m denotes the number of units of the item that are sold together.

The optimal coalition for this instance is $\{b_1, b_2, b_3\}$. But there exist no compatible optimal subcoalitions with any reservation cost division. By dividing r_3 to $r_3^1 = 3$,$r_3^2 = 3$, b_3 can always stay in the optimal subcoalitions. But there exist no division for r_1 and r_2 such that the optimal subcoalitions are compatible with respect to both b_1 and b_2.

Although the existence of compatible optimal subcoalitions is not guaranteed for the problems with general price functions(decreasing step functions), it is always true for the problems with linear price functions(the price decreases at a constant step when the quantity increases by one unit).

Definition 4 (Linear price function) *A linear price function $p_k(m) : Z^+ \to R^+$ for the item g_k is expressed as*

$$p_k(m) = -d_k \cdot m + a_k$$

where $d_k, a_k \in R^+$ and $m \leq a_k/(2d_k)$. (By bounding m from above we ensure that the purchasing cost $p_k(m) \cdot m$ is an increasing function of the quantity m.)

Claim 3 *Suppose the price functions are linear price functions, then there exist some reservation cost division RD such that the optimal subcoalitions $C_k^*(RD)$ are compatible.*

Before proving this claim we need to prove a lemma.

Lemma 1 *Let the collection of buyers $B = B_1 \cup B_2$. Let the optimal coalitions of B, B_1 be C, C^1 respectively . Then $C \supseteq C^1$ with linear price functions.*

Proof: Suppose $C^1 \not\subseteq C$. Let $C^0 = C^1 \cap C$ and $C'^0 = C^1 \setminus C^0$. Then $C'^0 \neq \emptyset$. Let $r_C = \sum_{b_n \in C} r_n$. From the definition of coalition value and linear price functions we have

$$v(C) - v(C^0) = r_{C \setminus C^0} - \sum_{k=1}^{K} [q_{C \setminus C^0}^k \cdot (-d_k \cdot q_{C \setminus C^0}^k + a_k) - 2q_{C^0}^k \cdot d_k \cdot q_{C \setminus C^0}^k] \tag{3}$$

and

$$v(C \cup C'^0) - v(C^1) = r_{C\setminus C^0} - \sum_{k=1}^{K}[q^k_{C\setminus C^0} \cdot (-d_k \cdot q^k_{C\setminus C^0} + a_k) - 2q^k_{C^1} \cdot d_k \cdot q^k_{C\setminus C^0}] \qquad (4)$$

Since $q^k_{C^1} \geq q^k_{C^0}$ and the strict inequality holds for at least one $g_i \in G$, by comparing Equation 3 and 4 we have

$$v(C \cup C'_0) - v(C^1) > v(C) - v(C^0).$$

$$v(C \cup C'_0) - v(C^1) > v(C) - v(C^1).$$

$$v(C \cup C'_0) > v(C).$$

This contradicts the optimality of C.□

Corollary 1 *With linear price functions the optimal coalition contains all the optimal coalitions of subsets of the buyers .*

Following Lemma 1 we have the proof of Claim 3.

Proof: By induction with the number of buyers N:

When $N = 1$ it holds straightforwardly.

Suppose the statement stands for $N \leq n$.

When $N = n + 1$: Let the optimal coalition of $B_0 = \{b_1, b_2, \dots, b_n\}$ be C^0. Let $B' = (B_0 \setminus C^0) \cup \{b_{n+1}\}$. Let the optimal coalition of $B = B_0 \cup \{b_{n+1}\}$ be C. From Lemma 1 we have $C \supseteq C^0$. Then

$$\begin{aligned} C \setminus C^0 &= argmax_{T \subseteq (B \setminus C^0)} v(C^0 \cup T) \\ &= argmax_{T \subseteq (B \setminus C^0)} \{v(C^0 \cup T) - v(C^0)\} \\ &= argmax_{T \subseteq (B \setminus C^0)} \{r_T - \sum_{k=1}^{K}[q^k_T(-d_k \cdot q^k_T + a_k - d_k \cdot 2q^k_{C^0})]\} \end{aligned}$$

Therefore $C \setminus C^0$ is an optimal coalition of $B \setminus C^0$ with the price functions $p_k^{'}(m) = -d_k \cdot m + a_k - d_k \cdot 2q_{C^0}^k$, $k = 1, \ldots, K$. This is a linear price function and from the assumption of inductions there exists some reservation cost division for all the buyers $b_i \in B \setminus C^0$ such that $(C \setminus C^0)_k$ is an optimal subcoalition among all the subsets of $(B \setminus C^0) \cap B_k$ with respect to each item g_k with the linear price function $p_k^{'}(m)$. With this reservation cost division $\{r_i^k\}$, $k = 1, \ldots, K$, $b_i \in B \setminus C^0$,

$$
\begin{aligned}
(C \setminus C^0)_k &= argmax_{T \subseteq (B \setminus C^0) \cap B_k} \{r_T^k - q_T^k(-d_k \cdot q_T^k + a_k - d_k \cdot 2q_{C^0}^k)\} \\
&= argmax_{T \subseteq (B_k \setminus (C^0)_k)} \{v_k((T \cup C^0)_k) - v_k((C^0)_k)\} \\
&= argmax_{T \subseteq (B_k \setminus (C^0)_k)} v_k((T)_k \cup (C^0)_k) \qquad (5)
\end{aligned}
$$

From Lemma 1 the optimal subcoalition C^k of B_k with respect to item g_k includes $(C^0)_k$ for every k. Equation 5 means that $(C)_k = (C \setminus C^0)_k \cup (C^0)_k$ is an optimal subcoalition with respect to g_k if $(C^0)_k$ is an optimal subcoalition of the buyers in $(C^0)_k$ with respect to g_k. Let the reservation cost division of C^0 to be that with which $(C^0)_k$ is optimal with respect to the item g_k among all the subsets of $(C^0)_k$ for every k.(From the assumption of inductions such a reservation cost division exists.) Then $(C)_k$ is an optimal subcoalition of the item g_k with the reservation cost division stated above and $(C)_k$, $k = 1, \ldots, K$ are compatible. $\square$

Although the existence of compatible optimal subcoalitions is not guaranteed for the systems with general price functions, the approach of virtual reservation cost transfer still helps to give a solution for the problem with general price functions, as shown in Section 7.

Based on the approach we need to answer the following questions:

- How to efficiently form an optimal subcoalition and distribute the payoff in the core
- How to transfer the virtual reservation cost among items to make the optimal subcoalitions compatible if they exist
- How to reduce the computational complexity and construct an approximation algorithm in polynomial time for the system with linear price functions
- How to construct an approximation algorithm in polynomial time for the system with general price functions based on the virtual reservation cost transfer approach

These problems are solved in Section 4, 5, 6 and 7 respectively. The notations with a "$'$" are used to denote the terms with the reservation cost division after a transfer.

4 Optimal subcoalition formation and payoff division

Optimal subcoalition formation is a basic component of the reservation cost transfer scheme. In [1] an efficient and accurate algorithm for subcoalition formation and stable payoff division is given for the situation where each buyer asks for one unit of the items. With some modifications the algorithm can be extended to the situation with multiple units.

4.1 Algorithm for optimal subcoalition formation

Claim 4 *Suppose b_i and b_j ask for the same quantity of g_k with virtual reservation cost r_i^k and r_j^k respectively, $r_i^k > r_j^k$. Let the optimal subcoalition of g_k be C_k^*. If $b_j \in C_k^*$, then $b_i \in C_k^*$.*

Proof: If $b_i \notin C_k^*$, construct a subcoalition $C_k^{'}$ by replacing b_j with b_i in C_k^*. This does not change the discount price of the item with the subcoalition since b_i and b_j ask for the same quantity of g_k. But $r_i^k > r_j^k$, hence $v_k(C_k^*) < v_k(C_k^{'})$ and this contradicts the optimality of C_k^*.$\square$

From Claim 4 to form an optimal subcoalition for the item g_k, we can first sort the buyers with the same bid quantity for g_k by their virtual reservation cost in a descending order. The candidates for the optimal subcoalition can be constructed by extracting from each list buyers from the prefix of the list. The subcoalition with the largest value among these candidate subcoalitions is the optimal subcoalition. The algorithm is stated in Algorithm 1. The complexity of the algorithm is analyzed in Claim 5.

Algorithm 1 (Optimal subcoalition formation)

Step 1: Calculate the virtual reservation price of each buyer for the item

$$p_n^k = r_n^k / q_n^k$$

Step 2: Order the buyers with the same bid quantity for the item by their virtual reservation prices descendantly, and form M lists $qu_1, \ldots, qu_M$ where M is the number of bid quantities for the item.

Step 3: Construct all possible collections formed by the buyers from prefixes of the lists and calculate their values with respect to the item. The collection with the largest value is the optimal subcoalition. □

Claim 5 *The complexity of Algorithm 1 is $O(N^M)$ where M is the number of different bidding quantities for the item*[2]*, N is the number of buyers requesting the item.*

Proof: The number of coalitions to be considered is $\prod_{m=1}^{M}(K_m + 1)$ where K_m is the length of qu_m, the number of buyers who desire m units of the item. Since $K_m \leq N$,

$$\prod_{m=1}^{M}(K_m + 1) \leq (N+1)^M$$

The algorithm has complexity $O(N^M)$.□

4.2 Payoff division of a subcoalition

Claim 6 gives a strategy to divide the payoff of the optimal subcoalition such that the payoff division is in the core of the subcoalition. Claim 7 gives the computational complexity of the strategy.

Claim 6 *Let $cost_k(C_k)$ denote the purchase cost to satisfy the requests of the members in C_k, i.e. $cost_k(C_k) = q^k_{C_k} \cdot p_k(q^k_{C_k})$. The payoff division $X_k(C_k)$ of the subcoalition C_k is in the core of C_k with*

$$x^i_k = \begin{cases} (p^k_i - h_{C_k}) \cdot q^k_i & (b_i \in \overline{C}_k) \\ 0 & (b_i \notin \overline{C}_k) \end{cases}$$

where h_{C_k} and $\overline{C}_k$ satisfy

[2]Based on the goods traded in electronic markets to personal buyers, for instance, books, electrics, parts and accessories, we can reasonably assume M be a small number.

$$cost_k(C_k) = (\sum_{b_i \in \overline{C}_k} q_i^k) \cdot h_{C_k} + \sum_{b_i \in C_k \setminus \overline{C}_k} p_i^k \cdot q_i^k$$

$$\overline{C}_k = \{b_i \in C_k | h_{C_k} \leq p_i^k\}$$

Proof: The conclusion follows from Proposition 1 in [1] by regarding a buyer b_n asking for q_n^k units of g_k with the reservation cost r_n^k as q_n^k buyers each one asking for 1 unit of g_k with the same reservation price r_n^k/q_n^k.□

The rule of payoff division for a subcoalition is shown in Figure 1. Each bin by solid lines represents the bid quantity of a buyer for the item g_k, and is divided by dotted lines into units. The area of the bins below h_{C_k} is equal to the purchasing cost of the subcoalition C_k for the item g_k. The shaded area in each bin is equal to the payoff assigned to that buyer.

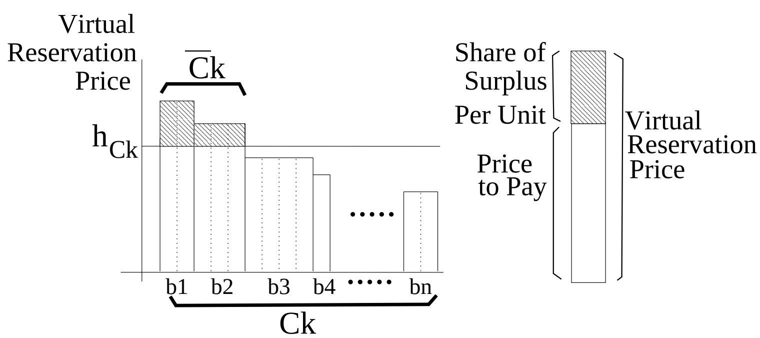

Figure 1: The payoff division rule for a subcoalition

Claim 7 *The payoff division scheme stated in Claim 6 can be implemented in $O(K \cdot N logN)$, where K is the number of items and N is the number of buyers.*

Proof: For one subcoalition C_k the payoff division can be implemented in three steps: First order the buyers by their virtual reservation price for the item g_k in a descending order $\{b^1, b^2, ..., b^J\}$. Second check the buyers one by one from the tail and choose the buyer b^{j^*} such that $\sum_{j \geq j^*+1} r_{b^j}^k + p_{b^{j^*+1}}^k \cdot \sum_{j < j^*+1} q_j^k < cost_k(C_k) \leq \sum_{j \geq j^*} r_{b^j}^k + p_{b^{j^*}}^k \cdot \sum_{j < j^*} q_j^k$. Third let $\overline{C}_k = \{b^j | j \leq j^*\}$ and $h_{C_k} = (cost_k(C_k) - \sum_{j > j^*} r_{b^j}^k)/q_{\overline{C}_k}^k$. The complexities of the three steps are $O(N\log N)$, $O(N)$ and $O(1)$ respectively. Therefore the complexity of payoff division for one subcoalition is $O(N\log N)$. When there are K items, we need to do once for each of the K subcoalitions and the complexity is $O(K \cdot N\log N)$.$\square$

4.3 In the sight of cooperative games

If we regard each buyer as a player this problem defines a cooperative game (B, v) where B is the set of players and $v(C)$, the value of a coalition $C \subset B$ is the characteristic function defined on every subset C of B.

The following definitions and theorems are noted in [2]

In a convex game the marginal contribution of a buyer b_n to a coalition not including b_n increases with expansion of the coalition.

Definition 5 (Convex Game) *A cooperative game (B, v) is convex if it satisfies one of the two following equivalent properties for all $b_n \in B$:*

for all $S, T \subset B \setminus \{b_n\}$:

$$\{S \subset T\} \Rightarrow \{v(S \cup \{b_n\}) - v(S) \leq v(T \cup \{b_n\}) - v(T)\}$$

and/or for all $S, T \subset B$:

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

Theorem 1 gives some ways to construct a payoff vector in the core for a convex game.

Definition 6 (Shapley value) *The Shapley value is a payoff vector which assigns a payoff*

$$\sum_{S \subset C \setminus b_i} (|S|!(n - |S| - 1)!/n!)(v(S \cup b_i) - v(S))$$

to each player b_i in the coalition C.

Theorem 1 *Let (B, v) be a convex game and C^* be an optimal coalition. Let C^* be ordered as $\{i_1, \ldots, i_n\}$ where n is the number of players in C^*. The payoff vector constructed by assigning marginal contributions for each item*

$$x_C(i_k) = v(i_1, \ldots, i_k) - v(i_1, \ldots, i_{k-1})$$

is in the core of C^*. Therefore the Shapley value is in the core, too.

For an optimal coalition C^* with n buyers, there are $n!$ ways to order the players in C^*. It is stated in [3] that the payoff vectors generated by the marginal contributions based on all the ordering construct the extreme points of the core of C^*, which is a convex space.

Based on the definitions and theorems above we can see that in our combinatorial coalition formation problem the set of buyers and the value function construct a convex game with linear price functions.

Claim 8 *Suppose linear price functions are applied. The set of buyers is B. The value function is v. Then (B, v) is a convex game.*

Proof: Denote by r_C the sum of reservation cost of all the players in $C \subset B$, by q_C^k the sum of the quantities asked by all the players in C for the item g_k. $r_C = \sum_{b_n \in C} r_n$, $q_C^k = \sum_{b_n \in C} q_n^k$. For any C', C'', B', $B'' \subset B$ and $C' \subset C''$, $B' \subset B''$, let $C = (C' \cup B'') \setminus (C' \cup B') = (C'' \cup B'') \setminus (C'' \cup B')$.

$$\begin{aligned}
v(C' \cup B'') - v(C' \cup B') &= r_C - \sum_{k=1}^{K} [p_k(q_{C' \cup B''}^k) \cdot q_{C' \cup B''}^k - p_k(q_{C' \cup B'}^k) \cdot q_{C' \cup B'}^k] \\
&= r_C + \sum_{k=1}^{K} [-a_k \cdot q_C^k + d_k((q_{C' \cup B''}^k)^2 - (q_{C' \cup B'}^k)^2)] \\
&= r_C + \sum_{k=1}^{K} [-a_k \cdot q_C^k + d_k(q_{C' \cup B''}^k + q_{C' \cup B'}^k) q_C^k]
\end{aligned}$$

Similarly

$$v(C'' \cup B'') - v(C'' \cup B') = r_C + \sum_{k=1}^{K} [-a_k \cdot q_C^k + d_k(q_{C'' \cup B''}^k + q_{C'' \cup B'}^k) q_C^k].$$

Since

$$q^k_{C'\cup B''} + q^k_{C'\cup B'} \leq q^k_{C''\cup B''} + q^k_{C''\cup B'}$$

we have

$$v(C' \cup B'') - v(C' \cup B') \leq v(C'' \cup B'') - v(C'' \cup B').$$

From Definition 5 (C^*, v) is a convex game.□

When the price functions are general decreasing step functions, (C^*, v) is not necessary a convex game since the purchasing cost of a coalition is not necessary submodular[3] on the coalitions.

Given linear price functions the payoff vectors generated by the marginal contribution and the Shapley value are also in the core of the optimal coalition. Compared to these our payoff division scheme proposed in Claim 6 has the following advantages:

- Symmetric: The payoff division proposed in Claim 6 does not depend on the ordering of players but only on the virtual reservation prices. As extreme points of the core the payoff vectors generated by the marginal contribution are dependent on the ordering of players and always give preference to the buyers ordered at the end.

- Low computational complexity: The computational complexity of the payoff vector proposed in Claim 6 is only $O(K \cdot N\log N)$(Claim 7) where K is the number of items and N is the number of buyers. But to compute the Shapley value we need to calculate the values of all coalitions of the players and the complexity is $O(2^N)$, although the Shapley value is at the centroid of the core([3]).

These points support the payoff division proposed in Claim 6 as distributing the payoff in a fairer way than the payoff vectors generated by the marginal contributions and in a more efficient way than the Shapley value.

[3]A function $f(\cdot)$ is submodular if $f(S_0 \cup S_1) - f(S_1) \leq f(S_0 \cup S_2) - f(S_2)$ for any $S_1 \supset S_2$.

5 Reservation cost transfer scheme

Definition 7 (Offer & Request) *Denote the optimal subcoalition of the item g_k by C_k^*.*

Offer Off_n^k of the buyer $b_n \in C_k^*$ with respect to the item g_k is the maximum amount of virtual reservation cost of b_n for g_k that can be reduced from r_n^k while keeping b_n in the optimal subcoalition of g_k and the reservation cost division of other buyers remains the same, i.e.,

$$Off_n^k = r_n^k - min\{r_n^k : b_n \in C_k^*(RD)\}$$

Request Req_n^k of the buyer $b_n \notin C_k^*$ with respect to the item g_k is the minimum amount of virtual reservation cost of b_n for g_k that needs to be added to r_n^k, such that b_n can join the optimal subcoalition of g_k while the reservation cost division of other buyers remains the same, i.e.,

$$Req_n^k = min\{r_n^k : b_n \in C_k^*(RD)\} - r_n^k$$

Requests and offers of a buyer b_n determine the range of virtual reservation cost of b_n to be transferred among items to make the optimal subcoalitions after the transfer for b_n compatible with respect to b_n.

Claim 9 *Suppose the optimal subcoalitions $C_1^*(RD), \ldots, C_K^*(RD)$ with the reservation cost division RD are not compatible with respect to the buyer b_n. The current reservation cost division of b_n is $\{r_n^1, \ldots, r_n^K\}$. Let $K_n^1 = \{g_k \in G_n | b_n \in C_k^*\}$ and $K_n^2 = \{g_k \in G_n | b_n \notin C_k^*\}$, $dif_n = \sum_{k:g_k \in K_n^1} Off_n^k - \sum_{k:g_k \in K_n^2} Req_n^k$ be the difference between the sum of the offers of b_n and the sum of the requests of b_n.*

(i) If $dif_n < 0$, then b_n cannot be involved in all the optimal subcoalitions he desires by a reservation cost transfer for b_n, but there exist some transfer for b_n by which b_n is excluded from all the optimal subcoalitions he desires.

(ii) If $dif_n > 0$, then b_n cannot be excluded from all the optimal subcoalitions he desires by a reservation cost transfer for b_n, but there exist some transfer for b_n by which b_n is involved in all the optimal subcoalitions he desires.

(iii) If $dif_n = 0$, then b_n can either be involved in or excluded from all the optimal subcoalitions he desires by a reservation cost transfer for b_n.

Proof:

(i) Suppose after a transfer for b_n, b_n is involved in all the subcoalitions he desires with the new reservation cost division $\{r_n^{1'}, \ldots, r_n^{K'}\}$. Then $r_n^k - r_n^{k'} \leq Off_n^k$ for $g_k \in K_n^1$ and $r_n^k + Req_n^k \leq r_n^{k'}$ for $g_k \in K_n^2$. The sum of the reservation costs of a buyer stays the same after the transfer, i.e., $\sum_{g_k \in K_n^1}(r_n^k - r_n^{k'}) = \sum_{g_k \in K_n^2}(r_n^{k'} - r_n^k)$. But $\sum_{g_k \in K_n^1}(r_n^k - r_n^{k'}) \leq \sum_{g_k \in K_n^1} Off_n^k$, $\sum_{g_k \in K_n^2}(r_n^{k'} - r_n^k) \geq \sum_{g_k \in K_n^2} Req_n^k$, therefore $\sum_{g_k \in K_n^1} Off_n^k \geq \sum_{g_k \in K_n^2} Req_n^k$. This contradicts the condition $dif_n < 0$.

There exist a set of numbers $\gamma_k \geq 0$ for $g_k \in G_n$ such that $\sum_{k=1}^{K} \gamma_k = -dif_n$. Let $r_n^{k'} = r_n^k - Off_n^k - \gamma_k$ for $g_k \in K_n^1$ and $r_n^{k'} = r_n^k + Req_n^k - \gamma_k$ for $g_k \in K_n^2$, then $\{r_n^{1'}, \ldots, r_n^{K'}\}$ defines a new reservation cost division for b_n with which b_n is excluded from all the optimal subcoalitions he desires.

(ii) It can be shown via a similar argument as in (i).

(iii) Let $r_n^{k'} = r_n^k - Off_n^k$ for $g_k \in K_n^1$ and $r_n^{k'} = r_n^k + Req_n^k$ for $g_k \in K_n^2$. The array $\{r_n^{1'}, \ldots, r_n^{K'}\}$ defines a new reservation cost for b_n with which there are multiple optimal subcoalitions for each item $g_k \in G_n$. The buyer b_n can be either involved in or excluded from all the optimal subcoalitions he desires.

□

The way to calculate requests and offers is stated in Claim 10. The offer Off_n^k is the difference between the value of the current optimal subcoalition C_k^* and the value of the optimal subcoalition without b_n. The request Req_n^k is the difference between the value of the current optimal subcoalition C_k^* and the value of the optimal subcoalition having b_n as a member.

Claim 10 *Let C_k^* be the optimal subcoalition of Item g_k with the reservation cost division $RD = \{r_n^k\}_{n=1,\ldots,N,\ k=1,\ldots,K}$. Suppose $b_n \in C_k^*$ and $b_l \notin C_k^*$. Let $C_k^{\bar{n}}$ be the optimal subcoalition of g_k among all those subcoalitions of g_k with the buyer set $B \setminus \{b_n\}$. Let C_k^l be the optimal subcoalition of g_k among all those subcoalitions of g_k having b_l as a member.*

(i) $Off_n^k = v_k(C_k^*) - v_k(C_k^{\bar{n}})$

(ii) $Req_l^k = v_k(C_k^*) - v_k(C_k^l)$

Proof: Use the notations with a "$'$" to denote the terms with the reservation cost division after the transfer(this is used throughout the following portions of the paper).

(i) Let the new reservation cost of b_n for g_k be $r_n^{k'} = r_n^k - \alpha$. b_n remains in the optimal subcoalition of g_k with the reservation cost $r_n^{k'}$. Since the reservation cost division of the buyers except b_n remains, the optimal subcoalition of g_k among all those subcoalitions of g_k having b_n as a member after the transfer is same as the optimal subcoalition of g_k before the transfer, i.e., $C_k^{n'} = C_k^*$. But the value of the coalition changes with the transfer, $v_k'(C_k^{n'}) = v_k(C_k^*) - r_n^k + r_n^{k'} = v_k(C_k^*) - \alpha$. For b_n to remain in the optimal subcoalition of g_k, $v_k'(C_k^{n'}) \geq v_k'(C_k^{\bar{n}'})$ need to be satisfied. But $C_k^{\bar{n}} = C_k^{\bar{n}'}$, so it follows that $\alpha \leq v_k(C_k^*) - v_k(C_k^{\bar{n}})$. Hence $Off_n^k = v_k(C_k^*) - v_k(C_k^{\bar{n}})$.

(ii) A similar argument as in (i) can be used.

□

In Example 2 since $v_1(C_1^{\bar{3}}) = v_1(\emptyset) = 0$,$v_1(C_1^2) = v_1(\{b_2, b_3\}) = -1$, $v_1(C_1^*) = v_1(\{b_3\}) = 0$, the offer of b_3 with respect to g_1 is $Off_3^1 = 0 - 0 = 0$ and the request of b_2 with respect to g_1 is $Req_2^1 = 0 - (-1) = 1$.

Claim 11 states that the offer of a member in an optimal subcoalition is an increasing function of the virtual reservation cost of other members in the optimal subcoalition for the item.

Claim 11 *Make a transfer for $b_i \in B_k$ such that $r_i^{k'} < r_i^k$. Suppose b_i and b_j are members of the optimal subcoalition of the item g_k both before and after the transfer. Then $Off_j^{k'} \leq Off_j^k$.*

Proof: From Claim 13 we have $C_k^{*'} = C_k^*$. Let the new reservation cost of b_i for g_k be $r_l^{k'} = r_l^k - \alpha$. Then $v_k'(C_k^{*'}) = v_k(C_k^*) - \alpha$. But $v_k'(C_k^{\bar{j}'}) \geq v_k(C_k^{\bar{j}}) - \alpha$ where the equality holds when $b_i \in C_k^{\bar{j}'}$. Therefore $v_k'(C_k^{*'}) - v_k'(C_k^{\bar{j}'}) \leq v_k(C_k^*) - v_k(C_k^{\bar{j}})$ and $Off_j^{k'} \leq Off_j^k$. □

Claim 12 states that with linear price functions the offer of a member in an optimal subcoalition is an increasing function and the request of a nonmember is a decreasing function of the virtual reservation cost of other buyers for the item.

Claim 12 *Suppose linear price functions are applied. Make transfer for $b_n \in B_k$ such that $r_n^{k'} > r_n^k$. Suppose b_i is a member while b_j is a nonmember of the optimal subcoalition of the item g_k both before and after the transfer. Then*

(i) $Off_i^{k'} \geq Off_i^k$

(ii) $Req_j^{k'} \leq Req_i^k$

Proof:

(i) From Corollary 1 $C_k^{\bar{i}} \subset C_k^*$ and $C_k^{\bar{i}'} \subset C_k^{*'}$. Discuss every possible condition:

(1) $b_n \in C_k^{\bar{i}}$.
Then $b_n \in C_k^{\bar{i}'}$ and $C_k^{\bar{i}'} = C_k^{\bar{i}}$. It follows that $b_n \in C_k^*$, $b_n \in C_k^{*'}$ and $C_k^* = C_k^{*'}$. Therefore $v_k'(C_k^{*'}) - v_k'(C_k^{\bar{i}'}) = v_k(C_k^*) - v_k(C_k^{\bar{i}})$ and $Off_i^{k'} = Off_i^k$.

(2) $b_n \notin C_k^{\bar{i}}$ and $b_n \notin C_k^{\bar{i}'}$.
Then $C_k^{\bar{i}'} = C_k^{\bar{i}}$ and $v_k'(C_k^{\bar{i}'}) = v_k(C_k^{\bar{i}})$. Since $v_k'(C_k^{*'}) \geq v_k(C_k^*)$, we have $v_k'(C_k^{*'}) - v_k'(C_k^{\bar{i}'}) \geq v_k(C_k^*) - v_k(C_k^{\bar{i}})$ and $Off_i^{k'} \geq Off_i^k$.

(3) $b_n \in C_k^*$ and $b_n \notin C_k^{\bar{i}}$ and $b_n \in C_k^{\bar{i}'}$.
Then $b_n \in C_k^{*'}$. Since it is a convex game and $C_k^{\bar{i}'} \supset C_k^{\bar{i}}$, $v_k(C_k^{\bar{i}'} \cup (C_k^* \setminus C_k^{\bar{i}})) - v_k(C_k^{\bar{i}'}) \geq v_k(C_k^*) - v_k(C_k^{\bar{i}})$. Then $v_k'(C_k^{\bar{i}'} \cup (C_k^* \setminus C_k^{\bar{i}})) - v_k'(C_k^{\bar{i}'}) \geq v_k(C_k^{\bar{i}}) - v_k(C_k^*)$. But $v_k'(C_k^{\bar{i}'} \cup (C_k^* \setminus C_k^{\bar{i}})) \leq v_k'(C_k^{*'})$. Therefore $v_k'(C_k^{*'}) - v_k'(C_k^{\bar{i}'}) \geq v_k'(C_k^*) - v_k(C_k^{\bar{i}})$ and $Off_i^{k'} \geq Off_i^k$.

(4) $b_n \notin C_k^*$ and $b_n \notin C_k^{*'}$.
Then $C_k^{*'} = C_k^*$ and $v_k'(C_k^{*'}) = v_k(C_k^*)$. It follows that $b_n \notin C_k^{\bar{i}}$ and $b_n \notin C_k^{\bar{i}'}$. From (2) $Off_i^{k'} \geq Off_i^k$.

(5) $b_n \notin C_k^*$ and $b_n \in C_k^{*'}$ and $b_n \notin C_k^{\bar{i}}$ and $b_n \in C_k^{\bar{i}'}$.
Since it is a convex game and $C_k^* \supset C_k^{\bar{i}}$, $v_k(C_k^* \cup (C_k^{\bar{i}'} \setminus C_k^{\bar{i}})) - v_k(C_k^*) \geq v_k(C_k^{\bar{i}'}) - v_k(C_k^{\bar{i}})$. Then $v_k'(C_k^* \cup (C_k^{\bar{i}'} \setminus C_k^{\bar{i}})) - v_k(C_k^*) \geq v_k'(C_k^{\bar{i}'}) - v_k(C_k^{\bar{i}})$. But $v_k'(C_k^* \cup (C_k^{\bar{i}'} \setminus C_k^{\bar{i}})) \leq v_k'(C_k^{*'})$. Therefore $v_k'(C_k^{*'} - v_k(C_k^*) \geq v_k'(C_k^{\bar{i}'}) - v_k(C_k^{\bar{i}})$ and $Off_i^{k'} \geq Off_i^k$.

(ii) From $C_k^j \supset C_k^*$ and $C_k^{j'} \supset C_k^j$, we have the conclusion following the same way.

□

The general reservation cost transfer scheme is stated in Algorithm 2. It starts with an initial reservation cost division and checks the buyers one by one. If the optimal subcoalitions are not compatible with respect to the buyer b_n, b_n makes a virtual reservation cost transfer such that the new optimal subcoalitions after the transfer are compatible with respect to b_n. It stops when the optimal subcoalitions are compatible with respect to all the buyers.

Algorithm 2 (General reservation cost transfer algorithm)

Step 0: Initialization

Make an initial reservation cost division $RD = \{r_n^k\}_{n=1,\ldots,N,\ k=1,\ldots,K}$ *such that* $\sum_{k=1}^{K} r_n^k = r_n$ *for all buyers* $b_n \in B$.

Step 1: Optimal subcoalition formation

Construct the optimal subcoalition for each item following Algorithm 1.

Step 2: Transfer

If there exist a buyer b_n *with whom the optimal subcoalitions are not compatible, change the division of* r_n *by some transfer unless all the offers and requests are zero, update the optimal subcoalitions so that they are compatible with respect to* b_n *with the new reservation cost division.*

Step 3: Termination judgment

If the optimal subcoalitions are compatible with respect to all the buyers, stop; else repeat Step 2.□

The reservation cost transfer for one buyer b_n affects not only the status(in or out of the optimal subcoalitions) of b_n but also the status of other buyers. The following claims help us to understand the stability of the optimal subcoalitions in the procedure of transfers.

Claim 13 *Denote by* $C_k^*(RD)$ *the optimal subcoalition of the item* g_k *with the reservation cost division* RD. *Make transfer for* $b_n \in C_k^*(RD)$ *such that* $b_n \in C_k^{*'}(RD')$ *where* $C_k^{*'}(RD')$ *is the optimal subcoalition of* g_k *with the new reservation cost division* RD'. *Then* $C_k^{*'}(RD') = C_k^*(RD)$.

Proof: Let $C_k^*(RD)$ be denoted by C_k and $C_k^{*'}(RD')$ be denoted by $C_k^{'}$. Denote by $v_k^{'}(\cdot)$ the value of subcoalitions with the reservation cost division $RD^{'}$ and by v_k the value of subcoalitions with RD. Suppose $C_k \neq C_k^{'}$ and $v_k^{'}(C_k^{'}) > v_k^{'}(C_k)$. Let $C_k^0 = C_k^{'} \cap C_k$. Let

$$\gamma = v_k(C_k) - v_k(C_k^0) \tag{6}$$

$$\gamma^{'} = v_k(C_k^{'}) - v_k(C_k^0) \tag{7}$$

Since $b_n \in C_k^{'} \cap C_k$ and b_n is the only buyer who has different reservation cost division between RD and $RD^{'}$,

$$\gamma = v_k^{'}(C_k) - v_k^{'}(C_k^0) \tag{8}$$

$$\gamma^{'} = v_k^{'}(C_k^{'}) - v_k^{'}(C_k^0) \tag{9}$$

Since $v_k(C_k) \geq v_k(C_k^{'})$ and $v_k^{'}(C_k^{'}) > v_k^{'}(C_k)$, we have $\gamma \geq \gamma^{'}$ from Equation 6 and 7, $\gamma < \gamma^{'}$ from Equation 8 and 9. Contradiction.$\Box$

From Claim 13 the reservation cost transfer of a member b_n in an optimal subcoalition does not cause change of the optimal subcoalition if b_n stays in it after the transfer.

If the sum of the offers of a buyer cannot cover the sum of his requests he has to leave all the optimal subcoalitions after one transfer for him. When the system has linear price functions the leaving of a buyer $b_n \in C_k^*$ from an optimal subcoalition $C_k^{*'}$ will not have other buyers $b_i \notin C_k^*$ that are not members of the subcoalition before the transfer join the optimal subcoalition $C_k^{*'}$ after the transfer(but it may have some other buyers $b_j \in C_k^*$ leave the optimal subcoalition $C_k^{*'}$ too). If the sum of the offers of a buyer covers the sum of his requests he will be involved in all the optimal subcoalitions he desires after one transfer for him. When the system has linear price functions the joining of a buyer $b_n \notin C_k^*$ into an optimal subcoalition $C_k^{*'}$ will not have other buyers $b_i \in C_k^*$ that are members of the subcoalition before the transfer leave the optimal subcoalition $C_k^{*'}$ after the transfer(but it may have some other buyers $b_j \notin C_k^*$ join into the new optimal subcoalition $C_k^{*'}$ too).

Claim 14 *Suppose $p_k(\cdot)$ is a linear price function and $b_n \in C_k^*(RD)$ with some reservation cost division RD. Make a transfer for b_n such that $b_n \notin C_k^{*'}(RD')$ where $C_k^{*'}(RD')$ is the optimal subcoalition with respect to the item g_k with the new reservation cost division RD'. Then $C_k^{*'}(RD') \subset C_k^*(RD)$.*

Proof: Let $C_k^*(RD)$ be denoted by C_k and $C_k^{*'}(RD')$ be denoted by C_k'. Denote by $v_k'(\cdot)$ the value of subcoalitions with the reservation cost division $RD' = \{r_n^{k'}\}_{n,k}$ and by v_k the value of subcoalitions with $RD = \{r_n^k\}_{n,k}$. Let $\bar{C}_k = C_k \cap C_k'$ and $\hat{C}_k = C_k' \setminus C_k$. Suppose $C_k' \not\subset C_k$, then $\hat{C}_k \neq \emptyset$.

From $v_k'(\hat{C}_k \cup \bar{C}_k) \geq v_k'(\bar{C}_k)$,

$$r_{\hat{C}_k}^k - q_{\hat{C}_k}^k(a_k - d_k \cdot q_{\hat{C}_k}^k) + 2q_{\hat{C}_k}^k \cdot d_k \cdot q_{\bar{C}_k}^k \geq 0 \tag{10}$$

Since $\bar{C}_k \subset C_k$, $b_n \in C_k \setminus \bar{C}_k$, and $q_n^k > 0$, we have $q_{C_k}^k > q_{\bar{C}_k}^k$. Substitute $q_{\bar{C}_k}$ in Equation 10 with $q_{C_k}^k$ and it follows that

$$r_{\hat{C}_k}^k - q_{\hat{C}_k}^k(a_k - d_k \cdot q_{\hat{C}_k}^k) + 2q_{C_k}^k \cdot d_k \cdot q_{\hat{C}_k}^k > 0 \tag{11}$$

The LSH of the inequality above is equal to $v_k(\hat{C}_k \cup C_k) - v_k(C_k)$. The inequality $v_k(\hat{C}_k \cup C_k) - v_k(C_k) > 0$ contradicts the optimality of the subcoalition C_k with the reservation cost division RD. □

Claim 15 *Suppose $p_k(\cdot)$ is a linear price function and $b_n \notin C_k^*(RD)$ with some reservation cost division RD. Make a transfer for b_n such that $b_n \in C_k^{*'}(RD')$ where $C_k^{*'}(RD')$ is the optimal subcoalition with respect to item g_k with the new reservation cost division RD'. Then $C_k^{*'}(RD') \supset C_k^*(RD)$.*

Proof: Similar as in Claim 14. □

Based on Claim 13, 14 and 15 we have the following claim:

Claim 16 *With linear price functions the general reservation cost transfer algorithm terminates with a set of compatible optimal subcoalitions.*

Proof: Let the system state be defined by a tuple $(\alpha_1, \ldots, \alpha_N)$, where $\alpha_n = (\alpha_n^1, \ldots, \alpha_n^K)$ with $\alpha_n^k \in \{0, 1\}$, $n = 1, \ldots, N$, $k = 1, \ldots, K$. $\alpha_n^k = 1$ if the buyer b_n is contained in the optimal subcoalition C_k^* of the item g_k, else $\alpha_n^k = 0$. Construct a directed graph composed of the all the nodes each one corresponding to a state. Two nodes are connected by an arc if and only if there exists one step of transfer for one buyer that transfers the sqystem state from the source node to the target node. Now prove that the system will not evolve by following infinitely a directed loop.

Suppose the system evolves following a directed loop L infinitely, and there are some arcs in this loop on which the amount of reservation cost transfered is not zero. Call these arcs non-zero arcs. If two arcs a and b are associate with the same buyer, one transfer reservation cost into item g_k, the other transfer out of g_k, we say they transfer in reverse direction for item g_k. For each non-zero arc $a \in L$, there exists at least another non-zero arc $a' \in L$ such that a and a' are associated with the same buyer but transfer the reservation cost in reverse directions for some items. Call arc b immediately succeeds arc a if the transfer on arc a causes the transfer on arc b. Denote $b \in succ(a)$ if b immediately succeeds a or there exists an arc $c \in succ(a)$ such that b immediately succeeds c. From Claim 14 and 15, if the transfer on one arc a causes the uncompatibility of another buyer b_n, then the corresponding transfer on arc b to have b_n compatible will not reverse the direction of transfer from that on arc a, so do all arcs in $succ(a)$. For example, if the transfer from item g_i to g_k through arc a for buyer b_m makes b_n uncompatible, then to recover the compatibility, b_n needs to transfer out of item g_k or into g_i, which can only cause further transfer out of g_i or into g_k for b_m if that makes b_m uncompatible in return. This implies there does not exit a path P between a and a' in which every arc $P(k)$, $k \neq 0$ immediately succeeds $P(k-1)$, where $P(k)$, $k \geq 0$ is the k-th arc in P from the head. But such a path should exist if we have infinite none zero transfer iterations, otherwise a and a' can not be both involved in the transfer procedure. Therefore we get a contradiction and following the transfer rule in Algorithm 2 the infinite iteration only happens when all the requests and offers are equal to zero for each of the buyers B' involved in the loop with the reservation cost division. This means there are multiple optimal subcoalitions with the same value but different formations for each item. When uncompatibility is found with respect to a buyer $b_n \in B'$, select(or discard) the optimal subcoalitions containing b_n for each $g_k \in G_n$. From Claim 15 and 14 a set of optimal subcoalitions that are compatible with respect to B' will be reached and the system will jump out of the loop.

Since the number of states is finite, a state that is compatible with respect to all the buyers can by reached in finite steps. $\Box$

The properties of Claim 14 and 15 are not necessary for the problems with general price functions. In Example 3, suppose the initial reservation cost division RD^0 be

b_n	r_n	r_n^1	r_n^2
b_1	3	1.25	1.75
b_2	3.5	1.75	1.75
b_3	6	3	3

The optimal subcoalitions with RD^0 are: $C_1^* = \{b_1, b_2, b_3\}$, $C_2^* = \{b_1, b_3\}$(or $\{b_2, b_3\}$). If b_1 joins C_2^* then b_2 leaves C_2^* and vice versa. The system evolution is trapped in the cycle composed of the two nodes: $((1,1),(1,0),(1,1))$ and $((1,0),(1,1),(1,1))$.

Therefore termination of the general algorithm is not guaranteed for the general systems. This is consistent with the conclusion that for those systems compatible optimal subcoalitions may not exist.

6 Algorithm with linear price functions

Although the general reservation price transfer mechanism leads to an optimal coalition and payoff division in the core of the coalition, polynomial computational complexity is not guaranteed. We need an approximation algorithm to decrease the complexity. The intuition is that once a buyer is excluded from all the optimal subcoalitions, the possibility that he will be involved in the final optimal coalition is very small. We maintain a subset of the buyers $\hat{B}$ which shrinks while excluding buyers. The coalition formation and payoff division is considered within $\hat{B}$ instead of B assuming the buyers out of $\hat{B}$ are not contained in the optimal coalition. In each iteration, a reservation cost transfer is made for one buyer such that the new optimal subcoalitions are compatible with him after the transfer. If the buyer is discarded by all the optimal subcoalitions he is excluded from $\hat{B}$, else the next buyer in $\hat{B}$ is visited. The coalition resulting from the algorithm is a set of buyers that is cohesive with the coalition value no less than the value of any of its subset and the payoff division is in the core of the coalition. The complexity of the algorithm is polynomial. An optional sub-iteration(Option 1) can be integrated into the algorithm to improve the performance(increase the chance of a buyer that would be excluded to be kept in $\hat{B}$) at the expense of increasing the computational complexity while keeping the complexity polynomial.

From Example 4 we can see that even if a buyer is rejected by all the optimal subcoalitions at some time of the transfer procedure it is still possible that he is involved in the optimal coalition.

Example 4 $G = \{g_1, g_2\}$, $B = \{b_1, b_2, b_3\}$,

b_n	q_n^1	q_n^2	r_n
b_1	1	1	6
b_2	1	1	11
b_3	1	1	14

The price discount functions for the two items are $p_1(m) = p_2(m) = -m + 8$, $m \leq 4$.

The optimal coalition is $\{b_1, b_2, b_3\}$ with value 1 and it can be induced by the compatible optimal subcoalitions with the reservation cost division:

b_n	r_n	r_n^1	r_n^2
b_1	6	3	3
b_2	11	5	6
b_3	14	7	7

But suppose the following reservation cost division appears in the procedure of transfer:

b_n	r_n	r_n^1	r_n^2
b_1	6	4	2
b_2	11	4	7
b_3	14	7	7

the two optimal subcoalitions are $C_1^* = \{b_3\}$ with value 0, and $C_2^* = \{b_2, b_3\}$ with value 2. The buyer b_1 is excluded from both optimal subcoalitions. If he is excluded then the final solution reached by the algorithm is not optimal.

The reason for the condition is that there exist dependency relations between the buyers: The joining of a buyer to an optimal subcoalition may have some other buyers also join it(Claim 15) and the leaving of a buyer from an optimal subcoalition may have some other buyers also leave it(Claim 14).

Definition 8 (Supporter & Dependant) *Suppose the buyers b_i and b_j are both included in the optimal subcoalition C_k. If dropping b_i causes the removal of b_j from the optimal subcoalition of g_k, then b_i is said to be a supporter of b_j and b_j is a dependant of b_i with respect to the item g_k.*

Even if a buyer is rejected by all the optimal subcoalitions temporarily it is possible that he will get some virtual price transferred from his dependants and accepted by all the optimal subcoalitions. This is like price flows from dependants to supporters. In Example 2 b_1 is a supporter of b_2 with respect to g_1 with the second reservation cost division. When b_2 transfers 1 from g_2 to g_1, b_1 and b_2 are both accepted by C_1 and then b_1 has an offer equal to 1 to be transferred to g_2 which makes him involved in C_2 too. The condition of price flow is considered in Option 1, which can be integrated into the main algorithm to improve the solution quality at the expense of increasing the computational complexity. Recognizing all of the supporters and dependants is intractable and the option of price flow can be realized approximately by the greedy transfer procedure as follows. With linear price functions the offer of a member in a subcoalition is non-decreasing and the request of a nonmember is non-increasing with the increasing of the virtual reservation cost of other buyers for the item(Claim 12). For a buyer b_n whose sum of offers cannot cover the sum of requests, the items desired are visited one by one. When an item g_k is visited, all the buyers desiring g_k transfer their extra virtual reservation cost(the offers) to g_k from other items, and the offers/requests of b_n are recalculated. The visit to the next item is stopped when the sum of offers covers the sum of requests, in which case a reservation cost division of b_n is constructed by some transfer to have b_n included in all the desired optimal subcoalitions. If all the items desired by b_n have been visited and the stop condition is not reached, b_n is excluded.

The algorithm of coalition formation for the problems with linear price functions is described in Algorithm 3:

Algorithm 3 (Coalition formation with linear price functions)

Step 0: Initialization

Make an initial reservation cost division $RD = \{r_n^k\}_{n=1,\ldots,N,\ k=1,\ldots,K}$ such that $\sum_{k=1}^{K} r_n^k = r_n$ for all buyers $b_n \in B$.

The set of buyers to be considered $\hat{B}$ is the set of all the buyers: $\hat{B} = B$.

Step 1: Optimal subcoalition formation

Construct the optimal subcoalition for each item following Algorithm 1 and go to Step 3.

Step 2: Transfer

For a buyer $b_n \in \hat{B}$ with whom the optimal subcoalitions are not compatible,

if the sum of the offers of r_n is no less than the sum of the requests of r_n,

change the division of r_n and update the optimal subcoalitions such that b_n is involved in all the desired optimal subcoalitions,

else

b_n is excluded: $\hat{B} = \hat{B} \setminus \{b_n\}$ or go to Option 1.

Step 3: Termination judgment

If the optimal subcoalitions are compatible, stop; else repeat Step 2.□

Option 1 (Price flow) *Let the item index be $k = 1$, the index in $\hat{B}$ be $i = 1$.*

Step 4: Let the index of the buyer $\hat{B}[i]$ be j. If $b_j \notin B_k$, go to Step 5; else for the buyer b_j make reservation cost transfer such that $r_j^k = r_j^k + \sum_{l \neq k, b_j \in C_l^*} Off_j^l$, $r_j^l = r_j^l - Off_j^l$ for $l \neq k$ and $b_j \in C_l^*$, go to Step 5.

Step 5: $i = i + 1$. If $i \leq |\hat{B}|$, go to Step 4; else update the optimal subcoalition C_k^*.

If the sum of the offers of r_n is no less than the sum of the requests of r_n,

change the division of r_n and update the optimal subcoalitions such that b_n is involved in all the desired optimal subcoalitions, go to Step 3.

else

go to Step 6.

Step 6: $k = k + 1$. If $k \leq K$, let $i = 1$ and go to Step 4; else b_n is excluded from $\hat{B}$ and go to Step 3. □

Claim 17 Claim 11*: The complexity of the Price Flow option with respect to one buyer is $O(K \cdot N^{M+1})$, where K is the number of items, N the number of buyers, M the number of different bid quantities for the items.*

Proof: The complexity to construct the optimal subcoalition for one item is $O(N^M)$. There are at most $K \cdot N$ iterations to construct the optimal subcoalition for one item in each round of Option 1.□

Claim 18 *The complexity of Algorithm 3 is:* $O(K \cdot N^{2+M})$ *without Option 1 and* $O(K \cdot N^{M+3})$ *with Option 1, where* K *is the number of items,* N *the number of buyers,* M *the number of different bid quantities for the items.*

Proof: The complexity of the sub-algorithms are:

Construct the optimal subcoalition for one item: $O(N^M)$.

Construct the optimal subcoalitions for all items: $O(K \cdot N^M)$.

Calculate the offer(request) of one buyer with one item: $O(N^M)$.

Calculate the offers(requests) of one buyer with all items: $O(K \cdot N^M)$.

With the number of buyers to be considered currently being $|\hat{B}| = n$ the largest number of iterations needed without Option 1 to exclude a buyer is $n - 1$. Therefore in the worst case the number of iterations needed without Option 1 is $(N-1) + (N-2) + \ldots + 1$ which is of complexity $O(N^2)$.

Therefore the complexity of the algorithm is $O(K \cdot N^M \cdot (N^2)) = O(K \cdot N^{2+M})$ without Option 1 and $O(K \cdot N^{M+1} \cdot (N^2)) = O(K \cdot N^{3+M})$ with Option 1.$\square$

7 Algorithm with general price discount functions

We can limit the number of iterations by reducing the number of buyers following the same intuition in Algorithm 3. But Algorithm 3 is not applicable in the case with general price functions since the number of iterations needed to exclude a buyer is not bounded by N any more and so the polynomial complexity is not guaranteed. The reason for the situation is that some buyers may be removed from the optimal subcoalition when a different buyer is added into it. Consequently it is possible that even after all the buyers have been visited no one had ever been rejected by all the optimal subcoalitions but the optimal subcoalitions have never been compatible either. To solve this problem, we make some modification of the transfer mechanism to reach a polynomial bound for the number of iterations. The idea is in each round to try to keep the compatibility with respect to the buyers that have been visited. For the buyer being visited consider the compatibility of the optimal subcoalitions one by one, that is to say, instead of making transfer from multiple to multiple items simultaneously make transfer from multiple items(source items) to one item(target item) at one time. If the transfer cannot make the buyer accepted by the optimal subcoalition of the target item, then the buyer is excluded, else consider the influence of the transfer to the members in the old optimal subcoalition of the target item. If some members that have been visited are removed then try to recover them one by one. One that cannot be recovered is excluded, otherwise if all are recovered, consider the next target item. Since decreasing the reservation cost of a subcoalition member will not cause increase of other member's offer(Claim 11), if the transfer is optimistic(let the amount transferred out equal the offer) the number of iterations for each buyer and each item is bounded by the number of buyers maintained. Claim 13 guarantees the stability of optimal subcoalitions of the source items. Therefore the number of iterations to visit one buyer is bounded by $K \cdot N$ and the number of iterations needed to exclude a buyer is bounded by $K \cdot N^2$.

It is not true that the optimal subcoalitions always exist for the instances with general price functions. Example 3 is such a counter example. In Example 3 the reason for the non-existence of compatible optimal subcoalitions is that the buyers a and b attract and resist each other.

Definition 9 (Attract) *We say the buyer b_i attracts b_j if when the optimal subcoalitions are compatible with respect to b_i, there exists some item $g_k \in G_i \cap G_j$ such that $b_i \in C_k^*$ implies $b_j \in C_k^*$, where C_k^* is the optimal subcoalition of the item g_k.*

Definition 10 (Resist) *We say the buyer b_i resists b_j if when the optimal subcoalitions are compatible with respect to b_i, $b_i \in C_{k_1}^*$ implies $b_j \notin C_{k_1}^*$, and $b_i \notin C_{k_2}^*$ implies $b_j \in C_{k_2}^*$ for some item $g_{k_1}, g_{k_2} \in G_i \cap G_j$, where C_k^* is the optimal subcoalition of the item g_k.*

When two buyers b_i, b_j attract and resist each other, there is no reservation cost division of b_i and b_j such that the optimal subcoalitions are compatible with respect to both b_i and b_j.

Claim 19 *There exist no reservation cost division of b_i and b_j such that the optimal subcoalitions C_k^*, $k = 1, \ldots, K$ are compatible with respect to both b_i and b_j if b_i and b_j attract and resist each other.*

Proof: Suppose such a reservation cost division of b_i and b_j exists and the optimal subcoalitions C_k^*, $k = 1, \ldots, K$ are compatible with respect to b_i and b_j. Let $C = \bigcup_{k=1}^{K} C_k^*$ and $s = C \cap \{b_i, b_j\}$. $s \neq \emptyset$ and $s \neq \{b_i, b_j\}$ since b_i and b_j resist each other. Suppose $s = \{b_i\}$

without loss of generality. Since b_i and b_j attract each other there exist one item g_k such that $b_j \in C_k^*$. This contradicts $b_j \notin C$. □

The buyer exclusion scheme will exclude both of the two buyers b_i and b_j that attract and resist each other and result in a cohesive coalition induced by compatible subcoalitions. However, by merging b_i and b_j into a single buyer we increase their chance to stay. In Example 3, if b_1 and b_2 merge as a single buyer b_4 that requests 2 units of both g_1 and g_2 with reservation cost 6.5, then with the reservation cost division

b_n	r_n^1	r_n^2	r_n
b_4	3	3.5	6.5
b_3	3	3	6

the optimal subcoalition of both g_1 and g_2 is $\{b_4, b_3\}$ and the coalition induced $\{b_4, b_3\}(\{b_1, b_2, b_3\})$ is the optimal one of the original problem.

An option of buyer merging can be integrated into the main algorithm. When a buyer is decided to be excluded in the main algorithm, use the option to judge if there is another buyer such that they attract and resist each other. If the answer is positive then merge them into a single buyer, and replace them with the new buyer, else exclude the buyer. In each case the number of buyers is decreased by one.

If a buyer b_n is produced by merging two buyers b_i and b_j, then b_n is called an artificial buyer, and b_i and b_j are called the component buyers of b_n. A component buyer of an artificial buyer could also be an artificial buyer. An artificial buyer can be regarded as a collection of the real buyers that construct it nestingly.

We can divide the payoff in a subcoalition following the same way in Claim 6 when artificial buyers are involved and then divide the payoff of an artificial buyer among its components by some way(for example, by dividing evenly). Let P be a partition of an optimal subcoalition C and P is composed of all the real buyers and artificial buyers in C. Then the resulting payoff division is not in the core of C but in the pseudo-core(Definition 11) of C with respect to P if we consider the real buyers.

Definition 11 (Pseudo-core) *A payoff division X_C for the coalition C is said to be in the pseudo-core with respect to $P = \{S_1, \ldots, S_L\}$, a partition of C, if for any $S' \subset \{S_1, \ldots, S_L\}$,*

$$v(S') \leq X_C(S')$$

When the partition P is composed of real buyers, the pseudo-core is identical to the core.

The payoff division for the coalition constructed by summing up the payoff for each buyer in the optimal subcoalitions is also in the pseudo-core of the coalition.

Algorithm 4 (Coalition formation with general price functions)

Step 0: Initialization

Make an initial reservation cost division $RD = \{r_n^k\}_{n=1,\ldots,N,\ k=1,\ldots,K}$ *such that* $\sum_{k=1}^{K} r_n^k = r_n$ *for all buyers* $b_n \in B$. *The set of buyers to be considered* $\hat{B}$ *is the set of all the buyers:* $\hat{B} = B$. *The set of buyers that have been visited and maintained* $\bar{B} = \emptyset$.

Step 1: Optimal subcoalition formation

Construct the optimal subcoalition for each item following Algorithm 1 and go to Step 3.

Step 2: Transfer

For a buyer $b_n \in \hat{B}$ *with whom the optimal subcoalitions are not compatible, for each item* g_k *that* $q_n^k > 0$ *and* $b_n \notin C_k^*$,

if the sum of offers of b_n *cannot cover* Req_n^k,

let $\hat{B} = \hat{B} \setminus \{b_n\}$ *or go to Option 2, and then go to Step 3;*

else

make the optimistic transfer for b_n *to* g_k *such that* $r_n^k = r_n^k + \sum_{j \neq k, b_n \in C_j^*} Off_n^j$, *and* $r_n^j = r_n^j - Off_n^j$ *for* $j \neq k, b_n \in C_j^*$, *update the optimal subcoalition* C_k^* *of* g_k.

If any buyer $b_i \in \bar{B}$ *is removed out of* C_k^*,

let $b_n = b_i$ *and repeat Step 2;*

else

visit the next item.

Let $\bar{B} = \bar{B} \cup \{b_n\}$ *and go to Step 3.*

Step 3: Termination judgment

If the optimal subcoalitions are compatible, stop; else repeat Step 2. □

Option 2 (Buyer merging) *For each buyer* $b_i \in \hat{B}$ *construct an optimal subcoalition* C_k^i *that includes* b_i *for each item* $g_k \in G_i$. *Let* $A_i = \bigcup_{k=1}^{K} C_k^i$ *and* $R_i = \bigcup_{k=1}^{K} (\hat{B} \setminus C_k^i) \cap B_k$. *If* *there is a buyer $b_i \in \hat{B}$ such that $b_i, b_n \in A_i \cap A_n$ and $b_i, b_n \in R_i \cap R_n$, merge b_i and b_n as an artificial buyer s, let $\hat{B} = \hat{B} \setminus \{b_i, b_n\} \cup \{s\}$; else $\hat{B} = \hat{B} \setminus \{b_n\}$.* $\square$

Claim 20 *The complexity of Option 2 is $O(K \cdot N^{M+1})$, where K is the number of items, N the number of buyers, M the number of different bid quantities for the items.*

Proof: For each buyer with respect to each item we need to construct an optimal subcoalition that does not include that buyer. Since the complexity to construct an optimal subcoalition is $O(N^M)$ when we have N buyers, the complexity of the option is $O(K \cdot N \cdot (N-1)^M)$, which is equal to $O(K \cdot N^{M+1})$.$\square$

Claim 21 *The complexity of Algorithm 4 is: $O(K \cdot N^{2+M})$ without Option 2, $O(K \cdot N^{M+3})$ with Option 2, where K is the number of items, N the number of buyers, M the number of different bid quantities for the items.*

Proof: With the number of buyers to be considered currently being $|\hat{B}| = n$, for one buyer $b \in \hat{B}$ there need at most $N - n + 1$ iterations of constructing the optimal subcoalition with respect to item g_k. Therefore in the worst case when Option 2 is not applied the complexity of the algorithm is $KN \cdot O(N^M) + K(N-1) \cdot O((N-1)^M) + \ldots + K \cdot O(1^M)$, which is equal to $O(KN^{2+M})$. When Option 2 is applied the complexity of the algorithm is $KN \cdot O(N^{M+1}) + K(N-1) \cdot O((N-1)^{M+1}) + \ldots + K \cdot O(1^{M+1})$, which is equal to $O(K \cdot N^{3+M})$.$\square$

8 Experiment

8.1 Instance generation

The framework of instance generation is partially inspired by [1]. An instance is characterized by three parameters: DS(Discount Slope) determines the magnitude of profit to form coalitions, RBMI(the Ratio of Buyers preferring Multiple Items) models the extent of complementarity among the items, RBBR(the Ratio of Buyers Bidding at the Retail Prices) reflects the extent of dependency among the buyers(to how much extent they need to form coalitions) to win their bids.

Price schedule generation

For general price functions let the length and height of each step be identical(the linear price function is a special case with the length of each step being one and the height of each step being d_k, the constant discount rate with the quantity of the item sold together increased by one unit). Let DS(Discount Slope) denote the ratio of the step height to the step length of the price function. The larger DS, the more beneficial to form coalitions. Denote by RP(the Retail Price) the highest price without discount. Assume the sum of quantities requested by all the buyers for one item is bounded from above by $UBSQ$(the Upper Bound of the Sum of Quantities requested by all the buyers for one item) and the quantity requested by one buyer for one item is bounded from above by $UBQB$(the Upper Bound of the Quantity requested by one Buyer for one item). $UBSQ$ and $UBQB$ satisfy $UBSQ = UBQB \cdot N$ where N is the number of buyers. By having $RP = 2 \cdot DS \cdot UBSQ$ we ensure that the entire purchasing cost for the item is an increasing function of the quantity requested for the item. Therefore a linear price function can be modeled by two parameters, DS and $UBSQ$. In addition to these two parameters, the Number of Steps(NS) is needed to model a general price function. NS is equal to the number of falls on the function curve. Define LDP(the Lowest Discount Price) to be equal to $RP/2$, which is the lowest price that could be reached assuming the price function was a continuous linear function with the slope $-DS$(the actual lowest price is no less than LDP since the actual price functions used are discrete functions of integer numbers). Figure 1 shows a sample general price function used in the experiment. For simplicity assume all the items have identical price schedules without decreasing the complexity of the problem.

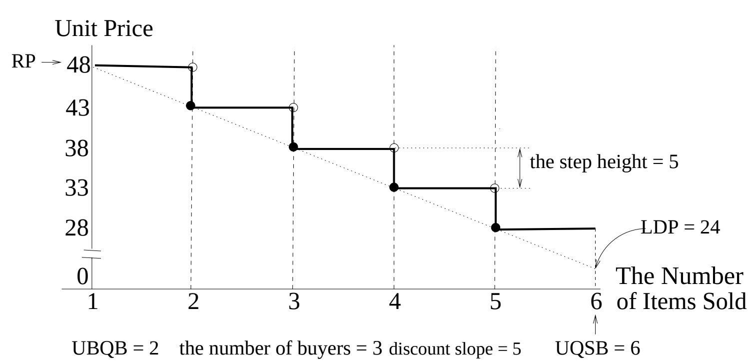

Figure 2: A sample price function

Bid generation

Suppose before the decision of the reservation cost each buyer b_n has a virtual reservation price p_n^k for each item g_k he desires. The reservation cost is decided by summing up the reservation prices multiplied by the quantity he requests for each item, i.e., $r_n = \sum_{n=1}^{N} p_n^k \cdot q_n^k$. Let the lower bound of the virtual reservation price be equal to LDP and the upper bound equal to RP. Let the ratio of buyers who have virtual reservation prices at RP equal to $RBBR$(the Ratio of Buyers Bidding at the Retail Prices). Small RBBR means many of the buyers cannot win their bids if they do not form coalitions. The virtual reservation prices of other buyers are randomly distributed between LDP and RP.

The distribution of preferences for multiple items is modeled by $RBMI$(the Ratio of Buyers preferring Multiple Items). $RBMI$ is a vector $\{rb_1, \ldots, rb_K\}$ where rb_k is the ratio of buyers who desire k items and $\sum_{k=1}^{K} rb_k = 1$. For example, when the number of items is 3, the number of buyers is 10, $RBMI = \{0.2, 0.3, 0.5\}$ means there are 2 buyers bid for 1 item, 3 buyers for 2 items, 5 buyers for 3 items. Large numbers at the end part of RBMI mean that many items are complementary for many buyers. The number of desired items of each buyer is randomly decided following the distribution consistent with $RBMI$. The quantities requested by each buyer for each item desired are generated randomly in the range $[1, UBQB]$.

8.2 Results

The simulation is based on the instances generated with combination of the parameter values listed in Table 1. For each set of parameters several(we took the number 3) instances are randomly generated. For each instance we construct the optimal coalition by exhaustive search, and the approximate solution by Algorithm 3 with or without Option 1 supposing linear price functions are used, and by Algorithm 4 with or without Option 2 supposing general price functions are used. The average coalition value of the instances with identical parameters is set as the coalition value for the condition with that set of parameters. The comparison is made among the optimal value and the value of the coalitions obtained by our algorithms with or without the options. The largest number of buyers is limited to 25 because of the complexity to compute the optimal value.

Comparison with respect to the number of buyers and the number of items:

We calculate the ratio of the approximate value over the optimal value and take the average of the ratios under all the conditions with the same number of buyers and number of items. Figure 3 shows the distribution of the average ratios with respect to the number of buyers

Table 1: Simulation Parameters

Parameter	Values
the number of buyers N	{5, 10, 15, 20, 25}
the number of items K	{3, 5, 7}
UBQB(the largest quantity to be requested for one item by a buyer)	{3}
DS(the discount slope)	{0.2, 0.4, 0.6, 0.8, 1}
NS(number of steps for general price functions)	4
RBMI (the ratio of buyers preferring multiple items)	{1.0, 0, 0};{0.7, 0.2, 0.1};{0.5, 0.3, 0.2} {0.333, 0.333, 0.334};{1.0, 0, 0, 0, 0}; {0.7, 0.2, 0.05, 0.03, 0.02}; {0.4, 0.2, 0.15, 0.15, 0.1} {0.2, 0.2, 0.2, 0.2, 0.2};{1.0, 0, 0, 0, 0, 0, 0}; {0.5, 0.2, 0.1, 0.05, 0.05, 0.05, 0.05}; {0.3, 0.2, 0.2, 0.1, 0.1, 0.05, 0.05} {0.14, 0.14, 0.14, 0.14, 0.14, 0.15,0.15};
RBBR(the ratio of buyers bidding at the retail price)	{0.1, 0.25, 0.4, 0.55, 0.7}

and the number of items with linear price functions, and Figure 4 shows the distribution with general price functions. From Figure 3 we can see that the value of the coalition generated by Algorithm 3 is very close to the optimal one when the system is equipped with linear price functions. Although from Figure 4 the performance of Algorithm 4 is worse for the systems equipped with general price functions, the approximate values still fall into 90 percent from the optimal ones with Option 2. The options improve the performance of the algorithms remarkably, especially when general price functions are used.

Comparison with respect to DS, RBBR and RBMI:

Since DS, RBMI and RBBR are three characteristic parameters of an instance, we would like to compare the coalition values on the three dimensions to see their impact on the performance of the algorithms. The average performance of the algorithms when the number of items is equal to 5 and the number of buyers is equal to 20 is shown in Figure 5, 6, 7, 8, 9 and 10 against

DS, RBBR and RBMI. We can see that the performance of the algorithms hurts from the increase of RBMI but is not monotone on DS or RBBR. The result is understandable since the algorithms aim to generate compatible subcoalitions. When more items are complementary to more buyers, it is more difficult to find compatible subcoalitions.

Average computation time

The algorithms were implemented by VC++ 6.0 and run on Windows/NT platform. All the experiments were performed on a Pentium 4/52M RAM personal computer. Figure 11 shows the average amount of time in CPU seconds that Algorithm 3 and 4 took to compute a suboptimal coalition, with $K = 5$, $UBQB = 2$.

9 Conclusion

In this paper we analyze the coalition formation of buyers in a electronic marketplace where sellers offer price discount based on the quantity of items sold together and buyers have preferences for combinations of items. Two approximation algorithms in polynomial time are presented for the conditions with linear or general price functions. A payoff division rule is proposed that is in the core with linear price functions and in the pseudo-core with general price functions. Experimental results show that the algorithms give good ratio to the optimal value.

In the situation considered in this paper, the buyers can only place one bid. More generally a buyer may have desire for multiple bundles of items and needs to be allowed to place a bid for each bundle. The relation between the bids can be OR or XOR. If there are exclusively OR relations between the bids of one buyer, then we can generate a dummy buyer for each bid of a buyer and this leads to a CCF problem with the dummy buyers. If some bids of a buyer are in XOR, then we can construct all the maximal sets of unconflicting bids(a set is maximal if there is no more qualified elements that can be added into it) for each buyer. The optimal coalition can be obtained by solving a CCF problem for each combination of the sets one from each buyer and choosing the optimal result.

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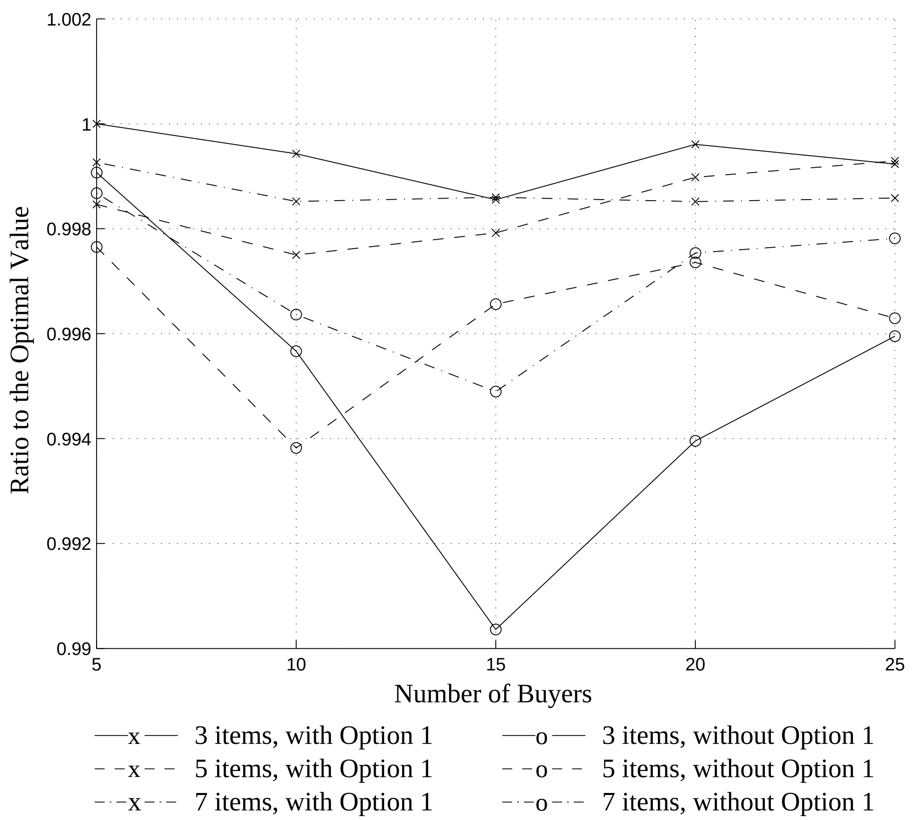

Figure 3: Compare the value of the coalition obtained by Algorithm 3 with or without Option 1 with the optimal value for the instances with linear price functions, with respect to the number of buyers and the number of items

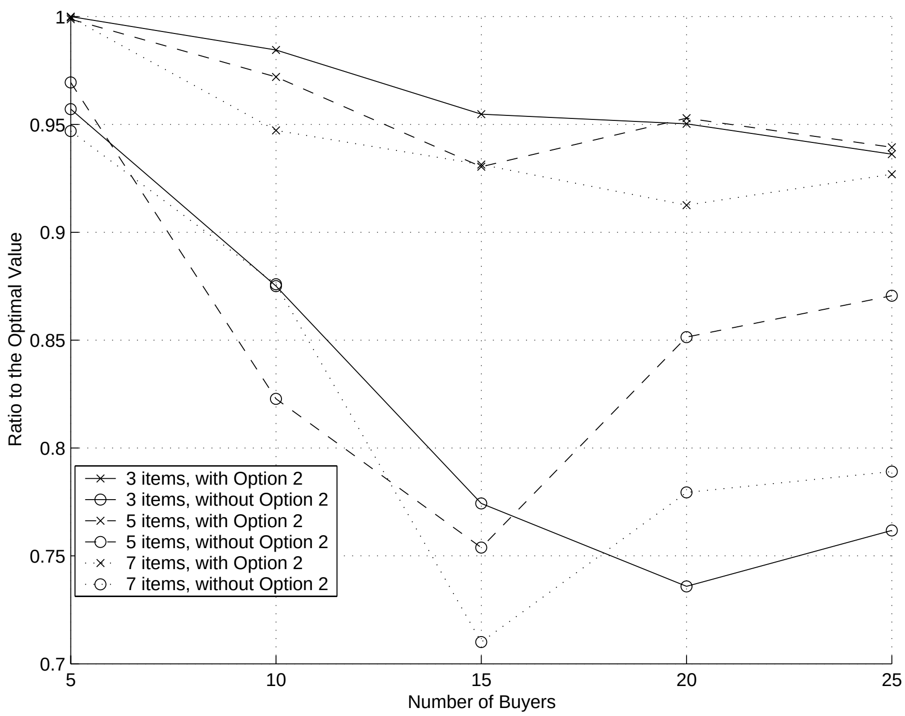

Figure 4: Compare the value of the coalition obtained by Algorithm 4 with or without Option 2 with the optimal value for the instances with general price functions, with respect to the number of buyers and the number of items

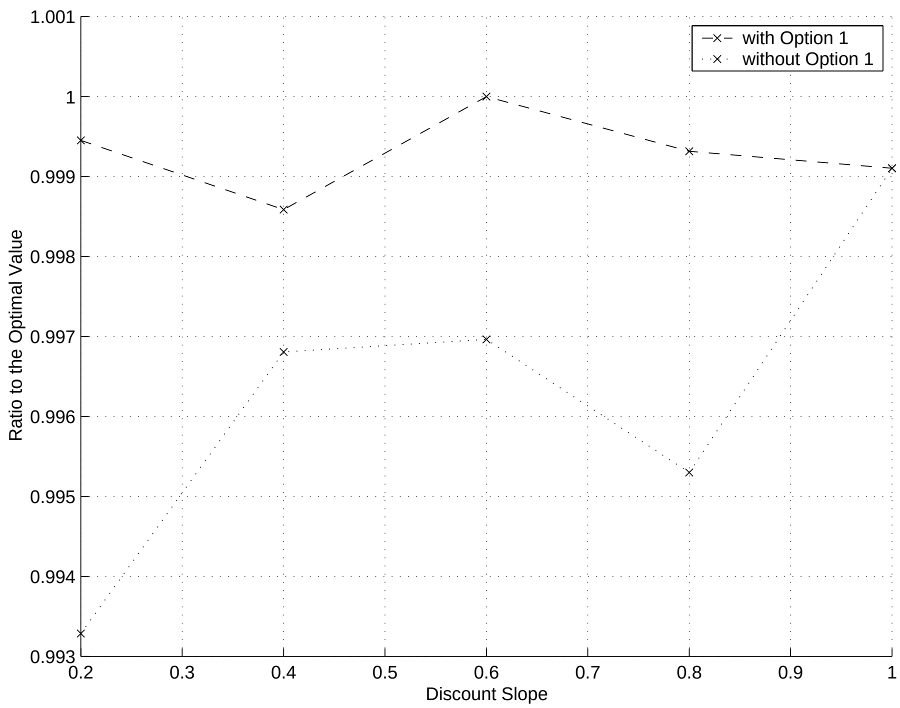

Figure 5: Compare the value of the coalition obtained by Algorithm 3 with or without Option 1 with the optimal value for the instances with linear price functions, with respect to the discount slope

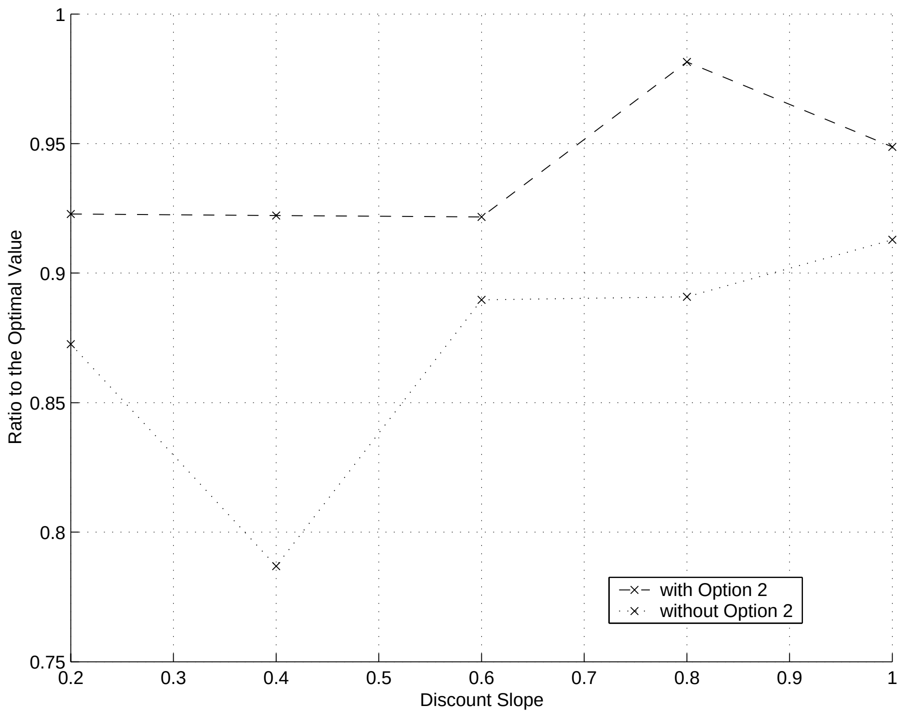

Figure 6: Compare the value of the coalition obtained by Algorithm 4 with or without Option 2 with the optimal value for the instances with general price functions, with respect to the discount slope

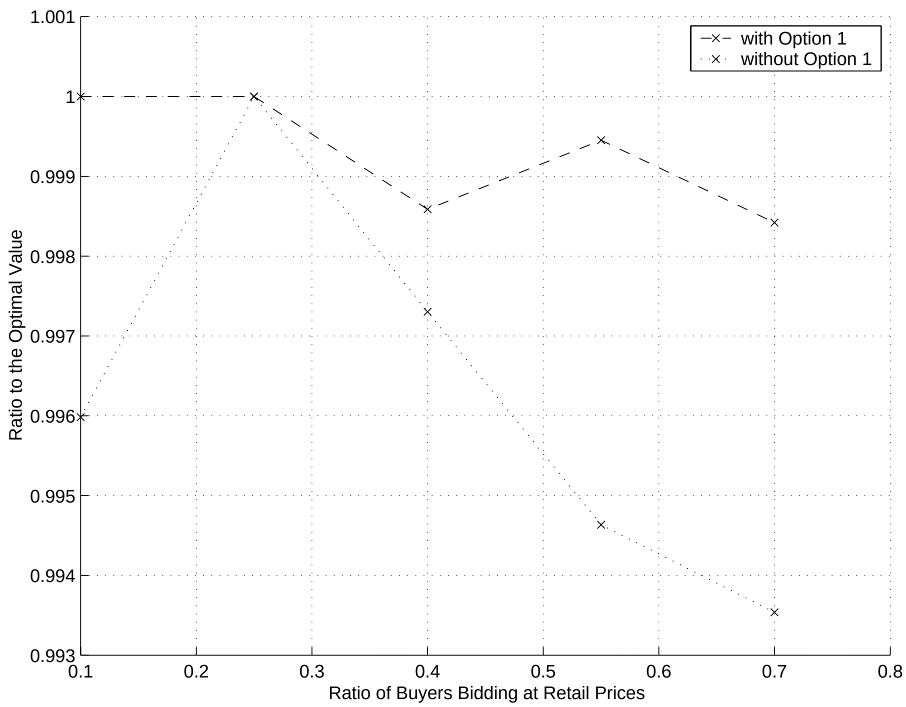

Figure 7: Compare the value of the coalition obtained by Algorithm 3 with or without Option 1 with the optimal value for the instances with linear price functions, with respect to the ratio of buyers bidding at the retail price

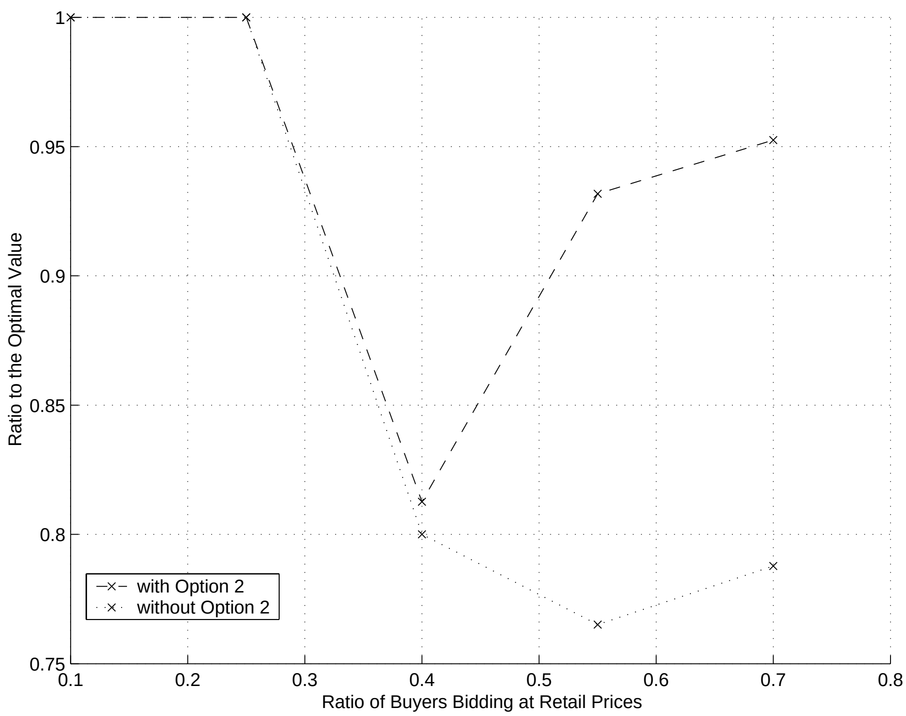

Figure 8: Compare the value of the coalition obtained by Algorithm 4 with or without Option 2 with the optimal value for the instances with general price functions, with respect to the ratio of buyers bidding at the retail price

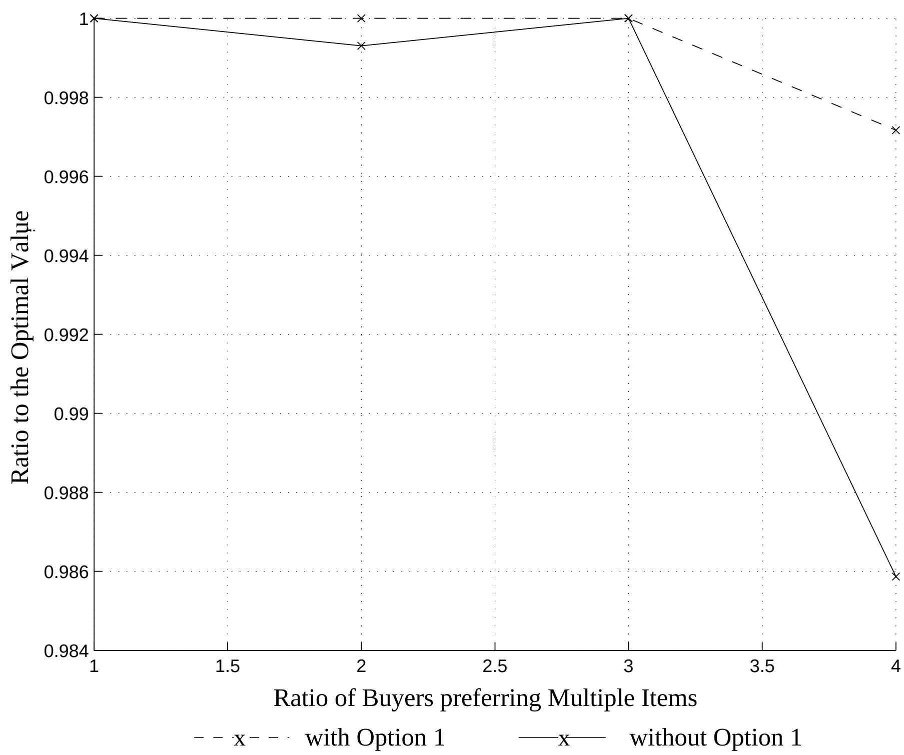

Figure 9: Compare the value of the coalition obtained by Algorithm 3 with or without Option 1 with the optimal value for the instances with linear price functions, with respect to the ratio of buyers preferring multiple items(one integer point on the horizontal axis represents one value of RBMI shown in Table 1)

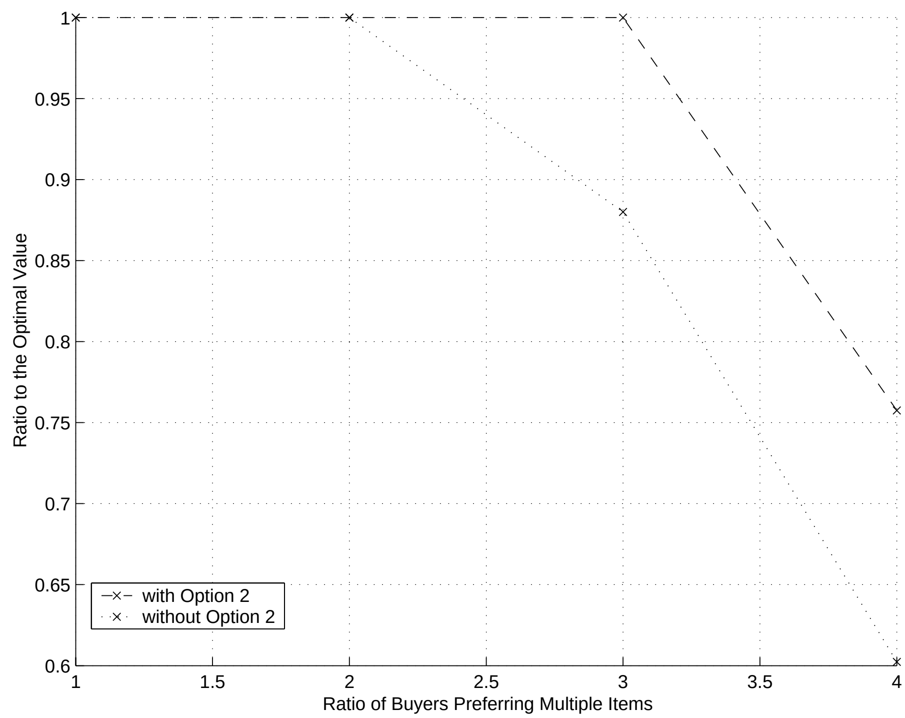

Figure 10: Compare the value of the coalition obtained by Algorithm 4 with or without Option 2 with the optimal value for the instances with general price functions, with respect to the ratio of buyers preferring multiple items(one integer point on the horizontal axis represents one value of RBMI shown in Table 1)

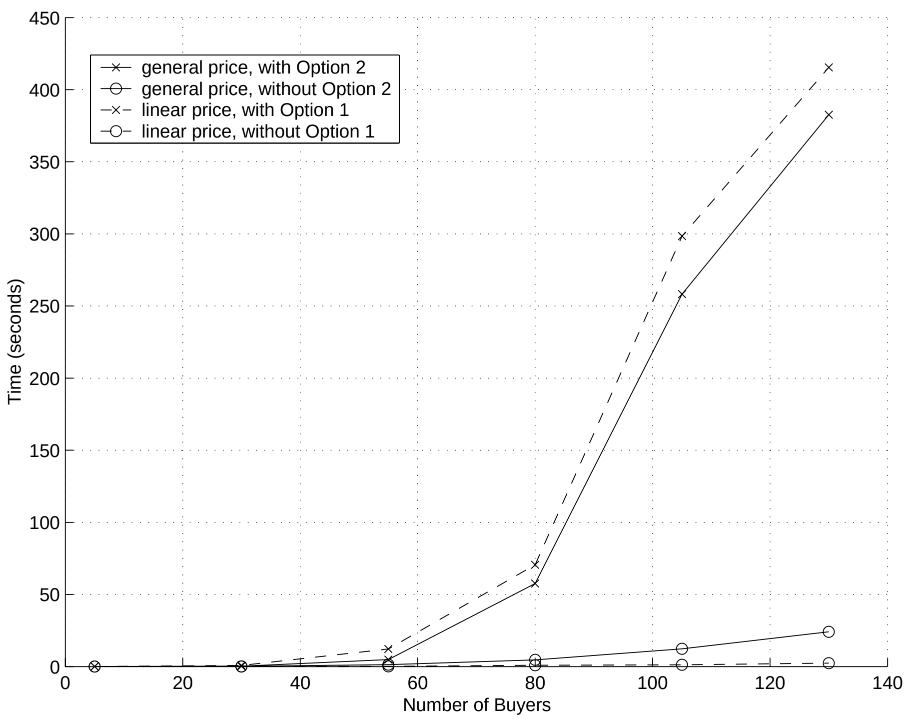

Figure 11: Average CPU time of computing the suboptimal coalitions($K = 5$, $UBQB = 2$)