# Towards a Formal Model for Multiresolution Spatial Maps


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**Abstract.** Topological and metric aspects of the multiresolution representation of geographic maps are considered. The combinatorial structure of maps is mathematically modelled through *abstract cell complexes*, and maps at different detail are related through continuous functions over such complexes. Metric aspects of multiresolution are controlled through the concept of homotopy. Two alternative multiresolution models are proposed, which are implicitly defined by a sequence of map simplifications that fulfil both topological and metric consistency rules.


## 1 Introduction

The representation of spatial data at different resolution in the context of a unified model is a topic of relevant interest in spatial information theory. Indeed, multiresolution modelling offers interesting capabilities for spatial representation and reasoning: from support to map generalisation and automated cartography [15], to efficient browsing over large GISs, to structured solutions in wayfinding and planning [25].

Current GISs do not offer much in multiresolution data handling: apart from some hierarchical capabilities in raster modelling, which are essentially based on structures and tools inherited from image processing, there is an almost total lack of features for handling and relating spatial data at different resolutions. In order to support GISs of future generations, it seems worthwhile to pursue the definition of a formal framework for multiresolution representation of spatial entities based on a topological model that offers explicit description of spatial objects, and efficient encoding/retrieval of spatial relations.

A fair amount of work has been done in the last few years in the direction of a formal approach to the description and manipulation of spatial entities and their relationships [8, 17, 9, 27, 16, 5, 28, 12], and a number of models have been proposed in the literature for giving a comprehensive representation of the geometric structure of plane geographic maps (see, e.g., [14, 7, 21, 27, 22, 5]). Different models are characterised mostly by their expressive power, defined by the degree of generality of objects and configurations that they are able to represent, and by the different data structures that they require to support the representation of such objects and configurations.

Some research has also been undertaken that address multiple representations of spatial data in the context of GIS, concerning either the development of data models [2, 3, 15], or the assessment of consistency among different representations [10, 11, 13]. The possibility of developing models that can support the multiresolution representation of maps through hierarchical structures based on trees of cells has been outlined in [15]. A first hierarchical model that is formally defined on a mathematical basis has been proposed in [1]: such model is described by a tree of maps at different resolutions, where each map is the refined description of a simple region of its parent node in the tree.

In this paper, we exploit mathematical principles from the theory of cell complexes to establish a formal basis for the definition of multiresolution maps. The scope of this work is limited to generic geographic objects represented in the context of two-dimensional maps. For the sake of clarity, we list assumptions and guidelines on which we rely, before introducing the technical content of the paper.

1. A plane map is composed of spatial entities of three classes, namely *points*, *lines*, and *regions* embedded in the Euclidean plane: we will make distinction between such *entities* and spatial (semantic) *objects* that they represent.
2. A broad classification of relationships intercurring among spatial entities is into *topological* relations, *metric* relations, and *order* relations. The three classes of relationships involve different geometric properties, and can be studied independently: here, we do not consider issues concerning order relations.
3. Spatial entities forming a thematic map have disjoint relative interiors, i.e., they form a partition of the domain of the map, therefore they are allowed to take only the subset of topological relations, which exclude intersection of interiors.
4. Two maps of the same area at different resolutions can differ in two basic aspects:
   a) *detail*: some objects can be represented only in the map at higher resolution; objects that appear in both maps can be represented by entities of lower complexity and/or dimension in the context of the map at lower resolution; any object represented by an entity in the map at lower resolution must be also represented by either an entity or a group of entities in the map at higher resolution (monotonicity of simplification);
   b) *precision*: the spatial extent and location of any object in a map is approximated by the extent and location of the entity representing the object itself: the higher the resolution, the lower the approximation error.

   Changes in detail involve topological aspects, while changes in precision involve metric aspects: hence, they can be studied separately.
5. Two maps of the same area at different resolution must be consistent, i.e., objects that appear in both maps must maintain compatible spatial relationships.

The main results stated in this paper are the following. The topological structure of a geographic map is completely captured by a purely combinatorial structure called an *abstract cell complex*, which is, by all means, the only possible topological space apt to represent the map topology. *Map simplification*, i.e., an operation that relates two consistent maps at different detail, can be expressed through a continuous mapping between the abstract cell complexes representing the two maps. Moreover, suitable rules permit to control such functions in order to guarantee that simplification occurs through gradual changes. The metric aspects concerning changes in precision can be controlled separately through the concept of line homotopy: this part extends preliminary results stated in [1]. Through the iterative application of simplification mappings that satisfy both topological and metric constraints it is possible to define a sequence of gradually simplified maps of the same region. This sequence implicitly provides means to organise the maps, together with the mappings relating them in the sequence, either in a multi-layer model, or in a hierarchical model described by a tree, which extends the model proposed in [1].

The rest of the paper is organised as follows. In Section 2 we give a mathematical characterisation of spatial entities, we review their topological relationships, and we give a formal definition of map. In Section 3 we introduce abstract cell complexes, we state the main results about their properties as topological spaces, and we show that they represent the whole combinatorial structure of maps. In Section 4 we show simplification rules and functions that can be used to relate maps at different detail. In Section 5 we focus on metric aspects by using $\varepsilon$-homotopies to relate spatial entities at different precision. In Section 6 we propose two possible multiresolution models that can be obtained through simplification functions. Finally, in Section 7 we address possible extensions and future work on this subject.

## 2 Spatial Entities, Relations, and Maps

As we stated in the Introduction (item 1), a plane map is composed of *points*, *lines*, and *regions* embedded in the Euclidean plane $\mathbb{R}^2$.

- A point is uniquely defined by its coordinates in a coordinate system on $\mathbb{R}^2$.
- A line is defined by a continuous function $l : I \to \mathbb{R}^2$, where $I = [0, 1]$ is the unit interval on the real line[1]; if $l(0) = l(1)$, line $l$ is said *closed*, otherwise it is said *open*; points $l(0)$ and $l(1)$ of an open line $l$ are called the *endpoints* of $l$; the set formed by an open line $l$ without its endpoints is called the *relative interior* of $l$, while the relative interior of a closed line is the line itself; a line is called *simple* if either $l$ is injective on $I$, or $l$ is closed and injective on $[0, 1)$.

[1] With abuse of notation, we will often use $l$ interchangeably to denote the parametric function of a line, and its image $l(I)$, which corresponds to the realisation of the line on the plane.

By Jordan theorem, a simple closed line separates $\mathbb{R}^2$ into two open sets: one set is bounded, called the *internal set*, and denoted $\mathrm{int}(l)$; the other set is unbounded, called the *external set*, and denoted $\mathrm{ext}(l)$.

A *chain* is a sequence of lines $c = (l_0, \ldots, l_{k-1})$ such that $\forall i = 1, \ldots, k-1, \quad l_{i-1}(1) = l_i(0)$. With abuse of notation, points $l_0(0)$ and $l_{k-1}(1)$ are called *endpoints* of $c$, only in case $l_0(0) \neq l_{k-1}(1)$; all other endpoints of lines forming $c$ are called *joints* of $c$. A chain $c$ admits itself a line parametrization:

$$c(x) = l_i(kx - i) \quad \text{for } x \in [\frac{i}{k}, \frac{i+1}{k}], \quad i = 0, \ldots, k-1$$

A chain is said *simple*, *open*, or *closed* if it is simple, open, or closed as a line, respectively.

- A region is characterised by a simple closed line (or chain) $l_0$, called the *outer boundary*, plus possibly a set of simple closed lines (or chains) $\{l_1, \ldots, l_k\}$, called the *inner boundaries*, such that: (i) no two such lines intersect; (ii) for $i = 1, \ldots, k$ line $l_i$ is contained in $\mathrm{int}(l_0)$ and in $\mathrm{ext}(l_j)$, $\forall\ 0 \neq j \neq i$. The region defined by $\{l_0, \ldots, l_k\}$ is the subset of $\mathbb{R}^2$ defined as

$$\cap_{i=1}^{k} \mathrm{ext}(l_i) \cap \mathrm{int}(l_0),$$

  i.e., it is the region interior to the outer boundary, and exterior to all inner boundaries. Inner boundaries define *holes* in the region. The number $\chi(r) = k+1$ of closed lines defining a region $r$ is called the *characteristic* of $r$. A region $r$ without holes has $\chi(r) = 1$, and it is said *simply connected*, while a region with holes has $\chi(r) > 1$, and it is said *multiply connected*. The *interior* and *closure* of a region $r$ correspond to the standard interior and closure of $r$ regarded as a set of $\mathbb{R}^2$ with the Euclidean topology. The *boundary* of $r$ is the subset of $\mathbb{R}^2$ covered by the lines defining $r$, corresponding to the difference between its closure and its interior. We will usually refer to a region $r$ by assuming it closed, otherwise we will call it explicitly *open region* $r$.

Topological relations between a pair of simply connected regions are usually classified according to the so-called 4-intersection relations [9], which are obtained combinatorially from a simple scheme involving the mutual intersections between interiors and boundaries. The following eight relations are possible: *disjoint, meet, contains, covers, equal, overlap, inside,* and *coveredBy*. For regions with holes the classification is obtained by combining the 4-intersections among the simply connected regions interior to the outer boundaries (i.e., those obtained by eliminating the holes), and the simply connected regions corresponding to the holes [12]. Topological relations can be extended easily to pairs of atomic entities, including also lines and points: in this case, also relations *bounds* and *boundedBy* are possible, between entities of different dimensions, and such that one is contained in the boundary of the other [5].

A map is often regarded as a disjoint covering of a portion $D$ of the plane $\mathbb{R}^2$, called the *domain* of the map, with a collection of atomic entities. A disjoint covering is a set of entities such that the relative interiors of any two such entities never intersect. We make a weaker assumption stated by the following constraints: no pair of open regions can intersect; no line can intersect the relative interior of a different line; no point can coincide with a different point, or intersect the relative interior of a line. Nevertheless, isolated open lines and points can be completely *inside* or *coveredBy* a region: such lines and points are called *features* of the region, and are different from the remaining lines and points of the map in being not part of the boundary of any region, and being not endpoints of any line, respectively[2] (see also [5]). We call such a covering a *weakly disjoint covering* (see Fig. 1 for an example of map satisfying a weakly disjoint covering).

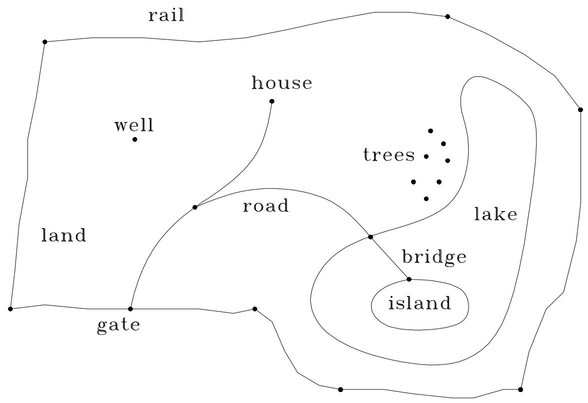


**Fig. 1.** An example of map: land, lake, and island are its regions; roads, trees, house, and well are features of land; bridge is a feature of lake; rail and gate form the boundary of land.

Given a generic set of atomic entities (which are the relevant entities to form a specific map, according to some semantics), a weakly disjoint covering of the portion of plane covered by such entities can be obtained by an overlay operation:

- whenever two open regions $r, r'$ intersect, then they are fragmented into three sets of regions: the connected components of $r \setminus r'$; the connected components of $r' \setminus r$; the connected components of $r \cap r'$;
- whenever the relative interior of a line $l$ intersects a point or another line $l'$, then $l$ is subdivided in two portions, joined at the intersection point.

Hence, a map can be represented as a triple $M = (P, L, R)$, where sets $P$, $L$, and $R$ contain the points, lines, and regions of $M$, respectively. Note that $M$ is completely characterised by $P$ and $L$, since each region of $R$ is understood as

[2] Some models of map require disjoint covering and do not accept features. We think that such a constraint is too strong to model real maps. On the other hand, it is possible to accept features while requiring disjoint covering if isolated lines (or chains) and points are considered as degenerated regions. This fact makes the definition of maps unwieldy, especially because any line *coveredBy* a region becomes part of the region boundary.

a maximal portion $r$ of the plane such that any two points interior to $r$ can be connected by a line that does not touch any line of $L$[3]. This definition of map either conforms to, or extends models proposed previously in the literature (e.g., [21, 22, 5]).

The possible relations between pairs of entities in a map are highly simplified, with respect to the generic case. In the context of a single map, we can only have the following situations:

- two distinct regions either are *disjoint*, or they *meet* at a common boundary (possibly a single point);
- given a line $l$ and a region $r$ we can have one of the following: $l$ and $r$ are *disjoint*; $l$ and $r$ *meet* at a single point $p$, where $p$ is an endpoint of $l$ and $p$ *bounds* $r$; $l$ *bounds* $r$; $l$ is *inside* $r$; $l$ is *coveredBy* $r$ (i.e., the relative interior of $l$ is inside $r$, and one endpoint of $l$ *bounds* $r$);
- given a point $p$ and a region $r$ we can have one of the following: $p$ and $r$ are *disjoint*; $p$ *bounds* $r$ (i.e., it is endpoint of two consecutive lines on the boundary of $r$); $p$ is *inside* $r$;
- given two distinct lines they either are *disjoint*, or they *meet* at a common endpoint;
- given a point $p$ and a line $l$ they are either *disjoint*, or $p$ *bounds* (i.e., it is an endpoint of) $l$;
- two distinct points are *disjoint*.

Note that for each non-symmetric relation listed above, the *converse* relation is verified too (e.g., $l$ is *coveredBy* $r$ if and only if $r$ *covers* $l$).

We give the concepts of *combinatorial boundary* and *combinatorial coboundary*, or *star*, of entities in a map $M$, where each entity is regarded as an atom. The combinatorial boundary differs from the topological boundary defined before in being composed of a collection of cells, rather than being a subset of $\mathbb{R}^2$: the relation between the combinatorial and topological boundary is readily seen from the definition. In the sequel of the paper we will always omit the adjective combinatorial, whenever no ambiguity arises [20].

- The combinatorial boundary of any point $p \in P$ is empty; the combinatorial boundary of a line $l \in L$, denoted $\partial l$, is formed by its two endpoints if $l$ is an open line, otherwise it is empty; the combinatorial boundary of a region $r \in R$, denoted $\partial r$, is formed by all lines and points of $M$ contained either in r or in the Euclidean boundary of $r$.
- The star of a point $p \in P$, denoted $*p$, is either formed by all lines of $L$, and regions of $R$ having $p$ on their boundary, or is the region containing $p$ in its interior, in case $p$ is an isolated point or lies on a lineal feature; the star of a line $l \in L$, denoted $*l$, is either formed by the (at most two) regions having $l$ as part of their boundary, or is the region containing $l$ in its interior, in case $l$ is a line feature; the star of a region $r \in R$ is $r$ itself.

[3] This definition includes as a region of $M$ also the infinite portion of plane surrounding the domain covered by the entities of $M$.

Given a map $M$, a chain $c$ of lines in $M$ is said a *free chain* if for any joint $p$ of $c$, the only lines in $*p$ belonging to $c$. In other words, $c$ meets no other lines of the map, except possibly at its endpoints.

On the basis of the concept of star, in the following section we make an abstraction over a map $M$, by considering only its combinatorial structure, while completely disregarding its metric structure. This fact will allow us to formally define and study separately all aspects of maps that are either purely combinatorial, or purely metric.

## 3 The Combinatorial Structure of Geographic Maps

The idea that geographic maps should be represented and studied as geometric cell complexes has been already stressed in the literature. Here, we handle the combinatorial aspects of maps through a structure that is more abstract than others proposed in the literature, such as simplicial complexes [7, 27], Plane Euclidean Graphs [5], or CW-complexes [22, 23]. Abstract cell complexes can capture the whole topological nature of maps, independently of their geometry [19, 20].

Let $C$ be a finite set, called the *set of cells*. An *abstract cell complex* $\Gamma = (C, \prec, \dim)$ with cells in $C$ is defined as follows:

- $\prec$ is a strict partial ordering on the elements of $C$ (i.e., $\prec$ is an irreflexive, antisymmetric, and transitive binary relation) called the *bounding relation*;
- $\dim : C \to \mathbb{N}$, called the *dimension function*, is such that

$$\gamma' \prec \gamma'' \Rightarrow \dim(\gamma') < \dim(\gamma'').$$

A cell $\gamma$ for which $\dim(\gamma) = k$ is called a *$k$-cell*. It is not restrictive to have dim such that there always exist some cells of dimension zero. A complex is called *$d$-dimensional* or a *$d$-complex* if $\max_{\gamma \in C}(\dim(\gamma)) = d$.

The *boundary*[4] of a cell $\gamma$ of $\Gamma$ is defined as $\partial\gamma = \{\xi \in C \mid \xi \prec \gamma\}$.

The *star* of a cell $\gamma$ of $\Gamma$ is defined as $*\gamma = \{\xi \in C \mid \gamma \prec \xi\} \cup \{\gamma\}$.

A *subcomplex* $\Gamma' = (C', \prec', \dim')$ of a given complex $\Gamma = (C, \prec, \dim)$ is a complex whose set $C'$ is a subset of C, and relation $\prec'$ and function $\dim'$ are restrictions of $\prec$ and dim to $C'$, respectively. A subcomplex $\Gamma'$ of $\Gamma$ is *open* if for every cell $\gamma$ of $\Gamma'$ all cells of the star of $\gamma$ in $\Gamma$ are also cells of $\Gamma'$. A subcomplex $\Gamma'$ of $\Gamma$ is *closed* if for every cell $\gamma$ of $\Gamma'$ all cells of the boundary of $\gamma$ in $\Gamma$ are also cells of $\Gamma'$. A subcomplex $\Gamma'$ of $\Gamma$ is *regular* if each cell $\gamma$ of $\Gamma'$ belongs to the star of some cell $\gamma'$ of $\Gamma'$ that has maximal dimension (possibly with $\gamma' = \gamma$). The *boundary* of a regular subcomplex $\Gamma'$ is the set of cells of $\Gamma'$ that belong to the boundary of some cell of $C \setminus C'$ in $\Gamma$; all other cells of $\Gamma'$ are called *internal* cells.

It follows from the definitions above that in order to define a subcomplex $\Gamma'$ of $\Gamma$ it suffices to define the corresponding subset $C'$ of the elements. All

[4] The overloading of terms *boundary* and *star* - that were already defined in the context of maps - is intentional here.

subcomplexes of $\Gamma$ may be regarded as subsets of $C$, therefore, it is possible to use the common formulae of the set theory to define intersections, unions, and complements of subcomplexes of one and the same complex $\Gamma$. The following proposition was proven in [19]:

**Proposition 3.1** *Let $\Gamma = (C, \prec, \dim)$ a cell complex. Let $T_\Gamma$ be the (finite) set of all open subcomplexes of $\Gamma$. Then $(C, T_\Gamma)$ is a separable topological space, and $B_\Gamma = \{*\gamma \mid \gamma \in C\}$ is a basis for $(C, T_\Gamma)$.*

It is easy also to see that for any cell $\gamma \in \Gamma$, its star $*\gamma$ is indeed the smallest neighbourhood of $\gamma$; hence, all cells with maximum dimension are open sets.

The fact that $\Gamma$ is a topological space is very important because allows us to exploit all results of topology - in particular, all results concerning mappings between topological spaces and homeomorphisms - to study the combinatorial structure of abstract cell complexes. Actually, an even stronger result states that each finite separable topological space is indeed an abstract cell complex [19]. For this reason the topology of cell complexes is called the *finite topology*: it is the only possible (non trivial) topology that one can consider on finite sets.

With this facts in mind, we observe that the family of all entities composing a map is indeed a finite set, and, thus, it should be possible to regard a map as an abstract cell complex. It is indeed straightforward to define an abstract cell complex on a map, which retains the whole combinatorial structure of $M$, while disregarding all its metric aspects. The proof of the following proposition is almost trivial, hence omitted.

**Proposition 3.2** *Let $M = (P, L, R)$ be a map. Let us define $C = P \cup L \cup R$. Let $\prec$ be a relation on $C$ defined as follows:*

$$x \prec y \Leftrightarrow x \subset y,$$

*where the symbol $\subset$ denotes containment between sets of $\mathbb{R}^2$. Let* $\dim$ *be a function defined as follows:*

$$\dim(x) = \begin{cases} 0 \text{ if } x \in P \\ 1 \text{ if } x \in L \\ 2 \text{ if } x \in R \end{cases}$$

*Then, $\Gamma = (C, \prec, \dim)$ is an abstract cell complex.*

From now on, when dealing with purely combinatorial aspects of maps, we will use interchangeably a map $M$, and its associated cell complex $\Gamma$, whenever no ambiguity arises. Also, we will speak of a *submap* of $M$ by meaning its associated subcomplex $\Gamma' \subset \Gamma$. Therefore, we will speak of points, lines, and regions of a complex, by meaning its cells of dimension zero, one, and two, respectively. Note that purely geometric concepts defined on spatial entities, such as simple line, relative interior, portion, internal and external set, outer and inner boundary, hole, interior, and closure, have no meaning in the context of an abstract complex. Nevertheless, some concepts like endpoints, open and closed line, chain, characteristic of a region, and, thus, simply and multiply connected region, have a straightforward translation in the context of an abstract complex. It is interesting to notice that in abstract cell complexes we are still able to say whether a region has holes or not, although we cannot decide which adjacent regions lie inside the holes, and which in the outer space: there is indeed no concept of outer space! The concept of characteristic is extended to a regular subcomplex (and its relative submap) by counting the number of closed chains forming the boundary of the subcomplex, as defined before.

By means of mappings and homeomorphisms between cell complexes we are able to characterise the similarity and equivalence of spatial maps. Let $\Gamma = (C, \prec, \dim)$ and $\Gamma' = (C', \prec', \dim')$ be two cell complexes. We call *mapping*[5] between $\Gamma$ and $\Gamma'$ an application $F : C \to C'$. With abuse of notation, we will indicate $F : \Gamma \to \Gamma'$.

A mapping $F$ defined as above is said *continuous* if for each $U$ open set of $\Gamma'$ (endowed with the finite topology) the inverse image $F^{-1}(U)$ is open in $\Gamma$ (also in the finite topology). In particular, $F$ is continuous in $\gamma \in \Gamma$ if $F^{-1}(*F(\gamma))$ is an open neighborhood of $\gamma$. Continuous mappings have nice properties such as preserving connectedness. Thus, if $\gamma, \xi$ *meet* in $\Gamma$, and $F$ is continuous, then $F(\gamma)$ and $F(\xi)$ cannot be *disjoint* in $\Gamma'$.

A one-to-one correspondence $F : \Gamma \to \Gamma'$ for which both $F$ and the inverse function $F^{-1}$ are continuous is a homeomorphism in the finite topology: in this case $\Gamma$ and $\Gamma'$ are said to be *isomorphic* [20]. The isomorphism between two complexes guarantees that their associated maps are combinatorially equivalent, i.e., that both the entities composing the maps, and their spatial relationships are in one-to-one correspondance[6].

## 4 Map Simplification

Different maps that correspond to non homeomorphic cell complexes are not topologically equivalent. However, we are interested in studying maps that can be converted into each other by means of reciprocal processes of simplification/refinement, in which details are either discarded or introduced while maintaining consistent the overall structure. In this section we will see that such maps can be related through continuous mappings.

First of all, we must informally understand what changes can involve simplifying a map $M$ into a less detailed map $M'$. The monotonicity assumption (item 4a in the Introduction) guarantees that for each entity in map $M'$ there must exist a corresponding entity in $M$; moreover, since each entity of $M$ must correspond to something either simpler or equivalent in $M'$, then a single entity

[5] In algebraic topology, an application between cell complexes is called a *map*. Here, we use the term *mapping* to avoid confusion with geographic maps, which are semantic objects whose structure is represented by cell complexes.

[6] In [13], *object homoemorphisms* and *relation homeomorphisms* between sets of spatial entities were defined. The isomorphism between cell complexes incorporates both the object homeomorphism, and the relation homeomorphism when spatial entities are considered in the context of a map.

of $M$ can correspond to at most one entity in $M'$. These simple observations guarantee that the map-to-map correspondance describing a simplification from $M$ to $M'$ can be described by a surjective mapping $F : \Gamma \rightarrow \Gamma'$, that will be called a *simplification mapping*, where $\Gamma$ and $\Gamma'$ are the cell complexes describing $M$ and $M'$, respectively.

Let us consider now a generic entity $e$ of a map $M$. We outline three possible basic simplifications of $e$ into $M'$:

- *preservation*: the object represented by $e$ appears also in $M'$ with the same dimension, and possibly with a simplified structure (e.g., if $e$ is a region, its corresponding entity in $M'$ will be also a region, possibly with a simplified boundary, or a smaller set of features, or a smaller characteristic).
- *reduction*: the object represented by $e$ appears also in $M'$, but with a lower dimension (e.g., if $e$ is a region, its corresponding entity in $M'$ could be a single point).
- *immersion*: the object represented by $e$ disappears in $M'$, i.e., it is immersed into some larger object.

If we translate the previous cases in terms of the simplification mapping we obtain:

- *preservation*: $\dim(F(\gamma)) = \dim(\gamma) \quad \wedge \quad \forall \tau \in \partial\gamma, \dim(F(\tau)) < \dim(\gamma)$;
- *reduction*: $\dim(F(\gamma)) < \dim(\gamma)$;
- *immersion*: $\dim(F(\gamma)) > \dim(\gamma) \quad \vee \quad (\dim(F(\gamma)) = \dim(\gamma) \quad \wedge$
$\exists \tau \in \partial\gamma : \dim(F(\tau)) = \dim(\gamma))$.

Note that when the image of a cell $\gamma$ maintains the same dimension as $\gamma$ itself, it may be either that $\gamma$ is maintained in the simplification, or that it is immersed into a "larger" cell. Indeed, when a whole submap of $M$ is simplified into a unique region $r'$ of $M'$, any cell $\gamma$ corresponding to a region of such submap must have as image $\gamma'$ corresponding to region $r'$. The discrimination between preservation and immersion is obtained through the condition on the boundary: indeed, if region $r$ is immersed into region $r'$, also one portion of the boundary of $r$ must be immersed inside $m'$.

Further constraints on simplification are concerning the consistency of topological relationships in the domain and codomain. For instance, objects that *meet* before simplification, cannot possibly be *disjoint* in the simplified map; also, a point that is *disjoint* from a region cannot jump *inside* the region in the simplified map (unless both the point and the region are immersed into something bigger); on the other hand, it is possible that two *disjoint* regions $r$ and $s$ will *meet* after simplification, if, for instance, a third region $t$ separating $r$ from $s$ is reduced to a line. A complete analysis of all possible cases arising from the simplification of pairs of entities is possible, on the basis of the admissible relationships between pairs of entities in a map (listed in Sect. 2). As one can easily guess, a result of such analysis is that a consistent simplification mapping must be continuous. Indeed, we need continuity if we want to guarantee that two objects that *meet* in the domain will not be *disjoint* in the codomain.

Although a complete analysis of possible cases is perhaps the most general approach to the formalization of consistency, it is quite involved and technical, hence not treated in this paper [6]. Here, we will rather give rules to define a continuous mapping, which reflect gradual changes, and can be used as constraints on the possible simplification mappings. So far, we have not given any constraint to guarantee the graduality of changes during simplification. The continuity of mappings guarantees consistency, but it is not sufficient to guarantee graduality: for instance, mapping of an arbitrarily complex structure into a map made of a single point is indeed continuous, but it is not much interesting! On the contrary, we wish to have simplification mappings that do not modify maps too abruptly, skipping meaningful intermediate representations.

We give rules only for the reductions of a single element that can happen without involving reductions of elements of higher dimension in its star, while reductions or immersions of elements of lower dimension are implied by the rules, and thus are defined inductively. The rules we list are not the only possible ones. Other consistent rules can be considered, depending on the application needs, still in the context of the same abstract framework. In the following, we use usual notations for regions, lines, and points, meaning their associated 2-, 1-, and 0-cells, respectively.

1. A line $l$ can reduce to a point $p'$ (Fig. 2a): in this case, also the endpoints of $l$ are mapped to $p'$; the region(s) of $*l$ simplify to elements of $*p'$. In synthesis:

$$F(l) = p' \;\Leftrightarrow\; (\forall p \in \partial l,\; F(p) = p').$$

2. A free open chain $c$ can reduce to an open line $l'$ (Fig. 2b): in this case, all lines and joints of $c$ are immersed into the same line $l'$, while the endpoints of $c$ are preserved into the endpoints of $l'$. In synthesis:

$$F(c) = l \;\Leftrightarrow (\forall l \text{ line of } c,\; F(l) = l')$$

$$\wedge$$

$$(\forall p \text{ joint of } c,\; F(p) = l') \quad \wedge \quad (\forall p \text{ endpoint of } c,\; F(p) \in \partial l').$$

   The case of a closed chain that can reduce to a closed line is completely analogous: the condition on endpoints is just ignored.
3. A free open chain $c$ can reduce to a point $p'$: this case is easily derived from the composition of the previous two, hence not detailed.
4. A simply connected region $r$ can reduce to a point $p'$ (Fig. 2c): in this case, features and elements of the boundary of $r$ are also reduced to $p'$. In synthesis:

$$F(r) = p' \;\Leftrightarrow\; \forall p, l \in \partial r,\; F(p) = F(l) = p'.$$

5. A simply connected region $r$ can reduce to an open line $l'$ (Fig. 2d-e): in this case, it must be possible to subdivide the boundary of $r$ into four consecutive free chains $c_{p_0}, c_{l_0}, c_{p_1}, c_{l_1}$, where $c_{p_0}$ and $c_{p_1}$ can possibly be degenerate to single points (Fig. 2e), such that $c_{l_0}$ and $c_{l_1}$ reduce to $l'$, and $c_{p_0}$ and $c_{p_1}$ reduce to the two endpoints of $l'$, respectively. Endpoints and joints of chains

reduce consistently to such chains, according to rules 2 and 3. Let $p_0'$ and $p_2'$ denote the endpoints of $l'$, and let symbol $\searrow$ denote a consistent reduction according to rule 2 or 3. In synthesis we have:

$$F(r) = l \;\Leftrightarrow\; (c_{l_0} \searrow l) \;\;\wedge\;\; (c_{l_1} \searrow l) \;\;\wedge\;\; (c_{p_0} \searrow p_0') \;\;\wedge\;\; (c_{p_1} \searrow p_1').$$

6. A regular submap $\bar{M}$ with characteristic $k$, containing more than one entity, can reduce to a single region $r$ with characteristic $k$ (Fig. 2f): in this case, all regions of $\bar{M}$ are immersed into $r$; all lines and point s internal to $\bar{M}$ are either immersed into $r$ or mapped to features of $r$ (through consistent mappings, according to rules 1, 2, 3); all lines and points forming the boundary of $\bar{M}$ are mapped to the boundary of $r$ (through consistent mappings, according to rules 1, 2, 3). The synthetic description of this transformation is involved and omitted here for brevity [6].

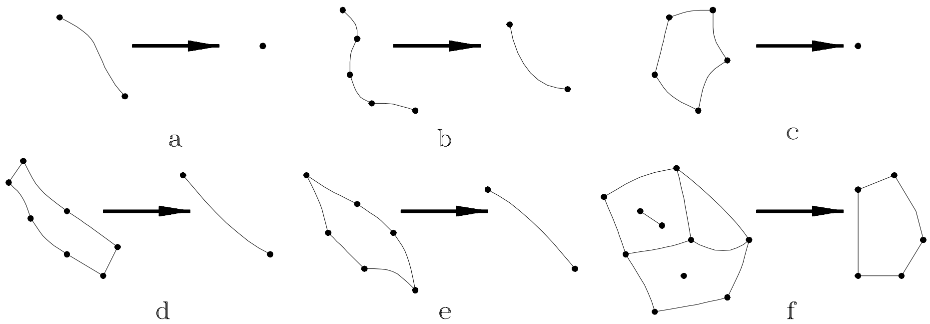


**Fig. 2.** Simplifications of entities and submaps: (a) from line to point; (b) from chain to line; (c) from region to point; (d,e) two different simplifications from region to line; (f) from submap to region.

It is easy to verify that a mapping $F : \Gamma \to \Gamma'$ satisfying the rules listed above will verify the following conditions on the inverse images of elements of $\Gamma'$.

- $\forall p'$ point of $\Gamma'$, $F^{-1}(p')$ is formed either by a single point, or by an edge plus its endpoints, or by a free chain plus its joints and endpoints, or by a simply connected region plus the elements of its boundary.
- $\forall l'$ line of $\Gamma'$, $F^{-1}(l')$ is formed either by a single line, or by the lines forming a free chain plus all joints of the chain, or by a simply connected region plus a portion of its boundary formed by two free chains without their endpoints, and such that the rest of the boundary is formed by two connected components, each of which is either a free chain or a point.
- $\forall r'$ region of $\Gamma'$, $F^{-1}(r')$ is either a region, or an open regular submap of $\Gamma$ having the same characteristic as $r'$.

By using such conditions it is easy (though tedious) to test that the inverse image of $*\gamma'$, for every $\gamma' \in \Gamma'$ is open in $\Gamma$, and hence, that a simplification $F$ satisfying rules 1-6 is continuous [6].

## 5 Metric Aspects of Multiresolution

The previous discussion on map similarity takes into consideration only the combinatorial structure of maps. However, the fact that two maps have consistent structures does not imply that they are also metrically consistent. While variations in the combinatorial structure are related to the concept of detail in a map, variations of its metric aspects are related to the concept of precision (see item 4 in the Introduction).

In [1], the precision of a map was measured through the horizontal error in representing the lines in the map, formalised through the mathematical concept of $\varepsilon$-homotopy. Here, we extend and formalise further such concept to measure the precision of any possible entity.

Intuitively, a geometric object $o$ is represented by an entity $e$ at precision $\varepsilon \geq 0$ if and only if $o$ is contained in a region obtained by "fattening" $e$ of $\varepsilon$, i.e., in $r_{e,\varepsilon} = \{x \mid d(x, e) \leq \varepsilon\}$, where $d$ denotes the Hausdorff distance in $\mathbb{R}^2$. Note that a "thin" (though possibly "long") region, such as a portion of road or river in a geographic map, can possibly be represented by a line; also, a region that is completely contained in a disc of radius $\varepsilon$ can be approximated by a single point. We will model such idea of approximation by specifying a continuous transformation that maps the object $o$ into the entity $e$ representing it without leaving the fattened region $r_{e,\varepsilon}$.

Let $f, g : I \to \mathbb{R}^2$ two continuous functions from the standard unit interval into the real plane. A *homotopy* between $f$ and $g$ is a continuous function $H : I \times I \to \mathbb{R}^2$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ [24]. Such homotopy $H$ defines a continuous deformation from $f$ to $g$ through all functions $H(\cdot, y)$, $y \in (0, 1)$. For $\varepsilon \geq 0$, a homotopy $H$ is called an $\varepsilon$-homotopy if

$$\forall x \in I,\ \forall y \in I,\quad d(H(x, y), H(x, 0)) < \varepsilon.$$

If $H$ is an $\varepsilon$-homotopy between functions $f$ and $g$, then $f$ and $g$ are said $\varepsilon$-homotopic. In particular, if $f$ and $g$ are two lines, this means that we can approximate $f$ by $g$ (or viceversa) with precision $\varepsilon$. Note that the $\varepsilon$-homotopy implies a condition much stronger than simply assuming $g$ inside the region fattening $f$: indeed, $\varepsilon$-homotopy requires that any portion of $g$ remains inside the fattened region of the corresponding portion of $f$; or, in other words, that two walkers synchronised to start together from $f(0)$ and $g(0)$, and to arrive together at $g(1)$ and $f(1)$ respectively, will always remain at a distance smaller or equal than $\varepsilon$. This fact excludes that $g$ could possibly be a line going back and for an arbitrary number of times inside the "tube" surrounding $f$ (see Fig. 3a), while it is admitted a fractal representation $g$ that has a length much higher than $f$, but remains locally near $f$ (see Fig. 3b).

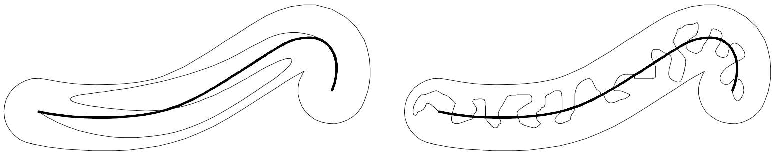

**Fig. 3.** Approximations of lines ($f$ is the thick line, $g$ is the thin line): (left) $g$ is not $\varepsilon$-homotopic to $f$; (right) $g$ is $\varepsilon$-homotopic to $f$.

If $f$ is a line and $g$ is a constant function, then the fact that $H$ is a $\varepsilon$-homotopy means that $f$ can be approximated by a single point at precision $\varepsilon$.

Approximations of regions can be defined by considering $\varepsilon$-homotopies of the lines (chains) defining their boundaries. A region $r$ can be approximated by a point $p$ at precision $\varepsilon$ if and only if the outer boundary of $r$ is $\varepsilon$-homotopic to $p$ (seen as a constant function). A region $r$ can be approximated by a line $l$ at precision $\varepsilon$ if and only if the outer boundary of $r$ can be subdivided into two chains such that each of them is $\varepsilon$-homotopic to $l$ . A region $r$ can be approximated by another region $s$ if and only if (i) the outer boundary of $r$ is $\varepsilon$-homotopic to the outer boundary of $s$, (ii) each inner boundary of $s$ is $\varepsilon$-homotopic to an inner boundary of $r$, and (iii) any inner boundary of $r$ that is not $\varepsilon$-homotopic to an inner boundary of $s$ is $\varepsilon$-homotopic to a point. A regular submap $\bar{M}$ can be approximated by a region $r$ at precision $\varepsilon$ if and only if the region covered by $\bar{M}$ is $\varepsilon$-homotopic to $r$.

Based on the above definitions, we can introduce the concept of $\varepsilon$-simplification of a map. Given two maps $M$ and $M'$, a simplification $F : M \to M'$ is an $\varepsilon$-*simplification* if and only if

$$\forall e' \in M' \quad e' \text{ approximates } F^{-1}(e') \text{ at precision } \varepsilon.$$

Hence, combinatorial and metric map simplifications, based respectively on variations of detail and of precision, are combined to obtain a simplified map that is consistent both in maintaining compatible topological and metric relations.

Note that the level of precision does not give by itself any condition on map elements that survive, and elements that are immersed, when simplifying $M$ into $M'$. This is due to the fact that the degree of detail and the level of precision of a map are not necessarily related with each other, since there is no quantifiable dependence between the scale of a map and the fact that a given entity appears in it or not. Indeed, the relation between the relevance of an entity and the scale of a map can depend both on the size of the entity and on the semantics associated to the map. In actual implementations of systems for map generalisation, combinatorial and metric criteria must be strictly interrelated, together with semantic criteria, and all such information must be integrated to guide map simplification. The detail reduction, whose consistence we have analysed by formal methods, must be actually driven by metric checks on the actual map features, while reductions based on metric evaluations can be applied only as long as they do not violate structural consistency. However, we think that it is very important to separate the two components when working on theoretic aspects of multirepresentation, because the two levels involve different problems, and can be better studied in different mathematical environments.

## 6 Multiresolution Models

Let $M_0$ be a map at the maximum available resolution, and let $M_1$ be a simplification of $M_0$ through a simplification mapping $F_1$. If we apply a simplification mapping $F_2 : M_1 \to M_2$, and so on iteratively, we obtain a whole sequences of upward compatible maps,

$$M_0 \xrightarrow{F_1} M_1 \xrightarrow{F_2} \ldots \xrightarrow{F_n} M_n,$$

corresponding to less and less detailed descriptions of the same area. Note that the composed mapping

$$F^i = F_1 \circ F_2 \circ \cdots \circ F_i : M_0 \to M_i,$$

$\forall i = 1, \ldots, n$, is itself continuous, and it describes a *macro-simplification* of $M_0$, which transforms it abruptly from the full resolution to the $i$-th degree of detail. If we are further given an increasing sequence of tolerances $\varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_n$, such that the composed mapping $F^i$, $\forall i = 1, \ldots, n$, is a $\varepsilon_i$-simplification of $M_0$, then our sequence is consistent both with detail and precision simplification, and it is called a *multiresolution sequence.*

Let $\mathcal{M} = \{M_i \mid i = 0 \ldots, n\}$ be the whole family of maps of a multiresolution sequence, and let $\mathcal{F} = \{F_i \mid i = 1 \ldots, n\}$ be the corresponding family of simplifications, as defined above. The pair $(\mathcal{M}, \mathcal{F})$ is called a *layered multiresolution model.* A layered multiresolution model represents the map of a domain at different levels of resolution through different independent models that cover the whole domain. Models that are adjacent in the sequence are related through simplification mappings, which can be implemented as "vertical" links that allow browsing the model through the different levels of resolution, while "horizontal" browsing happens in the context of a single map.

In order to make more manageable the whole structure, it is possible to define an alternative model, in which every map $\mathcal{M}$ can be subdivided into independent submaps covering disjoint portions of the domain. In such a model a portion of a map can be handled without needing to consider the map of the whole domain. This approach gives rise to a hierarchy of maps similar to that proposed in [1], which is described by a tree structure. The hierarchy can be inductively defined on the basis of the multiresolution sequence. Since maps represented with more detail and precision contain a higher number of elements, the tree results reversed with respect to the layered model, i.e., it has as root the map at coarsest resolution.

Let $M_n$ be the root node. The tree is defined inductively on the basis of the inverse image of elements inside each node. Given a generic node $M_{i,j}$, which is a submap of $M_i$ (initially, we have $M_{n,1} = M_n$), we consider each element $e_k$

of $M_{i,j}$ whose inverse image through $F_i^{-1}$ contains more than one cell. For each such element $e_k$ we build a node[7]

$$M_{i-1,k} = \overline{F_i^{-1}(e_k)},$$

where $\overline{F_i^{-1}(e_k)}$ denotes the closure of $F_i^{-1}(e_k)$ (i.e., the inverse image is completed with its boundary). According to the usual terminology, $M_{i-1,k}$ is a *child* of $M_{i,j}$, labeled through $e_k$. Let $\mathcal{N} = \{M_n, M_{n-1,1}, \dots, M_{0,h}\}$ be the whole family of submaps generated inductively as above, and let us define

$$\mathcal{E} = \{(M_{i,j}, M_{i-1,k}) \mid M_{i-1,k} \text{ is child of } M_{i,j}\}.$$

The pair $(\mathcal{N}, \mathcal{E})$ is called a *hierarchical multiresolution model.*

The child relationship expresses the *refinement* of (a portion of) a map, while the inverse *parent* relationship expresses a simplification. The root $M_n$ represents the map over the whole domain, at precision $\varepsilon_n$; all nodes at level $i$ represent portions of the map at the same resolution $\varepsilon_i$.

The model is richer than the hierarchical model described in [1], since the nodes of the tree are not simply refinements of regions, but they can also be non-regular submaps refining points and lines through chains, lines, and regions, as described in Section 4.

## 7 Concluding Remarks

We have given a formal framework that permits to study the representation of maps at multiple resolution by mathematical tools that keep combinatorial and geometric aspects separated. Abstract cell complexes and mappings appear promising means for handling the relations between maps at different detail, while $\varepsilon$-homotopies help handling multiple precision.

On the basis of such framework we have defined compatibility rules for mappings that define gradual simplifications of maps into less detailed ones. We have combined such mappings with metric aspects in order to obtain map transformations that take into account both combinatorial and metric aspects of multiresolution. We have derived multiresolution models that admit a natural and elegant definition on the basis of a sequence of successive simplification mappings. Such models are more general than other multiresolution models previously proposed.

We believe that this study can be widely developed towards a more synthetic and precise interpretation of integrated combinatorial and metric aspects of multiresolution, in order to obtain a sound theory for multiresolution representations.

Although the scope of this paper is limited to two-dimensional geographic maps, the same concepts and tools can be applied in the context of multidimensional representations based on decompositions of domains into cells. We plan to extend our study in the future to structures in arbitrary dimension.

[7] We assume here that $k$ is an index that uniquely identifies an element of $M_i$.

We have intentionally left out of the scope of this paper all issues concerning data structures and computational aspects of multiresolution. Based on former experience on multiresolution models for terrains [4], we believe that a clear formalization and a sound theory are essential for an efficient implementation of models. The literature offers several schemes of data structures for multidimensional cell complexes, that seem suitable as a starting point to elaborate the necessary data structures to implement our representation framework. From the computational point of view, constructing a multiresolution representation implies classical problems of map simplification, hence requiring a variety of topological, metric and semantic bindings. We believe that our formalization helps providing topological and metric constraints to support logical deduction in the framework of hybrid systems for map simplification. We plan to tackle data structures and computational aspects in future work.

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