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The big Mother of all Dualities 2: Macaulay Bases

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Abstract We present some interesting computational applications of Macaulay's notion of *inverse systems* and *Noether equations*. In particular we discuss an algorithm by Macualay which computes the forgotten notion (introduced by Emmy Noether) of *reduced irreducible decomposition* for ideals of the polynomial ring.

Keywords Lasker-Noether Decomposition, Macaulay Inverse Systems

Mathematics Subject Classification (2000) 13E10, 13E05, 13P10

1 Introduction

This paper is the second in a series of survey articles on results which introduced duality tools in computer algebra and follows [1], reported on Möller's Algorithm. Its aim is to present some interesting computational applications of Macaulay's notions of *inverse systems* and *Noether equations*.

After introducing general notation (§ 2), we recall Macaulay's ideas, his notion of Noether equations as a tool for describing $\mathfrak{m}$-closed ideals (§ 3) and the module structure imposed on inverse systems (§ 4).

We then discuss Macaulay's duality between $\mathfrak{m}$-closed ideals and modules of Noether equations (§ 5), together with Gröbner's interpretation of Macaulay's results in terms of differential equations (§ 7) and we deduce the relations with Leibniz's (§ 6) and Taylor's Formulas (§ 8)

We formalize and specialize the duality discussed in the previous paper [1], proposing a characterization of zero-dimensional ideals in a polynomial ring in terms of what we label *Macaulay Bases* (§ 9),

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i.e. the collections of the Noether equations of the primary components. We also discuss a compact representation of them (§ 13), which is already implicitly presented in Macaulay's book (cf. [13],§ 69), and the relations between Macaulay Bases and Gröbner Bases (§ 10).

In order to illustrate the most relevant applications of Macaulay's ideas, we need to recall (§ 11) a forgotten idea by Emmy Noether, namely she proved a stronger version of the well-known Lasker–Noether Decomposition Theorem, that is: each ideal has an *irredundant reduced primary decomposition* (see Rem 11.2, $VIII$); such irredundant reduced decomposition can be characterized also for embedded primary ideals, unfortunately strongly depending on the frame of coordinates, so no uniqueness result can be stated. Gröbner (cf. [9], pp. 177–178) explicitly suggested to apply an improved version of an algorithm by Macaulay (§ 12) in order to successfully compute the irreducible reduced decomposition of an ideal: both the examples by Hentzel and the approach used by Gröbner clarify that such notion strongly depends on a frame of coordinates.

We conclude this survey pointing, without proofs, to some further results:

- a good complexity algorithm which allows to evaluate a polynomial into a Macaulay basis (§ 14.1)
- a good complexity algorithm which allows to compute Noether equations of a primary ideal (§ 14.2)
- a combinatorial algorithm which deduces the (finite) *Gröbner éscalier* (i.e. the set of terms which are not maximal terms of members of the given ideal) of a zero-dimensional ideal from its Macaulay basis. (§ 14.3)
- a theorem which merges Lazard Theorem [11], Möller's Algorithm [4,1] Gianni–Kalkbrener Theorem [8,10] and Cerlienco–Mureddu Correspondence [5,6] giving a strong description of the structure of the Gröbner basis and of the dual basis of a zero-dimensional ideal (§ 14.3).

2 General notation

Throughout the paper
$\mathbf{k}$ is a field,
for all $n \in \mathbb{N}$, $\{X_1, \dots, X_n\}$ is a (finite) set of indeterminates,
$\mathcal{P} := \mathbf{k}[X_1, \dots, X_n] \subset \mathbf{k}[[X_1, \dots, X_n]] =: \mathcal{S}$,
$\mathfrak{a} \subset \mathcal{P}$ is an ideal and
$\mathfrak{m} := (X_1, \dots, X_n) \subset \mathcal{P}$ is the maximal ideal at the origin O.
Moreover, we let $\mathcal{T} := \{X_1^{a_1} \dots X_r^{a_n} : (a_1, \dots, a_n) \in \mathbb{N}^n\}$ and, for all $d \in \mathbb{N}$,
$\mathcal{T}_d := \{\tau \in \mathcal{T} : deg(\tau) = d\}$ and $\mathcal{T}(d) := \{\tau \in \mathcal{T} : deg(\tau) \leq d\}$.
For any $\tau \in \mathcal{T}$ and $1 \leq h \leq n$ with $X_h \mid \tau$, the term $\frac{\tau}{X_h}$ is called *h-predecessor* of τ and for all $j \in \{1, \dots, n\}$ the term $X_j\tau$ is called *j-successor* of τ.

Each (non-zero) $f \in \mathcal{P}$ can be uniquely expressed either as polynomial in X_n over $\mathbf{k}[X_1, \dots, X_{n-1}]$

$$f = \sum_{i=0}^{\deg(f)} g_i X_n^i, g_i \in \mathbf{k}[X_1, \dots, X_{n-1}], (g_{\deg(f)} \neq 0), \tag{1}$$

or, if $\mathbf{B}$ is any $\mathbf{k}$-basis of $\mathcal{P}$, as linear combination of elements in $\mathbf{B}$

$$f = \sum_{\beta \in \mathbf{B}} c(f, \beta)\beta = \sum_{i=1}^{s} c(f, \beta_i)\beta_i : c(f, \beta_i) \in k^*, \beta_i \in \mathbf{B}. \tag{2}$$

The $\mathbf{B}$-*support of* f is the finite set

$$\mathbf{S_B}(f) := \{\beta \in \mathbf{B} : c(f, \beta) \neq 0\}.$$

In particular, if $\mathbf{B} = \mathcal{T}$ instead of $\mathbf{S}_\mathcal{T}(f)$ we simply write $\mathbf{S}(f)$ and also consider

$$\tilde{\mathbf{S}}(f) := \{t \in \mathcal{T} : t \mid \tau, \text{ for some } \tau \in \mathbf{S}(f)\}.$$

For any $f = \sum_{\tau \in \mathcal{T}} c(f, \tau)\tau \in \mathcal{S}$ is called *support of* f also the, possibly infinite, set
$\mathbf{S}(f) := \{\tau \in \mathcal{T} : c(f, \tau) \neq 0\}$.

If $<$ is a semigroup ordering on $\mathcal{T}$, then, for any $f \in \mathcal{P}$, the $t_i \in \mathbf{S}(f)$ can be chosen so that $t_1 > \cdots > t_s$ and call
$\mathbf{T}_<(f) := t_1$ is called *maximal term of f*, $\mathrm{lc}(f)_< := c(f, t_1)$ *leading cofficient of f*.
For each set $G \subset \mathcal{P}$ we consider the set

$$\mathbf{T}_<\{G\} := \{\mathbf{T}_<(g) : g \in G\}, \text{ and}$$

$$\mathbf{T}_<(G) := \{\tau\mathbf{T}_<(g) : \tau \in \mathcal{T}, g \in G\}$$

the monomial ideal it generates.
For each ideal $\mathfrak{a} \subset \mathcal{P}$, the minimal basis of the monomial ideal $\mathbf{T}_<(\mathfrak{a}) = \mathbf{T}_<\{\mathfrak{a}\}$ is denoted $\mathbf{G}(\mathfrak{a})$. Moreover,

$$\mathbf{N}_<(\mathfrak{a}) := \mathcal{T} \setminus \mathbf{T}_<(\mathfrak{a})^1,$$

$$\begin{aligned}\mathbf{B}_<(\mathfrak{a}) &:= \{X_h t : 1 \le h \le n, t \in \mathbf{N}_<(\mathfrak{a})\} \setminus \mathbf{N}_<(\mathfrak{a}) \\ &= \mathbf{T}_<(\mathfrak{a}) \cap (\{1\} \cup \{X_h t : 1 \le h \le n, t \in \mathbf{N}_<(\mathfrak{a})\}),\end{aligned}$$

$$\mathbf{C}_<(\mathfrak{a}) := \{t \in \mathbf{N}_<(\mathfrak{a}) : X_h t \in \mathbf{T}_<(\mathfrak{a}), \text{ for all } h\},$$

are respectively called *Gröbner éscalier* (or *souséscalier), border set* and *corner set* of $\mathfrak{a}$ w.r.t. $<$;
we also set $\mathbf{k}[\mathbf{N}_<(\mathfrak{a})] := \mathrm{Span}_\mathbf{k}(\mathbf{N}_<(\mathfrak{a}))$ and $\mathbf{k}[[\mathbf{N}_<(\mathfrak{a})]] \subset \mathcal{S}$ the vector subspace consisting of all the series $f \in \mathcal{S}$ with $\mathbf{S}(f) \subseteq \mathbf{N}_<(\mathfrak{a})$.

If no confusion can arise, we will usually omit the dependence on $<$, simply writing $\mathbf{T}\{\cdot\}, \mathbf{T}(\cdot), \mathbf{N}(\cdot), \mathbf{k}[\mathbf{N}(\cdot)]$, etc.

For each $f \in \mathcal{P}$, there is [2,3] a unique *canonical form*

$$g := \mathrm{Can}(f, \mathfrak{a}, <) = \sum_{\tau \in \mathbf{N}(\mathfrak{a})} \gamma(f, \tau, <)\tau = \sum_{\tau \in \mathbf{N}(\mathfrak{a})} \gamma(t, \tau, \mathbf{N}(\mathfrak{a}))\tau \in \mathbf{k}[\mathbf{N}(\mathfrak{a})] \tag{3}$$

such that

$$f - g \in \mathfrak{a} \text{ and, if } t \in \mathcal{T},\ t < \tau \Longrightarrow \gamma(t, \tau, <) = 0. \tag{4}$$

A *Gröbner basis* [2,3] of $\mathfrak{a}$ is any set $G \subset \mathfrak{a}$ such that $\mathbf{T}(G) = \mathbf{T}(\mathfrak{a})$, i.e. $\mathbf{T}\{G\}$ generates the monomial ideal $\mathbf{T}(\mathfrak{a})$; the *reduced Gröbner basis* [2,3] of $\mathfrak{a}$ is the set $\mathcal{G}(\mathfrak{a}) := \{\tau - \mathrm{Can}(\tau, \mathfrak{a}) : \tau \in \mathbf{G}(\mathfrak{a})\}$; the *border basis* [14] of $\mathfrak{a}$ is the set $\mathcal{B}(\mathfrak{a}) := \{\tau - \mathrm{Can}(\tau, \mathfrak{a}) : \tau \in \mathbf{B}(\mathfrak{a})\}$.
Finally, for each 0-dimensional ideal $\mathfrak{a} \subset \mathcal{P}$, its *degree* or *multiplicity* is:

$$\deg(\mathfrak{a}) := \#\mathbf{N}(\mathfrak{a}).$$

Denoting $\mathcal{P}^* := \mathrm{Hom}_\mathbf{k}(\mathcal{P}, \mathbf{k})$ the $\mathbf{k}$-vector space of $\mathbf{k}$-linear functionals, each $\ell \in \mathcal{P}^*$ is characterized by its values on any $\mathbf{k}$-basis $\mathbf{B}$ of $\mathcal{P}$, namely for each $f \in \mathcal{P}$ we have, by the $\mathbf{k}$-linearity of $\ell \in \mathcal{P}^*$:

$$\ell(f) = \sum_{\beta \in \mathbf{B}} c(f, \beta)\ell(\beta).$$

In particular if $\mathbf{B} = \mathcal{T}$, then, each $\ell \in \mathcal{P}^*$ can be encoded by means of the series $\sum_{t \in \mathcal{T}} \ell(t)t \in \mathcal{S}$ in such a way that to each series $\sum_{t \in \mathcal{T}} \gamma(t)t \in \mathcal{S}$ is associated the $\mathbf{k}$-linear functional $\ell \in \mathcal{P}^*$ defined, on each $f = \sum_{t \in \mathcal{T}} c(f, t)t \in \mathcal{P}$ by:

$$\ell(f) := \sum_{t \in \mathcal{T}} c(f, t)\gamma(t).$$

$\mathcal{P}^*$ has a natural $\mathcal{P}$-module structure associating to each $\ell \in \mathcal{P}^*$ and $f \in \mathcal{P}$

$$(\ell f) \in \mathcal{P}^* \text{ defined by } (\ell f)(g) := \ell(fg), \text{ for all } g \in \mathcal{P}.$$

Two sets $\mathbb{L} := \{\ell_1, \ldots, \ell_s\} \subset \mathcal{P}^*$ and $\mathbf{q} = \{q_1, \ldots, q_s\} \subset \mathcal{P}$ are said to be:

[1] Note that $\mathbf{N}_<(\mathfrak{a})$ is an *order ideal*, i.e. a subset $\mathbf{N} \subset \mathcal{T}$ satisfying $st \in \mathbf{N} \Longrightarrow t \in \mathbf{N}$ for all $s, t \in \mathcal{T}$; we also note that $\mathbf{N} \subset \mathcal{T}$ is an order ideal iff $\mathbf{I} := \mathcal{T} \setminus \mathbf{N}$ is a semigroup ideal, and conversely $\mathbf{I} \subset \mathcal{T}$ is a semigroup ideal iff $\mathbf{N} := \mathcal{T} \setminus \mathbf{I}$ is an order ideal. Moreover, if $\mathbf{I} \subset \mathcal{T}$ is a semigroup ideal with a slight abuse of language, it may happen that we use the same letter $\mathbf{I}$ to denote the monomial ideal it generates in $\mathcal{P}$.

- *triangular* if $\ell_i(q_j) = 0$, for each $i < j$;
- *biorthogonal* if $\ell_i(q_j) = 0$, for each $i \neq j$.

For **k**-vector subspaces $L \subset \mathcal{P}^*$ and $P \subset \mathcal{P}$, see ([14,15,1]) we let:

$$\mathfrak{P}(L) := \{g \in \mathcal{P} : \ell(g) = 0, \forall \ell \in L\}, \tag{5}$$

$$\mathfrak{L}(P) := \{\ell \in \mathcal{P}^* : \ell(g) = 0, \forall g \in P\}, \tag{6}$$

having:

$$P \subset \mathfrak{P}(\mathfrak{L}(P)) \quad \text{(resp.} \quad L \subset \mathfrak{L}(\mathfrak{P}(L)),$$

and, more precisely, it holds $P = \mathfrak{P}(\mathfrak{L}(P))$, without any assumption on P, while $L = \mathfrak{L}(\mathfrak{P}(L))$, only if L is finite dimensional.
Moreover, for **k**-vector subspaces $P, P_1, P_2 \subset \mathcal{P}$ and $L, L_1, L_2 \subset \mathcal{P}^*$ it holds:

1. P is an ideal iff $\mathfrak{L}(P)$ is a $\mathcal{P}$-module, L is a $\mathcal{P}$-module iff $\mathfrak{P}(L)$ is an ideal.
2. $P_1 \subset P_2$ implies $\mathfrak{L}(P_1) \supset \mathfrak{L}(P_2)$ and $L_1 \subset L_2$ implies $\mathfrak{P}(L_1) \supset \mathfrak{P}(L_2)$.
3. $\mathfrak{L}(P_1 + P_2) = \mathfrak{L}(P_1) \cap \mathfrak{L}(P_2)$ and $\mathfrak{P}(L_1 + L_2) = \mathfrak{P}(L_1) \cap \mathfrak{P}(L_2)$.
4. $\mathfrak{L}(P_1 \cap P_2) \supset \mathfrak{L}(P_1) + \mathfrak{L}(P_2)$ and $\mathfrak{P}(L_1 \cap L_2) \supset \mathfrak{P}(L_1) + \mathfrak{P}(L_2)$.

If $P_1, P_2 \subset \mathcal{P}$ are 0-dimensional ideals and $L_1, L_2 \subset \mathcal{P}^*$ are finite dimensional, then in 4. equalities hold. Actually, also hold:

5. $\mathfrak{L}(\sum_\rho \mathfrak{a}_\rho) = \cap_\rho \mathfrak{L}(\mathfrak{a}_\rho)$ and $\mathfrak{P}(\sum_\rho L_\rho) = \cap_\rho \mathfrak{P}(L_\rho)$, with no assumption on $\mathfrak{a}_{\rho\in\mathbb{N}} \subset \mathcal{P}$ and $L_{\rho\in\mathbb{N}} \subset \mathcal{P}^*$,
6. $\mathfrak{L}(\cap_\rho \mathfrak{a}_\rho) \supseteq \sum_\rho \mathfrak{L}(\mathfrak{a}_\rho)$ and $\mathfrak{P}(\cap_\rho L_\rho) \supseteq \sum_\rho \mathfrak{P}(L_\rho)$, where strict inclusion can hold also if $\mathfrak{a}_{\rho\in\mathbb{N}} \subset \mathcal{P}$ are zero-dimensional ideals and $L_{\rho\in\mathbb{N}} \subset \mathcal{P}^*$ finite **k**-dimensional $\mathcal{P}$-modules.

3 Macaulay notation

For any polynomial (or series) $f \in \mathcal{S}$,

- $L(f)$ is its *lowest degree non-zero homogeneous component*,
- $\text{ord}(f) := \deg(L(f))$ is its *order* or *underdegree*.

Moreover, $\{\zeta_\tau : \tau \in \mathcal{T}\}$ is an (infinite) set of indeterminates and $\mathbf{k}[\zeta_\tau]_{\tau\in\mathcal{T}} \subset \mathbf{k}[[\zeta_\tau]]_{\tau\in\mathcal{T}}$.

A *dialytic equation of* $\mathfrak{a}$ is any linear combination

$$\sum_{\tau\in\mathcal{T}} a_\tau \zeta_\tau \in \mathbf{k}[\zeta_\tau]_{\tau\in\mathcal{T}} \text{ satisfying } \sum_{\tau\in\mathcal{T}} a_\tau \tau \in \mathfrak{a}.$$

For each $v \in \mathcal{T}$, the $v-$ *derivative* of the dialytic equation $\sum_{\tau\in\mathcal{T}} a_\tau \zeta_\tau$ is the dialytic equation $\sum_{\tau\in\mathcal{T}} a_\tau \zeta_{\tau v}$ corresponding to the ideal member

$$\sum_{\tau\in\mathcal{T}} a_\tau \tau v = v \sum_{\tau\in\mathcal{T}} a_\tau \tau \in \mathfrak{a}.$$

The *modular equations* or *inverse functions* of $\mathfrak{a}$ are the equations identically satisfied by the coefficients of each and every member of $\mathfrak{a}$, i.e. the elements

$$\sum_{\tau\in\mathcal{T}} c_\tau \zeta_\tau \in \mathbf{k}[[\zeta_\tau]]_{\tau\in\mathcal{T}} \text{ with } \sum_{\tau\in\mathcal{T}} c_\tau a_\tau = 0 \text{ for all } \sum_{\tau\in\mathcal{T}} a_\tau \tau \in \mathfrak{a} \subset \mathcal{P}.$$

The notions of lowest degree component, under-degree (or order) etc. are implicitly extended to dialytic equations and inverse functions.

If $\mathfrak{a}$ is an ideal (resp. homogeneous ideal), then the set of all inverse functions up to (resp. of) degree d and the set consisting of all dialytic equations up to (resp. of) the same degree are conjugate systems of linear equations (i.e. the solutions of either system give the coefficients of the other one).

To each inverse function $\sum_{\tau\in\mathcal{T}} c_\tau \zeta_\tau \in \mathbf{k}[[\zeta_\tau]]_{\tau\in\mathcal{T}}$ we can associate the linear functional $\gamma \in \mathcal{P}^*$ defined by $\gamma(\tau) = c_\tau$ (encoded, see previous §, by the series $\sum_{\tau\in\mathcal{T}} c_\tau \tau \in \mathbf{k}[[X_1, \ldots, X_n]]$), conversely each series $\sum_{\tau\in\mathcal{T}} c_\tau \tau$ is associated to the inverse function $\sum_{\tau\in\mathcal{T}} c_\tau \zeta_\tau$.

Macaulay proposed a more illuminating notation and expressed such modular equation as the Laurent series
$$\sum_{\tau\in\mathcal{T}} c_\tau \tau^{-1} = \sum_{(a_1,\dots,a_n)\in\mathbb{N}^n} c_{a_1\dots a_n} X_1^{-a_1}\cdots X_n^{-a_n} \in \mathbf{k}[[X_1^{-1},\dots,X_n^{-1}]].$$

The *inverse system* of the ideal $\mathfrak{a}$ is the set of all negative power series $\sum_{\tau\in\mathcal{T}} c_\tau\tau^{-1}$ which are inverse functions of $\mathfrak{a}$.
Note that, in contrast to dialytic equations (involving only a finite number of variables ζ_τ), in general the inverse functions $\sum_{\tau\in\mathcal{T}} c_\tau\zeta_\tau = \sum_{\tau\in\mathcal{T}} c_\tau\tau^{-1}$ can have an infinite number of variables ζ_τ with nonzero coefficient c_τ.

In the set of inverse functions, Laurent series which are just polynomials are characterized, as follows:

Definition 3.1 (Macaulay) An inverse function $\sum_{\tau\in\mathcal{T}} c_\tau\tau^{-1}$ for which there exists $\gamma\in\mathbb{N}$ such that if $\deg(\tau)>\gamma \Longrightarrow c_\tau=0$, is called *Noetherian equation.*

For any inverse function E, representing a Noetherian equation of degree d, and every $f\in\mathcal{P}$ we have:

- $\mathrm{ord}(f)>d\Longrightarrow E(f)=0$ and, more generally,
- $E(f)=E(g)$ for $g=\mathrm{Can}(f,\mathfrak{m}^{d+1})\in\mathrm{Span}_{\mathbf{k}}(\mathcal{T}(d))$, so that E is a modular equation for $\mathfrak{m}^{d+1}$

and the set of all modular equations of $\mathfrak{a}$ having degree bounded by d coincides with the set of all modular equations of $\mathfrak{a}+\mathfrak{m}^{d+1}$.

Since for each $\mathfrak{m}$-primary ideal $\mathfrak{q}\subset\mathcal{P}$ there exists some $\rho\in\mathbb{N}^*$ (the *characteristic number* of $\mathfrak{q}$) such that $\mathfrak{q}\supset\mathfrak{m}^\rho$, for each $\tau\in\mathcal{T}$ with $\deg(\tau)\geq\rho$ it results $\tau\in\mathfrak{m}^\rho\subset\mathfrak{q}$, therefore each inverse function $\sum_{\tau\in\mathcal{T}} c_\tau\tau^{-1}$ of $\mathfrak{q}$ has $c_\tau=0$ for all $\tau\in\mathcal{T},\deg(\tau)\geq\rho$, i.e. it is a Noetherian equation of degree bounded by $\rho-1$.

For each $\tau\in\mathcal{T}$, a $\mathbf{k}$-linear functional $M(\tau)\in\mathcal{P}^*$ is defined by:

$$M(\tau)(f) := c(f,\tau) \quad \text{for all } f=\sum_{t\in\mathcal{T}} c(f,t)t\in\mathcal{P}. \tag{7}$$

We set $\mathbb{M}:=\{M(\tau):\tau\in\mathcal{T}\}\subset\mathcal{P}^*$ and we consider $\mathrm{Span}_{\mathbf{k}}(\mathbb{M})\subset\mathcal{P}^*$, denoting, for each $\ell:=\sum_{\tau\in\mathcal{T}} c(\tau,\ell)M(\tau)\in\mathrm{Span}_{\mathbf{k}}(\mathbb{M})$, *support of* ℓ the finite set:

$$\mathbf{S}(\ell):=\{\tau\in\mathcal{T}: c(\tau,\ell)\neq 0\}.$$

For every $f:=\sum_{t\in\mathcal{T}} a_t t\in\mathcal{P}$ and $\ell:=\sum_{\tau\in\mathcal{T}} c_\tau M(\tau)\in\mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ we have:

$$\ell(f)=\sum_{t\in\mathcal{T}} a_t c_t = \sum_{t\in\mathbf{S}(\ell)\cap\mathbf{S}(f)} a_t c_t.$$

Therefore $\mathrm{Span}_{\mathbf{k}}(\mathbb{M})\subset\mathcal{P}^*$ is the set of all the Noetherian equations. In particular for each $\mathfrak{m}$-primary ideal $\mathfrak{q}$, we have $\mathfrak{L}(\mathfrak{q})\subset\mathrm{Span}_{\mathbf{k}}(\mathbb{M})$.

Denote, for each vector subspace $\Lambda\subset\mathrm{Span}_{\mathbf{k}}(\mathbb{M})$,

$$\mathfrak{I}(\Lambda)^2 := \{f\in\mathcal{P}:\ell(f)=0,\ \forall\ \ell\in\Lambda\}$$

and, for each vector subspace $P\subset\mathcal{P}$,

$$\mathfrak{M}(P):=\mathfrak{L}(P)\cap\mathrm{Span}_{\mathbf{k}}(\mathbb{M})=\{\ell\in\mathrm{Span}_{\mathbf{k}}(\mathbb{M}):\ell(f)=0,\ \forall\ f\in P\}. \tag{8}$$

Any semigroup ordering[3] $<$ on $\mathcal{P}$ induces an ordering on $\mathbb{M}$ defined by:

$$M(\tau)\leq M(\omega) \iff \tau\leq\omega.$$

Notice that whenever dialytic equations (i.e. polynomials) are ordered according to their degree, the corresponding inverse functions are ordered according to their order (or under-degree) and conversely.

In order to extend the notation of Buchberger Theory to inverse functions one can consider any semigroup ordering and not just well-orderings.

[2] Remarking that $\mathrm{Span}_{\mathbf{k}}(\mathbb{M})\subset\mathcal{P}^*$ we also note the equality $\mathfrak{I}(\Lambda)=\mathfrak{P}(\Lambda)$ and point out that in the sequel mostly this second notation will be used.

[3] Not necessarily a termordering!

Definition 3.2 *For every*

$$\ell := \sum_i c_i M(\tau_i) \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M}) \text{ with } c_i \in k \setminus \{0\}, \tau_i \in \mathcal{T}, \tau_1 < \tau_2 < \cdots < \tau_i < \cdots$$

$\mathbf{T}_<(\ell) := \tau_1$ *is the* leading term *of* ℓ *,*
$\mathrm{ord}(\ell) := \min_i(\deg(\tau_i))$ *is the* order *(or* under-degree*) of* ℓ *,*
$\deg(\ell) := \max_i(\deg(\tau_i))$ *is the* degree *of* ℓ*.*
For any $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$*, we set*
$\mathbf{T}_<\{\Lambda\} := \{\mathbf{T}_<(\ell), \ell \in \Lambda\}, \quad \mathbf{N}_<\{\Lambda\} := \mathcal{T} \setminus \mathbf{T}_<\{\Lambda\}$
and again, when no confusion can arise, we will omit $<$.

■

For a degree-compatible term-ordering $<$, $\mathrm{ord}(\ell) = \deg(\mathbf{T}_<(\ell)), \forall\, \ell \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$.

4 Stability

For each $j \in \{1, \ldots, n\}$, σ_j, ρ_j, $\lambda_j \in End_{\mathbf{k}}(\mathrm{Span}_{\mathbf{k}}(\mathbb{M}))$ are defined as follows:

$$\sigma_j(M(\tau)) := \sigma_{X_j}(M(\tau)) = \begin{cases} M(\omega) & \text{if } \tau = X_j\omega \\ 0 & \text{if } X_j \nmid \tau \end{cases} \quad \forall\, \tau \in \mathcal{T};$$

$$\rho_j(M(\tau)) := \rho_{X_j}(M(\tau)) = M(X_j\tau) \quad \forall\, \tau \in \mathcal{T};$$

$$\lambda_j(M(\tau)) = \begin{cases} M(\tau) & \text{if } X_j \mid \tau \\ 0 & \text{if } X_j \nmid \tau \end{cases} \quad \forall\, \tau \in \mathcal{T}.$$

Remark that

$$\begin{aligned} \sigma_j\rho_j &= \mathrm{Id}, \quad \forall\ j, \\ \rho_j\sigma_j &= \lambda_j, \quad \forall\ j, \\ \sigma_k\rho_j &= \rho_j\sigma_k, \ \forall\ j,k, j \neq k. \end{aligned}$$

As for each i, j we have $\sigma_j\sigma_i = \sigma_i\sigma_j$, for each $t \in \mathcal{T}$ is inductively defined a $\sigma_t \in End_{\mathbf{k}}(\mathrm{Span}_{\mathbf{k}}(\mathbb{M}))$, by $\sigma_{X_j t} := \sigma_{X_j}\sigma_t$, so that for each $\tau, \omega \in \mathcal{T}$ we have:

$$\sigma_\tau(M(\omega)) = \begin{cases} M(\upsilon) & \text{if } \omega = \tau\upsilon \\ 0 & \text{if } \tau \nmid \omega. \end{cases}$$

Therefore, for each $f = \sum_i c_i t_i \in \mathcal{P}$, also a $\sigma_f \in End_{\mathbf{k}}(\mathrm{Span}_{\mathbf{k}}(\mathbb{M}))$ is uniquely defined by $\sigma_f(\ell) := \sum_i c_i \sigma_{t_i}(\ell)$.

Letting, for all $f \in \mathcal{P}$, $\ell \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$,

$$\ell f := \sigma_f(\ell),$$

the $\mathbf{k}$-vector space $\mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ is naturally endowed with a $\mathcal{P}$-module structure.

Remark also that, for each $\ell \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ and each $f \in \mathcal{P}$, $\sigma_f(\ell)$ is exactly the f-derivative of ℓ.

Lemma 4.1 *Given any* $\ell \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$*,* $f \in \mathcal{P}$ *and* i*, it holds:*

$$\ell(X_i f) = \sigma_i(\ell)(f).$$

Proof. Notice that for each $t \in \mathcal{T}$ we have $\mathbf{S}(tf) = t\mathbf{S}(f) := \{t\tau : \tau \in \mathbf{S}(f)$. Writing $f := \sum_{t\in\mathcal{T}} a_t t$ and $\ell := \sum_{\tau\in\mathcal{T}} c_\tau M(\tau)$, we have that for all $t \in \mathbf{S}(X_i f)$ it holds $X_i \mid t$ and $c(t, X_i f) = a_\tau$, where τ is the i-predecessor of t.

Since each $t \in \mathbf{S}(\ell) \cap \mathbf{S}(X_i f)$ is the i-successor of some $\tau \in \mathbf{S}(f) \cap \mathbf{S}(\sigma_i(\ell))$, and $X_i\tau \in \mathbf{S}(X_i f) \cap \mathbf{S}(\ell)$ holds for all $\tau \in \mathbf{S}(f) \cap \mathbf{S}(\sigma_i(\ell))$, we have our contention because

$\ell(X_i f) = \sum_{t\in\mathbf{S}(\ell)\cap\mathbf{S}(X_i f)} c(t, X_i f) c(t, \ell)$ and
$\sigma_i(\ell)(f) = \sum_{\tau\in\mathbf{S}(\sigma_i(\ell))\cap\mathbf{S}(f)} c(\tau, f) c(\tau, \sigma_i(\ell))$.

■

Definition 4.2 *A* $\mathbf{k}$*-vector subspace* $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ *is called:*
- X_j-stable *if* $\sigma_j(\ell) \in \Lambda$, *for each* $\ell \in \Lambda$;
- stable *if* $\sigma_f(\ell) \in \Lambda$, *for each* $\ell \in \Lambda$ *and* $f \in \mathcal{P}$. ■

Lemma 4.3 *Given vector subspaces* $\Lambda, \Lambda_1, \Lambda_2 \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ *we have:*
1. *for any change of coordinates* $\{Y_1, \dots, Y_n\}$, *are equivalent conditions:*
   - Λ *is stable,*
   - Λ *is* X_j*-stable, for each* j,
   - Λ *is* Y_i*-stable, for each* i;
2. *if* $\Lambda \neq \{0\}$ *is stable then* $M(1) \in \Lambda$;
3. *if* Λ_1 *and* Λ_2 *are stable, then also* $\Lambda_1 \cap \Lambda_2$ *and* $\Lambda_1 + \Lambda_2$ *are so.*

■

Theorem 4.4 *For any finite dimensional vector subspace* $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M}) \subset \mathcal{P}^*$, *are equivalent conditions:*
1. Λ *is stable,*
2. *the vector space* $\mathfrak{P}(\Lambda)$ *is an ideal and* $\mathfrak{P}(\Lambda) \subset \mathfrak{m}$.

Proof. 1. $\Longrightarrow$ 2. For each $\ell \in \Lambda$, $f \in \mathfrak{P}(\Lambda)$ and i, we have $\sigma_i(\ell) \in \Lambda$ and by Lemma 4.1 $\ell(X_i f) = \sigma_i(\ell)(f) = 0$. This proves that

$$X_i f \in \mathfrak{P}(\Lambda),\ \forall\ f \in \mathfrak{P}(\Lambda)\ \text{ and } i,\ \text{ i.e. } \mathfrak{P}(\Lambda) \text{ is an ideal.}$$

Moreover, since Λ is stable, by Lemma 4.3 we have $M(1) \in \Lambda$ so that

$$f(\mathbf{0}) = M(1)(f) = 0,\ \forall\ f \in \mathfrak{P}(\Lambda)\ \text{ i.e. } \mathfrak{P}(\Lambda) \subset \mathfrak{m}.$$

2. $\Longrightarrow$ 1. Since $\Lambda \subset \mathcal{P}^*$ is finite dimensional we have $\Lambda = \mathfrak{L}\mathfrak{P}(\Lambda)$.
For each $f \in \mathfrak{P}(\Lambda)$, $\ell \in \Lambda$ and i, since $\mathfrak{P}(\Lambda)$ is an ideal we have $X_i f \in \mathfrak{P}(\Lambda)$ so that $\sigma_i(\ell)(f) = \ell(X_i f) = 0$ and

$$\sigma_i(\ell) \in \mathfrak{L}\mathfrak{P}(\Lambda) = \Lambda.$$

■

5 Gröbner Duality

Proposition 5.1 *Given vector subspaces* $\mathfrak{a} \subset \mathcal{P}$ *and* $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$, *it holds:*
1. $\Lambda \subset \mathfrak{M}\mathfrak{P}(\Lambda)$ *and, if* Λ *is finite dimensional, then equality holds;*
2. $\mathfrak{a} \subset \mathfrak{P}\mathfrak{M}(\mathfrak{a})$.

Proof. 1. We have $\Lambda \subset \mathfrak{L}\mathfrak{P}(\Lambda)$ so that

$$\Lambda = \Lambda \cap \mathrm{Span}_{\mathbf{k}}(\mathbb{M}) \subset \mathfrak{L}\mathfrak{P}(\Lambda) \cap \mathrm{Span}_{\mathbf{k}}(\mathbb{M}) = \\ = \mathfrak{L}(\mathfrak{P}(\Lambda)) \cap \mathrm{Span}_{\mathbf{k}}(\mathbb{M}) = \mathfrak{M}(\mathfrak{P}(\Lambda)).$$

If Λ is finite dimensional, then $\Lambda = \mathfrak{L}\mathfrak{P}(\Lambda)$ and so equality substitutes inclusion.
2. Since $\mathfrak{M}(\mathfrak{a}) \subset \mathfrak{L}(\mathfrak{a})$ we have $\mathfrak{P}(\mathfrak{M}(\mathfrak{a})) \supset \mathfrak{P}(\mathfrak{L}(\mathfrak{a}))$; so that

$$\mathfrak{a} \subset \mathfrak{P}\mathfrak{L}(\mathfrak{a}) \subset \mathfrak{P}\mathfrak{M}(\mathfrak{a}).$$

■

For each $\rho \in \mathbb{N}$, denoting $\nabla_\rho := \mathrm{Span}_{\mathbf{k}}(\{M(\tau)(\cdot) : \tau \in \mathcal{T}(\rho - 1)\})$, we have:

Lemma 5.2 *For each* $\rho \in \mathbb{N}$ *it holds:*
- $\mathfrak{P}(\nabla_\rho) = \mathfrak{m}^\rho$,
- $\mathfrak{M}(\mathfrak{m}^\rho) = \mathfrak{L}(\mathfrak{m}^\rho) = \nabla_\rho$.

Proof. Trivially we have

$$\mathfrak{P}(\nabla_\rho) \supset \mathfrak{m}^\rho \quad \text{and } \mathfrak{L}(\mathfrak{m}^\rho) \supset \mathfrak{M}(\mathfrak{m}^\rho) \supset \nabla_\rho,$$

and the equalities follow since $\dim_{\mathbf{k}}(\nabla_\rho) = \binom{n}{\rho} = \deg(\mathfrak{m}^\rho)$. ■

Corollary 5.3 *For each* $\mathfrak{m}$*-primary ideal* $\mathfrak{q}$ *it holds:*
- $\mathfrak{M}(\mathfrak{q}) = \mathfrak{L}(\mathfrak{q})$,
- $\mathfrak{q} = \mathfrak{P}\mathfrak{M}(\mathfrak{q})$;

Proof. Since $\mathfrak{q}$ is $\mathfrak{m}$-primary we have $\mathfrak{q} \supset \mathfrak{m}^\rho$ for some $\rho \in \mathbb{N}$ and

$$\mathfrak{L}(\mathfrak{q}) \subset \mathfrak{L}(\mathfrak{m}^\rho) = \nabla_\rho \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M}),$$

so that $\mathfrak{M}(\mathfrak{q}) = \mathfrak{L}(\mathfrak{q})$. Hence

$$\mathfrak{q} = \mathfrak{P}L(\mathfrak{q}) = \mathfrak{P}M(\mathfrak{q}).$$

■

Proposition 5.4 *Given a finite-dimensional stable vector subspace* $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$*:*
- $\mathfrak{P}(\Lambda) \subset \mathfrak{m}$ *is an* $\mathfrak{m}$*-primary ideal,*
- $\dim_{\mathbf{k}}(\Lambda) = \deg(\mathfrak{P}(\Lambda))$.

Proof. From Theorem 4.4 $\mathfrak{P}(\Lambda) \subset \mathfrak{m}$ is an ideal, Λ finite dimensional implies that there exists $\rho \in \mathbb{N}$ with $\Lambda \subset \nabla_\rho$, thus $\mathfrak{P}(\Lambda) \supset \mathfrak{m}^\rho$ is a primary ideal.

Also $\dim_{\mathbf{k}}(\Lambda) = \deg(\mathfrak{P}(\Lambda))$. ■

Proposition 5.5 *For each* $\mathfrak{m}$*-primary ideal* $\mathfrak{q}$ *it holds:*
- $\mathfrak{M}(\mathfrak{q})$ *is stable;*
- $\dim_{\mathbf{k}}(\mathfrak{M}(\mathfrak{q})) = \deg(\mathfrak{q})$.

Proof. As $\mathfrak{q} = \mathfrak{P}\mathfrak{M}(\mathfrak{q})$ by 5.3, Theorem 4.4 grants that $\mathfrak{M}(\mathfrak{q})$ is stable. Also, since $\mathfrak{M}(\mathfrak{q}) = \mathfrak{L}(\mathfrak{q})$ we have

$$\dim_{\mathbf{k}}(\mathfrak{M}(\mathfrak{q})) = \dim_{\mathbf{k}}(\mathfrak{L}(\mathfrak{q})) = \deg(\mathfrak{P}L(\mathfrak{q})) = \deg(\mathfrak{q}).$$

■

Lemma 5.6 *Given* $\mathfrak{m}$*-primary ideals* $\mathfrak{q}_1$ *and* $\mathfrak{q}_2$ *and finite dimensional stable vector subspaces* $\Lambda_1, \Lambda_2 \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$, *it holds:*
1. $\mathfrak{q}_1 \subset \mathfrak{q}_2 \Longrightarrow \mathfrak{M}(\mathfrak{q}_1) \supset \mathfrak{M}(\mathfrak{q}_2)$ *and* $\Lambda_1 \subset \Lambda_2 \Longrightarrow \mathfrak{P}(\Lambda_1) \supset \mathfrak{P}(\Lambda_2)$;
2. $\mathfrak{M}(\mathfrak{q}_1 + \mathfrak{q}_2) = \mathfrak{M}(\mathfrak{q}_1) \cap \mathfrak{M}(\mathfrak{q}_2)$ *and* $\mathfrak{P}(\Lambda_1 + \Lambda_2) = \mathfrak{P}(\Lambda_1) \cap \mathfrak{P}(\Lambda_2)$;
3. $\mathfrak{M}(\mathfrak{q}_1 \cap \mathfrak{q}_2) = \mathfrak{M}(\mathfrak{q}_1) + \mathfrak{M}(\mathfrak{q}_2)$ *and* $\mathfrak{P}(\Lambda_1 \cap \Lambda_2) = \mathfrak{P}(\Lambda_1) + \mathfrak{P}(\Lambda_2)$. ■

All the above facts can be summarized as follows:

Remark 5.7 *The maps* $\mathfrak{P}(\cdot)$ *and* $\mathfrak{M}(\cdot)$ *(respectively restriction of* $\mathfrak{P}(\cdot)$ *to* $\mathfrak{m}$*-primary ideals and* $\mathfrak{L}(\cdot)$ *to finite dimensional stable* $\mathbf{k}$*-vector subspaces) are mutually inverse by 5.1 and 5.3. They actually give a biunivocal, inclusion reversing, correspondence between the set of the* $\mathfrak{m}$*-primary ideals* $\mathfrak{q} \subset \mathcal{P}$ *and the set of the finite dimensional stable vector subspaces* $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$.

Moreover, for each $\mathfrak{m}$*-primary ideal* $\mathfrak{q} \subset \mathcal{P}$ *we have* $\deg(\mathfrak{q}) = \dim_{\mathbf{k}}(\mathfrak{M}(\mathfrak{q}))$ *and, for any finite dimensional stable vector subspace* $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ *we have* $\dim_{\mathbf{k}}(\Lambda) = \deg(\mathfrak{P}(\Lambda))$. ■

Proposition 5.8 *For* $\rho \in \mathbb{N}$, *let* $\mathfrak{q}_\rho$ *be* $\mathfrak{m}$*-primary ideals and* $\Lambda_\rho \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ *be finite dimensional stable vector subspaces. Then*
1. $\mathfrak{M}(\sum_\rho \mathfrak{q}_\rho) = \cap_\rho \mathfrak{M}(\mathfrak{q}_\rho)$ *and* $\mathfrak{P}(\sum_\rho \Lambda_\rho) = \cap_\rho \mathfrak{P}(\Lambda_\rho)$;
2. $\mathfrak{M}(\cap_\rho \mathfrak{q}_\rho) = \sum_\rho \mathfrak{M}(\mathfrak{q}_\rho)$ *and* $\mathfrak{P}(\cap_\rho \Lambda_\rho) = \sum_\rho \mathfrak{P}(\Lambda_\rho)$.

Proof. Clearly 1. is a consequence of 5. of §2, as for 2. it follows from of 6. of §2, by definition of $\mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ and $\mathfrak{M}(-)$. ■

Lemma 5.9 *Given a (not necessarily finite-dimensional) stable vector subspace $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$, for each $\rho \in \mathbb{N}$, let $\Lambda_\rho := \Lambda \cap \nabla_\rho$. Then we have:*
1. $\Lambda_1 \subset \cdots \subset \Lambda_\rho \subset \Lambda_{\rho+1} \cdots \subset \Lambda$, *and so* $\mathfrak{P}(\Lambda_1) \supset \cdots \supset \mathfrak{P}(\Lambda_\rho) \supset \mathfrak{P}(\Lambda_{\rho+1}) \supset \cdots \supset \mathfrak{P}(\Lambda)$,
2. $\Lambda = \sum_\rho \Lambda_\rho$, *and so* $\mathfrak{P}(\Lambda) = \cap_\rho \mathfrak{P}(\Lambda_\rho)$,
3. $\mathfrak{P}(\Lambda)$ *is an* $\mathfrak{m}$*-closed ideal and* $\Lambda = \mathfrak{M}\mathfrak{P}(\Lambda)$.

Proof. Clearly 1. and 2. are trivial. Ad 3., we have $\mathfrak{P}(\Lambda) = \cap_\rho(\mathfrak{P}(\Lambda) + \mathfrak{m}^\rho)$, namely

$$\mathfrak{P}(\Lambda) = \cap_\rho \mathfrak{P}(\Lambda_\rho) = \cap_\rho \mathfrak{P}(\Lambda \cap \nabla_\rho) = \\ = \cap_\rho(\mathfrak{P}(\Lambda) + \mathfrak{P}(\nabla_\rho)) = \cap_\rho(\mathfrak{P}(\Lambda) + \mathfrak{m}^\rho);$$

on the other hand

$$\Lambda = \sum_\rho \Lambda_\rho = \sum_\rho \mathfrak{M}\mathfrak{P}(\Lambda_\rho) = \mathfrak{M}(\cap_\rho \mathfrak{P}(\Lambda_\rho) = \mathfrak{M}P(\Lambda).$$

∎

Proposition 5.10 *For each* $\mathfrak{m}$*-closed ideal* $\mathfrak{a} \subset \mathcal{P} \subset \mathcal{S}$, *it holds:*
- $\mathfrak{a} = \mathfrak{P}\mathfrak{M}(\mathfrak{a})$;
- $\mathfrak{M}(\mathfrak{a})$ *is stable.*

Proof. Considering, for every $\rho \in \mathbb{N}$, the $\mathfrak{m}$-primary ideal $\mathfrak{a}_\rho := \mathfrak{a} + \mathfrak{m}^\rho$, we have

$$\mathfrak{a} = \cap_\rho \mathfrak{a}_\rho = \cap_\rho \mathfrak{P}\mathfrak{M}(\mathfrak{a}_\rho) = \mathfrak{P}\left(\sum_\rho \mathfrak{M}(\mathfrak{a}_\rho)\right) = \mathfrak{P}\mathfrak{M}\left(\cap_\rho \mathfrak{a}_\rho\right) = \mathfrak{P}M(\mathfrak{a}).$$

Let $\ell \in \mathfrak{M}(\mathfrak{a})$ and let $\rho - 1 = \deg(\ell)$, we have $\ell \in \nabla_\rho = \mathfrak{M}(\mathfrak{m}^\rho)$ and therefore

$$\ell \in \mathfrak{M}(\mathfrak{a}) \cap \mathfrak{M}(\mathfrak{m}^\rho) = \mathfrak{M}(\mathfrak{a} + \mathfrak{m}^\rho);$$

since $\mathfrak{M}(\mathfrak{a} + \mathfrak{m}^\rho)$ is stable, for each $f \in \mathcal{P}$,

$$\sigma_f(\ell) \in \mathfrak{M}(\mathfrak{a}) \cap \mathfrak{M}(\mathfrak{m}^\rho) \subset \mathfrak{M}(\mathfrak{a}).$$

∎

Theorem 5.11 *The mutually inverse maps* $\mathfrak{P}(\cdot)$ *and* $\mathfrak{M}(\cdot)$ *give a biunivocal, inclusion reversing, correspondence between the set of* $\mathfrak{m}$*-closed ideals* $\mathfrak{a} \subset \mathcal{P} \subset \mathcal{S}$ *and the set of stable vector subspaces* $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$. ∎

6 Leibniz Formula

Proposition 6.1 *For any* $f, g \in \mathcal{P}$ *and* $\omega \in \mathcal{T}$ *it holds:*

$$M(\omega)(fg) = \sum_{\substack{\upsilon \in \mathcal{T} \\ \upsilon\tau = \omega}} M(\upsilon)(f)M(\tau)(g).$$

Proof. Let

$$\begin{aligned} f &= \textstyle\sum_{\upsilon \in \mathbf{S}(f)} c(f,\upsilon)\upsilon &&= \textstyle\sum_{\upsilon \in \mathbf{S}(f)} M(\upsilon)(f)\upsilon, \\ g &= \textstyle\sum_{\tau \in \mathbf{S}(g)} c(g,\tau)\tau &&= \textstyle\sum_{\tau \in \mathbf{S}(g)} M(\tau)(g)\tau, \\ fg &= \textstyle\sum_{\omega \in \mathbf{S}(fg)} c(fg,\omega)\omega &&= \textstyle\sum_{\omega \in \mathbf{S}(fg)} M(\omega)(fg)\omega \end{aligned}$$

for each $\omega \in \mathcal{T}$, we have

$$\begin{aligned} M(\omega)(fg) = c(fg,\omega) &= \sum_{\substack{\upsilon \in \mathbf{S}(f) \\ \upsilon\tau = \omega}} c(f,\upsilon)c(g,\tau) \\ &= \sum_{\substack{\upsilon \in \mathbf{S}(f) \\ \upsilon\tau = \omega}} M(\upsilon)(f)M(\tau)(g). \end{aligned}$$

∎

Corollary 6.2 (Leibniz-type Formula) *For any* $f, g \in \mathcal{P}$ *and* $\ell \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ *it holds:*

$$\ell(fg) = \sum_{\upsilon \in \mathbf{S}(f)} M(\upsilon)(f)\sigma_\upsilon(\ell)(g).$$

■

We point out that Lemma 4.1 is nothing but a particular case of Corollary 6.2.

Proposition 6.3 (Möller–Stetter) *Given any* $\mathbf{k}$*-basis* $\{\ell_1, \dots, \ell_s\}$ *of a finite dimensional stable vector space* $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ *and any finite basis* $\{g_1, \dots, g_t\}$ *of an ideal* $\mathfrak{a} \subset \mathcal{P}$*. Then*

$$\ell_i(g_j) = 0, \forall i, j \Longrightarrow \ell(f) = 0, \forall \ell \in \Lambda, f \in \mathfrak{a}.$$

Proof. Let $f = \sum_{j=1}^t f_j g_j \in \mathfrak{a}$ and let $\ell \in \Lambda$. Then, for each $\upsilon \in \mathcal{T}$, $\sigma_\upsilon(\ell) \in \Lambda$ as Λ is stable. Therefore, for all j and $\upsilon \in \mathcal{T}$, $\sigma_\upsilon(\ell)(g_j) = 0$ and by the Leibniz-type Formula

$$\ell(f) = \sum_{j=1}^t \ell(f_j g_j) = \sum_{j=1}^t \sum_{\upsilon \in \mathbf{S}(f_j)} M(\upsilon)(f_j)\sigma_\upsilon(\ell)(g_j) = 0$$

■

Corollary 6.4 *With the same notation as above*

$$\ell_i(g_j) = 0, \forall i, j \Longrightarrow \Lambda \subset \mathfrak{M}(\mathfrak{a}).$$

■

7 Differential inverse functions at the origin

A nice interpretation of the set $\mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ of all the Noetherian equations at the origin[4] in terms of differential operators was proposed by Gröbner, assuming (as we will do throughout the section) $\mathrm{char}(\mathbf{k}) = 0$.

For each $(i_1, \dots, i_n) \in \mathbb{N}^n$, setting $\tau := X_1^{i_1} \dots X_n^{i_n}$, we denote by

$$D(\tau) := D(i_1, \dots, i_n) : \mathcal{P} \to \mathcal{P}$$

the differential operator:

$$D(\tau) := D(i_1, \dots, i_n) = \frac{1}{i_1! \cdots i_n!} \frac{\partial^{i_1 + \cdots + i_n}}{\partial X_1^{i_1} \cdots \partial X_n^{i_n}}.$$

Also, for $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathcal{T}$, and $t := X_1^{e_1} \cdots X_n^{e_n} \in \mathcal{T}$ such that $\tau \mid t$ so that $d_i \le e_i$, we will use the following shorthand

$$\binom{t}{\tau} := \binom{e_1}{d_1} \cdots \binom{e_n}{d_n}.$$

Proposition 7.1 *[14] Let* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathcal{T}$*, and* $t := X_1^{e_1} \cdots X_n^{e_n} \in \mathcal{T}$*. Then*

$$D(\tau)(t) := \begin{cases} \binom{t}{\tau} X_1^{e_1 - d_1} \dots X_n^{e_n - d_n} & \mathit{if} \tau \mid t, \\ 0 & \mathit{if}\ \tau \nmid t \end{cases}.$$

■

[4] Noetherian equation at a point means ‘at the maximal ideal corresponding to the point’.

Denoting $\mathbb{D} := \{D(\tau) : \tau \in \mathcal{T}\}$, for each $\delta := \sum_{\tau\in\mathcal{T}} c_\tau D(\tau)(\cdot) \in \mathrm{Span}_{\mathbf{k}}\,\mathbb{D}$ we set $\mathbf{S}(\delta) := \{\tau \in \mathcal{T} : c_\tau \neq 0\}$ and $\tilde{\mathbf{S}}(\delta) := \{t \in \mathcal{T} : t \mid \tau,\ \text{for some}\, \tau \in \mathbf{S}(\delta)\}$.
Remark that, for each $\tau \in \mathcal{T}$, $D(\tau)(\cdot)(0,\dots,0) = M(\tau)$, so that if we set $\mathrm{ev} : \mathrm{Span}_{\mathbf{k}}(\mathbb{D}) \to \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ the morphism defined by $\mathrm{ev}(D(\tau)) = M(\tau)$ for each $\tau \in \mathcal{T}$ we have

$$\mathrm{ev}(\delta)(\cdot) = \delta(\cdot)(0,\dots,0) = \sum_{\tau\in\mathcal{T}} c_\tau M(\tau)(\cdot)\ \forall\delta := \sum_{\tau\in\mathcal{T}} c_\tau D(\tau)(\cdot) \in \mathrm{Span}_{\mathbf{k}}\,\mathbb{D}$$

so that the set

$$\{\delta(\cdot)(0,\dots,0) : \delta \in \mathrm{Span}_{\mathbf{k}}(\mathbb{D})\} \subset \mathcal{P}^*$$

coincides with the set of all the Noetherian equations at the origin and, in particular, for each $\mathfrak{m}$-primary ideal $\mathfrak{q}$, we have

$$\mathfrak{L}(\mathfrak{q}) \subset \{\delta(\cdot)(0,\dots,0) : \delta \in \mathrm{Span}_{\mathbf{k}}(\mathbb{D})\}.$$

We impose on $\mathbb{D}$ the same semigroup ordering $<$ induced on $\mathbb{M}$ so that

$$D(\tau) \le D(\omega) \iff M(\tau) \le M(\omega) \iff \tau \le \omega$$

and we set

$$\mathbf{T}_<(\delta) := \mathbf{T}_<(\mathrm{ev}(\delta)), \quad \mathrm{ord}(\delta) := \mathrm{ord}(\mathrm{ev}(\delta)), \quad \deg(\delta) := \deg(\mathrm{ev}(\delta)).$$

Letting $D(\tau_1)\cdot D(\tau_2) := D(\tau_1\tau_2)$ we impose on $\mathbb{D}$ also a semigroup structure which is isomorphic to the one of $\mathcal{T}$:

Proposition 7.2 *[14] For $\upsilon := X_1^{d_1}\cdots X_n^{d_n}$, and $\tau := X_1^{e_1}\cdots X_n^{e_n}$, we have*

$$D(\upsilon)\left(D(\tau)(\cdot)\right) = \binom{\upsilon\tau}{\tau} D(\upsilon\tau)(\cdot).$$

■

Setting,
$\sigma_\tau(D(\omega)) = \begin{cases} D(\upsilon) & \text{if } \omega = \tau\upsilon \\ 0 & \text{if } \tau \nmid \omega \end{cases}$ for each $\tau, \omega \in \mathcal{T}$
$\sigma_f(\delta) = \sum_i c_i \sigma_{t_i}(\delta)$ for each $f = \sum_i c_i t_i \in \mathcal{P}, \delta \in \mathrm{Span}_{\mathbf{k}}(\mathbb{D})$
we get $\sigma_f : \mathrm{Span}_{\mathbf{k}}(\mathbb{D}) \to \mathrm{Span}_{\mathbf{k}}(\mathbb{D})$ for all $f \in \mathcal{P}$.

Definition 7.3 *A vector subspace $\Delta \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{D})$ is called*
- X_j-stable *if for each $\delta \in \Delta, \sigma_j(\delta) \in \Delta$;*
- stable *if for each $\delta \in \Delta$ and each $f \in \mathcal{P}, \sigma_f(\delta) \in \Delta$.* ■

8 Taylor Formula and Gröbner Duality

Letting $\mathsf{b} := (b_1,\dots,b_n) \in k^n$, $\mathfrak{m}_{\mathsf{b}} := (X_1 - b_1,\dots,X_n - b_n) \subset \mathcal{P}$,
$\lambda_{\mathsf{b}} : \mathcal{P} \to \mathcal{P}$ the translation $\lambda_{\mathsf{b}}(X_i) = X_i + b_i$, for all i; we have $\lambda_{\mathsf{b}}(\mathfrak{m}_{\mathsf{b}}) = \mathfrak{m}$ and, for each $\mathfrak{m}_{\mathsf{b}}$-closed ideal $\mathfrak{a}_{\mathsf{b}}$, $\mathfrak{a} := \lambda_{\mathsf{b}}(\mathfrak{a}_{\mathsf{b}})$ is an $\mathfrak{m}$-closed ideal. Therefore

$$\{\ell\lambda_{\mathsf{b}}(\cdot) : \ell \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M})\} = \{\delta(\cdot)(\mathsf{b}) : \delta \in \mathrm{Span}_{\mathbf{k}}(\mathbb{D})\} \subset \mathcal{P}^*$$

is the set of all the Noetherian inverse equations w.r.t. $\mathfrak{m}_{\mathsf{b}}$-closed ideals and, in particular

$$\mathfrak{L}(\mathfrak{q}_{\mathsf{b}}) \subset \{\ell\lambda_{\mathsf{b}}(\cdot) : \ell \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M})\},$$

for each $\mathfrak{m}_{\mathsf{b}}$-primary ideal $\mathfrak{q}_{\mathsf{b}}$.

Remark 8.1 *[15] Let $\mathfrak{p} \subset \mathcal{P}$ be an (n-r)-dimensional prime ideal, up to a suitable change of coordinates, we may assume $\mathfrak{p} \cap \mathbf{k}[X_{r+1}, \dots, X_n] = \{0\}$, i.e. $\mathsf{p} := \mathfrak{p}\mathsf{k}(X_{r+1}, \dots, X_n)[X_1, \dots, X_r]$ maximal ideal. Let $\mathsf{pQ} = \cap_{i=1}^{s} \mathsf{n}_i$ be a prime decomposition in $\mathsf{Q} := \Omega(\mathsf{k})[X_1, \dots, X_n]$ (where $\Omega(\mathsf{k})$ is the universal field of $\mathbf{k}$), with $\mathsf{p} = \mathsf{n}_i \cap \mathcal{P}$, for each i. If $\mathsf{a}_i := (a_{i1}, \dots, a_{in}) \in \Omega(k)^n$ is the root for which*

$$\mathsf{n}_i = (X_1 - a_{i1}, \dots, X_n - a_{in}),$$

then, via the translation $\lambda_{\mathsf{a}_i} : \mathsf{Q} \to \mathsf{Q}$, we are in the situation discussed above. In particular
- *the set $\{\ell\lambda_{\mathsf{a}_i}(\cdot) : \ell \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M})\} \subset \mathsf{Q}^*$ consists of all the Noetherian inverse equations w.r.t. n_i-closed ideals;*
- *if $\mathfrak{q} \subset \mathcal{P}$ is $\mathfrak{p}$-primary, then $\mathsf{q} := \mathfrak{q}k(X_{r+1}, \dots, X_n)[X_1, \dots, X_r]$ is a p-primary ideal and $\mathsf{qQ} = \cap_{i=1}^{s} \mathsf{s}_i$ is a decomposition into simple[5] primary components satisfying:*
  - *$\sqrt{\mathsf{s}_i} = \mathsf{n}_i$,*
  - *$\mathsf{q} = \mathsf{s}_i \cap \mathcal{P}$ for each i,*
  - *$\mathfrak{L}(\mathsf{s}_i) \subset \{\ell\lambda_{\mathsf{a}_i}(\cdot) : \ell \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M})\}$;*
- *if j is a $\mathfrak{p}$-closed ideal, then for $\mathsf{J} := \mathsf{j}k(X_{r+1}, \dots, X_n)[X_1, \dots, X_r]$ one has $\mathsf{JQ} = \cap_{i=1}^{s} \mathsf{J}_i$ where J_i is n_i-closed and $\mathsf{j} = \mathsf{J}_i \cap \mathcal{P}$, for each i.* ■

Lemma 8.2 *For each $\mathsf{b} := (b_1, \dots, b_n) \in k^n$ and $f := \sum_{i=1}^{\mu} c(f, t_i)t_i \in \mathcal{P}$, it holds:*

$$c(\tau, \lambda_{\mathsf{b}}(f)) = M(\tau)\lambda_{\mathsf{b}}(f) = D(\tau)\lambda_{\mathsf{b}}(f)(0, \dots, 0) = D(\tau)(f)(\mathsf{b}).$$

■

Corollary 8.3 (Taylor formula) *For each $\mathsf{b} := (b_1, \dots, b_n) \in k^n$ and each $f := \sum_{i=1}^{\mu} c(f, t_i)t_i \in \mathcal{P}$, it holds*

$$\begin{aligned}\lambda_{\mathsf{b}}(f) &= f(X_1 + b_1, \dots, X_n + b_n)\\ &= \sum_{\tau \in \mathcal{T}} D(\tau)(f)(\mathsf{b})\tau.\end{aligned}$$

■

Let us denote, for each vector subspace $\Delta \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{D})$,

$$\mathfrak{I}_{\mathsf{b}}(\Delta) := \{f \in \mathcal{P} : \delta(f)(\mathsf{b}) = 0, \forall \delta \in \Delta\}$$

and, for each vector subspace $P \subset \mathcal{P}$,

$$\mathfrak{D}_{\mathsf{b}}(P) := \{\delta \in \mathrm{Span}_{\mathbf{k}}(\mathbb{D}) : \delta(f)(\mathsf{b}) = 0, \forall f \in P\}.$$

We point out that if $\mathsf{b} = 0$, then we will simply write $\mathfrak{I}(\Delta)$ and $\mathfrak{D}(P)$, noticing also that $\mathfrak{D}(\mathfrak{a}) = \mathfrak{M}(\mathfrak{a})$ for all $\mathfrak{a} \subset \mathcal{P} \subset \mathcal{S}$, $\mathfrak{m}$-closed ideal.

Corollary 8.4 *Let $\Delta \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{D})$ be any vector subspace.*
Then, the following conditions are equivalent:
- *Δ is stable,*
- *$\Lambda := \mathrm{ev}(\Delta)$ is stable,*
- *the vectorspace $\mathfrak{I}_{\mathsf{b}}(\Delta)$ is an ideal and $\mathfrak{I}_{\mathsf{b}}(\Delta) \subset \mathfrak{m}_{\mathsf{b}}$.*

Proof. Clearly 1. $\iff$ 2..
The equivalence with 3. is a consequence of the obvious equality

$$\delta(f)(\mathsf{b}) = \delta\lambda_{\mathsf{b}}(f)(0, \dots, 0) = \mathrm{ev}(\delta)\lambda_{\mathsf{b}}(f)$$

and Theorem 4.4 ■

Lemma 8.5 *For any stable vector space $\Delta \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{D})$, it holds $\mathfrak{I}_{\mathsf{b}}(\Delta) = \lambda_{\mathsf{b}}^{-1}(\mathfrak{P}(\mathrm{ev}(\Delta)))$.*

[5] A primary ideal is called simple if its corresponding maximal ideal is linear.

Proof. Denoting $\Lambda := \mathrm{ev}(\Delta)$, we have

$$\begin{aligned}\mathfrak{I}_{\mathsf{b}}(\Delta) &= \{f \in \mathcal{P} : \delta(f)(\mathsf{b}) = 0, \forall \delta \in \Delta\}\\ &= \{f \in \mathcal{P} : \mathrm{ev}(\delta)\lambda_{\mathsf{b}}(f) = 0, \forall \delta \in \Delta\}\\ &= \{\lambda_{\mathsf{b}}^{-1}(g) : g \in \mathcal{P}, \mathrm{ev}(\delta)(g) = 0, \forall \delta \in \Delta\}\\ &= \lambda_{\mathsf{b}}^{-1}\left(\{g : g \in \mathcal{P}, \ell(g) = 0, \forall \ell \in \Lambda\}\right)\\ &= \lambda_{\mathsf{b}}^{-1}(\mathfrak{P}(\Lambda))\\ &= \lambda_{\mathsf{b}}^{-1}(\mathfrak{P}(\mathrm{ev}(\Delta))).\end{aligned}$$

■

Lemma 8.6 *For $P \subset \mathcal{P}$, it holds $\mathfrak{D}_{\mathsf{b}}(\lambda_{\mathsf{b}}^{-1}(P)) = \mathrm{ev}^{-1}(\mathfrak{M}(P))$.*

Proof. It holds

$$\begin{aligned}\mathfrak{D}_{\mathsf{b}}(\lambda_{\mathsf{b}}^{-1}(P)) &= \{\delta \in \mathrm{Span}_{\mathbf{k}}(\mathbb{D}) : \delta(f)(\mathsf{b}) = 0, \forall f \in \lambda_{\mathsf{b}}^{-1}(P)\}\\ &= \{\delta \in \mathrm{Span}_{\mathbf{k}}(\mathbb{D}) : \delta\lambda_{\mathsf{b}}^{-1}(g)(\mathsf{b}) = 0, \forall g \in P\}\\ &= \{\delta \in \mathrm{Span}_{\mathbf{k}}(\mathbb{D}) : \mathrm{ev}(\delta)\lambda_{\mathsf{b}}(\lambda_{\mathsf{b}}^{-1}(g)) = 0, \forall g \in P\}\\ &= \{\delta \in \mathrm{Span}_{\mathbf{k}}(\mathbb{D}) : \mathrm{ev}(\delta)(\cdot) \in \mathfrak{L}(P)\}\\ &= \{\delta \in \mathrm{Span}_{\mathbf{k}}(\mathbb{D}) : \mathrm{ev}(\delta)(\cdot) \in \mathfrak{L}(P) \cap \mathrm{Span}_{\mathbf{k}}(\mathbb{M})\}\\ &= \{\delta \in \mathrm{Span}_{\mathbf{k}}(\mathbb{D}) : \mathrm{ev}(\delta)(\cdot) \in \mathfrak{M}(P)\}\\ &= \mathrm{ev}^{-1}(\mathfrak{M}(P)).\end{aligned}$$

■

Corollary 8.7 *Each $\mathfrak{m}_{\mathsf{b}}$-closed ideals $\mathfrak{a}_{\mathsf{b}} \subset \mathcal{P}$ and each stable vector subspaces $\Delta \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{D})$ satisfy*

$$\mathfrak{I}_{\mathsf{b}}\mathfrak{D}_{\mathsf{b}}(\mathfrak{a}_{\mathsf{b}}) = \mathfrak{a}_{\mathsf{b}} \quad \text{and} \quad \mathfrak{D}_{\mathsf{b}}\mathfrak{I}_{\mathsf{b}}(\Delta) = \Delta.$$

Proof. We have

$$\begin{aligned}\mathfrak{I}_{\mathsf{b}}\mathfrak{D}_{\mathsf{b}}(\mathfrak{a}_{\mathsf{b}}) &= \lambda_{\mathsf{b}}^{-1}(\mathfrak{P}(\mathrm{ev}(\mathfrak{D}_{\mathsf{b}}(\mathfrak{a}_{\mathsf{b}}))))\\ &= \lambda_{\mathsf{b}}^{-1}(\mathfrak{P}(\mathrm{ev}\,\mathrm{ev}^{-1}(\mathfrak{M}(\lambda_{\mathsf{b}}(\mathfrak{a}_{\mathsf{b}})))))\\ &= \lambda_{\mathsf{b}}^{-1}(\mathfrak{P}\mathfrak{M}(\lambda_{\mathsf{b}}(\mathfrak{a}_{\mathsf{b}})))\\ &= \lambda_{\mathsf{b}}^{-1}\lambda_{\mathsf{b}}(\mathfrak{a}_{\mathsf{b}})\\ &= \mathfrak{a}_{\mathsf{b}}\end{aligned}$$

and

$$\mathfrak{D}_{\mathsf{b}}\mathfrak{I}_{\mathsf{b}}(\Delta) = \mathfrak{D}_{\mathsf{b}}(\lambda_{\mathsf{b}}^{-1}(\mathfrak{P}(\mathrm{ev}(\Delta)))) = \mathrm{ev}^{-1}(\mathfrak{M}(\mathfrak{P}(\mathrm{ev}(\Delta)))) = \mathrm{ev}^{-1}\,\mathrm{ev}(\Delta) = \Delta.$$

■

This allows to conclude that

Theorem 8.8 (Gröbner) *The mutually inverse maps $\mathfrak{I}_{\mathsf{b}}(\cdot)$ and $\mathfrak{D}_{\mathsf{b}}(\cdot)$ give a biunivocal, inclusion reversing, correspondence between the set of the $\mathfrak{m}_{\mathsf{b}}$-closed ideals $\mathfrak{a}_{\mathsf{b}} \subset \mathcal{P}$ and the set of the stable vector subspaces $\Delta \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{D})$.*

Moreover, to any $\mathfrak{m}_{\mathsf{b}}$-primary ideals $\mathfrak{q}_{\mathsf{b}} \subset \mathcal{P}$ corresponds a finite dimensional stable **k***-vector subspace so that $\deg(\mathfrak{q}_{\mathsf{b}}) = \dim_{\mathbf{k}}(\mathfrak{D}_{\mathsf{b}}(\mathfrak{q}_{\mathsf{b}}))$; and to any finite dimensional stable vector subspaces $\Delta \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{D})$ corresponds an $\mathfrak{m}_{\mathsf{b}}$-primary ideal so that $\dim_{\mathbf{k}}(\Delta) = \deg(\mathfrak{I}_{\mathsf{b}}(\Delta))$.* ■

We recall here the classical Leibniz Formula in order to stress how applying ev one can motivate the use of this name for Proposition 6.1 and its corollary

Lemma 8.9 (Leibniz Formula) *For any $f, g \in \mathcal{P}$ and $\omega \in \mathcal{T}$ it holds*

$$D(\omega)(fg) = \sum_{\substack{\upsilon \in \tilde{\mathbf{S}}(f) \\ \upsilon\tau=\omega}} D(\upsilon)(f)D(\tau)(g)$$

■

Proposition 8.10 *For any $f, g \in \mathcal{P}$ and any $\delta \in \mathrm{Span}_{\mathbf{k}}(\mathbb{D})$ it holds*

$$\delta(fg) = \sum_{\upsilon \in \tilde{\mathbf{S}}(f)} D(\upsilon)(f)\sigma_\upsilon(\delta)(g).$$

■

Corollary 8.11 *For all $f \in \mathcal{P}, \delta \in \mathrm{Span}_{\mathbf{k}}(\mathbb{D}), 1 \leq i \leq r$, it holds*

$$\delta(X_i f) = X_i\delta(f) + \sigma_{X_i}(\delta)(f).$$

Corollary 8.12 *Given any $\mathsf{b} \in k^n$* and $\mathfrak{m}_{\mathsf{b}} \subset \mathcal{P}$, *for any $\delta \in \mathrm{Span}_{\mathbf{k}}(\mathbb{D})$, it holds*

$$\delta(X_i f)(\mathsf{b}) = b_i\delta(f)(\mathsf{b}) + \sigma_i(\delta)(f)(\mathsf{b}).$$

■

Notice that by applying ev to Corollary 8.11 and Proposition 8.10 we get exactly what stated in Lemma 4.1 and Corollary 6.2.

Corollary 8.13 (Möller–Stetter) *Given*
any $\mathbf{k}$-basis $\{\delta_1, \ldots, \delta_s\}$ of a stable vector subspace $\Delta \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{D})$,
$\mathsf{b} := (b_1, \ldots, b_n) \in k^n$ *and* $\mathfrak{m}_{\mathsf{b}} \subset \mathcal{P}$,
an ideal $\mathfrak{a} \subset \mathcal{P}$ and any finite basis $\{g_1, \ldots, g_t\}$ of $\mathfrak{a}$.

$$\text{If } \delta_i(g_j)(\mathsf{b}) = 0\,\forall i, j, \text{ then } \delta(f)(\mathsf{b}) = 0, \forall \delta \in \Delta, f \in \mathfrak{a}.$$

Corollary 8.14 *With the same notation as above.*

$$\text{If } \delta_i(g_j)(\mathsf{b}) = 0\,\forall i, j, \text{ then } \Delta \subset \mathfrak{D}_{\mathsf{b}}(\mathfrak{a}).$$

■

9 Macaulay Bases

Given a semigroup ordering $<$ on $\mathcal{T}$ and an $\mathfrak{m}^6$-closed ideal $\mathfrak{a} \subset \mathcal{P} \subset \mathcal{S}$, for each $t \in \mathcal{T}$ let $\gamma(t, \tau, <)$ be the coefficient corresponding to $\tau \in \mathbf{N}(\mathfrak{a})$ in the canonical form $Can(t, \mathfrak{a}, <)$ of t (see (3)).
Labelling the elements in $\mathbf{N}(\mathfrak{a})$, for each $\tau_i \in \mathbf{N}(\mathfrak{a})$, we let

$$\ell(\tau_i) := M(\tau_i) + \sum_{t \in \mathbf{T}(\mathfrak{a})} \gamma(t, \tau_i, <)M(t),$$

and we will show that $\mathfrak{M}(\mathfrak{a}) = \mathrm{Span}_{\mathbf{k}}\{\ell(\tau_i), \tau_i \in \mathbf{N}(\mathfrak{a})\}$.
Notice that $\ell(\tau) \in \mathfrak{M}(\mathfrak{a})$ requires in particular $\ell(\tau) \in \mathbf{k}[(\mathbb{M}])$ which holds iff $\#\{t : \gamma(t, \tau_i, <) \neq 0, \tau_i \in \mathbf{N}(\mathfrak{a})\} < \aleph_0$, by (4) clearly this is granted if the set $\{t \in \mathcal{T} : t > \tau\}$ is finite.

In order to have duality between $\mathfrak{P}(-)$ and $\mathfrak{M}(-)$ (i.e. to deal with functionals which are polynomials (in $\mathrm{Span}_k(\mathbb{M}) = \mathbf{k}[\mathbb{M}]$ and not series in $\mathbf{k}[[\mathbb{M}]]$) we may choose on $\mathcal{T}$ a Hironaka/standard

[6] Where, as usual, $\mathfrak{m} = (X_1, \ldots, X_n))$ so that, in particular, $1 \in \mathbf{N}(\mathfrak{a})$.

inf-limited[7] ordering $<$. In this setting to Gröbner bases correspond the so-called *Hironaka/standard bases* (which deal with series instead of polynomials) and the notion of leading term is the one related to standard (and not Gröbner) bases.

Remark that in this context the ideal we obtain results to be given in terms of a standard (not Gröbner) basis. Note that, as a Hironaka's basis of an ideal returns its $\mathfrak{m}$-closure, the restrictions on both $\mathfrak{m}$-closed ideals and inf-limited ordering are quite natural and strictly related in the theory we have developped here.

Finally, notice that, letting $f_t := t - \sum_{\tau_j<t} \gamma(t, \tau_j, <)\tau_j$, for all $t \in \mathbf{T}(\mathfrak{a})$, the set $\{f_t : t \in \mathbf{T}(\mathfrak{a})\}$ is a dialytic array (i.e. a $\mathbf{k}$-linear basis of $\mathfrak{a}$) and we have:

Proposition 9.1 *With the notation above, it holds:*

$$\ell(\tau)(f_t) = 0, \forall t \in \mathbf{T}(\mathfrak{a}), \tau \in \mathbf{N}(\mathfrak{a}).$$

Proof. Our contention is true as for all $t \in \mathbf{T}(\mathfrak{a}), \tau \in \mathbf{N}(\mathfrak{a}) \implies M(\tau)(t) = 0$, and similarly for all $\upsilon \in \mathbf{T}(\mathfrak{a})$, $\tau \in \mathbf{N}(\mathfrak{a}) \implies M(\tau)(\upsilon) = 0$, moreover $M(\tau)(\tau_j) = 0$ for all $\tau_j \neq \tau \in \mathbf{N}(\mathfrak{a})$, so that

$$\begin{aligned}\ell(\tau)(f_t) &= M(\tau)(f_t) + \sum_{\upsilon\in\mathbf{T}(\mathfrak{a})} \gamma(\upsilon, \tau, <)M(\upsilon)(f_t)\\ &= M(\tau)(t - \sum_{\tau_j<t} \gamma(t, \tau_j, <)\tau_j) +\\ &\quad \sum_{\upsilon\in\mathbf{T}(\mathfrak{a})} \gamma(\upsilon, \tau, <)M(\upsilon)(t - \sum_{\tau_j<t} \gamma(t, \tau_j, <)\tau_j)\\ &= -\gamma(t, \tau, <) + \gamma(t, \tau, <) = 0\end{aligned}$$

■

Corollary 9.2 *With the notation above, it holds:*

$$\mathfrak{M}(\mathfrak{a}) = \mathrm{Span}_{\mathbf{k}}\{\ell(\tau_i), \tau_i \in \mathbf{N}(\mathfrak{a})\}.$$

■

Moreover, restricting ourselves (as done in most of our examples) either to $\mathfrak{m}$-primary ideals, or to ideals homogeneous w.r.t. the valuation v_{w}, associated to the weight function $\mathsf{w} := (w_1, \ldots, w_n) \in \mathbb{R}^n, w_i > 0$, we have:

Corollary 9.3 *Given $\rho \in \mathbb{N}$, for each τ_i, $\deg(\tau_i) < \rho$, denoting*

$$\ell^\rho(\tau_i) := M(\tau_i) + \sum_{\substack{t\in\mathbf{T}(\mathfrak{a})\\ \deg(t)<\rho}} \gamma(t, \tau_i, <)M(t), \quad \textit{then}$$

- $\mathbf{N}(\mathfrak{a} + \mathfrak{m}^\rho) = \{\tau_i \in \mathbf{N}(\mathfrak{a}), \deg(\tau_i) < \rho\}$,
- $\mathfrak{M}(\mathfrak{a} + \mathfrak{m}^\rho) = \mathrm{Span}_{\mathbf{k}}\{\ell^\rho(\tau_i), \tau_i \in \mathbf{N}(\mathfrak{a}), \deg(\tau_i) < \rho\}$.

■

Definition 9.4 *Referring to Definition* 3.1 *a basis* $\{\ell_1, \ell_2, \ldots, \ell_i, \ldots\}$ *of a stable vector subspace* $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ *is called* Macaulay basis *of Λ w.r.t. $<$ if*

- $\mathbf{T}\{\Lambda\} := \{\mathbf{T}(\ell_i)\} \subset \mathcal{T}$ *is an order ideal;*
- $\ell_i = M(\mathbf{T}(\ell_i)) + \sum_{\upsilon\in\mathbf{N}(\Lambda)} \xi(\upsilon, \mathbf{T}(\ell_i))M(\upsilon)$, $\forall i$ *and suitable* $\xi(\upsilon, \mathbf{T}(\ell_i)) \in k$. ■

---

[7] i.e. an ordering on $\mathcal{T}$ such that

- $X_i < 1, \forall\ i$
- for each decreasing infinite sequence in $\mathcal{T}, \tau_1 > \tau_2 > \ldots > \tau_r > \ldots$ and each $\tau \in \mathcal{T}$ there is $r \in \mathbb{N}$ such that $\tau_r < \tau$.

Corollary 9.5 *With the notation above, if we set* $\Lambda := \mathfrak{M}(\mathfrak{a})$ *it holds*
- $\{\ell(\tau_i), \tau_i \in \mathbf{N}(\mathfrak{a})\}$ *is a Macaulay basis of* Λ,
- $\mathbf{T}\{\Lambda\} = \mathbf{N}(\mathfrak{a})$. ∎

Proof. For each i and each $t \in \mathbf{T}(\mathfrak{a})$, we have

$$\gamma(t, \tau_i, <) \neq 0 \Longrightarrow t > \tau_i,$$

and so $\mathbf{T}(\ell(\tau_i)) = \tau_i$. ∎

Example 9.6 *Given the* $\mathfrak{m}$*-closed ideal* $\mathfrak{a} := (X_2 - X_1^2, X_3 - X_1^3, \ldots, X_n - X_1^n)$, *which is homogeneous w.r.t. the valuation*

$$v_{\mathsf{w}} : \mathcal{T} \mapsto \mathbb{R}, \text{ such that } v_{\mathsf{w}}(X_i) = i, \forall\, i \in \{1, \ldots, n\},$$

letting $\ell_j := \sum_{\substack{\tau \in \mathcal{T} \\ v_{\mathsf{w}}(\tau) = j}} M(\tau), \ \forall\, j \in \mathbb{N},$ *it is easy to verify that, for each* $\rho \in \mathbb{N}$:

$$\begin{aligned} \mathfrak{a} + \mathfrak{m}^\rho &= (X_1^\rho, X_2 - X_1^2, X_3 - X_1^3, \ldots, X_n - X_1^n), \\ \deg(\mathfrak{a} + \mathfrak{m}^\rho) &= \rho, \\ \mathfrak{M}(\mathfrak{a} + \mathfrak{m}^\rho) &= \mathrm{Span}_{\mathbf{k}}\{\ell_j, 0 \leq j < \rho\}, \\ \mathfrak{M}(\mathfrak{a}) &= \mathrm{Span}_{\mathbf{k}}\{\ell_j, j \in \mathbb{N}\}. \end{aligned}$$

Moreover, if $<$ *denotes the refinement of* v_{w} *by the lexicographical ordering induced by* $X_1 \prec \cdots \prec X_n$,
- *for each* $\rho \in \mathbb{N}$, $(X_1^\rho, X_2, X_3, \ldots, X_n) = \mathbf{T}(\mathfrak{a} + \mathfrak{m}^\rho)$;
- *for each* $\rho \in \mathbb{N}$, $\{X_1^\rho, X_2 - X_1^2, X_3 - X_1^3, \ldots, X_n - X_1^n\}$, *is the Gröbner basis of* $\mathfrak{a} + \mathfrak{m}^\rho$ *w.r.t.* $<$;
- *for each* $i \in \mathbb{N}$, $\mathbf{T}(\ell_i) = X_1^i$;
- *for each* $\rho \in \mathbb{N}$, $\mathbf{T}\{\mathfrak{M}(\mathfrak{a} + \mathfrak{m}^\rho)\} = \{1, X_1, \ldots, X_1^{\rho-1}\}$;
- *the Gröbner basis of* $\mathfrak{a}$ *w.r.t.* $<$ *is* $\{X_2 - X_1^2, X_3 - X_1^3, \ldots, X_n - X_1^n\}$;
- $\mathbf{N}(\mathfrak{a}) = \{X_1^j, j \in \mathbb{N}\} = \mathbf{T}\{\mathfrak{M}(\mathfrak{a})\}$. ∎

10 Macaulay Bases and Gröbner Representations

Proposition 10.1 *If* $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ *is any stable vector subspace, then also* $\mathrm{Span}_{\mathbf{k}}(M(\mathbf{T}\{\Lambda\}))$ *(where* $M(\mathbf{T}\{\Lambda\}) := \{M(\tau) : \tau \in \mathbf{T}\{\Lambda\}\}$*) is so.*
Moreover, if $\{\ell_i, 1 \leq i \leq s\}$ *is a Macaulay basis of* Λ, *then* $\{M(\mathbf{T}(\ell_i)), 1 \leq i \leq s\}$ *is a Macaulay basis of* $\mathrm{Span}_{\mathbf{k}}(M(\mathbf{T}\{\Lambda\}))$.

Proof. For each $\ell \in \Lambda$ either $\sigma_i(\mathbf{T}(\ell)) = 0$ or $\sigma_i(\mathbf{T}(\ell)) = \mathbf{T}(\sigma_i(\ell))$ as, by assumption, Λ is stable. ∎

Proposition 10.2 *Given a stable vector subspace* $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$, *let:*
- $\{\ell_1, \ell_2, \ldots, \ell_i, \ldots\}$ *be its Macaulay basis w.r.t.* $<$, *where, for each* i,
  $\ell_i = M(\tau_i) + \sum_{v \in \mathbf{N}(\Lambda)} \xi(v, \tau_i) M(v), \quad \tau_i = \mathbf{T}(\ell_i);$
- $\{t_1, \ldots, t_s\}$ *be the minimal basis of the monomial ideal* $\mathbf{N}(\Lambda) \subset \mathcal{P}$;
- $g_j := t_j - \sum_{\tau_i \in \mathbf{T}\{\Lambda\}} \xi(t_j, \tau_i) \tau_i$, *for each* j.

Then $(g_1, \ldots, g_s)$ *is the Gröbner basis of* $\mathfrak{P}(\Lambda)$ *w.r.t.* $<$.

Proof. It is sufficient to show that

$$\begin{aligned} \ell_i(g_j) &= M(\tau_i)(g_j) + \sum_{v \in \mathbf{N}(\Lambda)} \xi(v, \tau_i) M(v)(g_j) \\ &= -\xi(t_j, \tau_i) M(\tau_i)(\tau_i) + \xi(t_j, \tau_i) M(t_j)(t_j) = 0. \end{aligned}$$

∎

Let $<$ be any semigroup ordering on $\mathcal{T}$,
$\mathfrak{q} \subset \mathcal{P}$ an $\mathfrak{m}$-primary ideal,
$\mathbf{N}(\mathfrak{q}) := \{\tau_1, \ldots, \tau_s\}$, and
$\ell_i := \ell(\tau_i) := M(\tau_i) + \sum_{t \in \mathbf{T}(\mathfrak{q})} \gamma(t, \tau_i, <) M(t) \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ as above;
then:

Proposition 10.3 *With the above notation, $\Lambda := \mathrm{Span}_{\mathbf{k}}\{\ell_1, \ldots, \ell_s\}$ and $\mathbf{N}(\mathfrak{q})$ are biorthogonal.* ∎

Note, (see also Corollary 12.2), that $\tau_i < \tau_j$, for all $i < j$, does not imply $\Lambda_i := \mathrm{Span}_{\mathbf{k}}\{\ell_1, \ldots, \ell_i\}$ is a $\mathcal{P}$-module for each i. For instance consider the following:

Example 10.4 *Let $\mathcal{P} := \mathbf{k}[X_1, X_2]$,*
$<$ any termordering on $\mathcal{T}$ such that $X_2 > X_1^2$,
$\mathfrak{a} := (X_2^2 - X_1^2, X_1X_2, X_1^3)$, so that $\mathbf{N}(\mathfrak{a}) := \{1, X_1, X_1^2, X_2\}$, and
$\ell_1 = \ell(1) = M(1)$, $\ell_2 = \ell(X_1) = M(X_1)$, $\ell_3 = \ell(X_1^2) = M(X_1^2) + M(X_2^2)$,
$\ell_4 = \ell(X_2) = M(X_2)$.
Then $\Lambda_3 := \mathrm{Span}_{\mathbf{k}}\{\ell_1, \ell_2, \ell_3\}$, is not a $\mathcal{P}$-module as $\mathfrak{P}(\Lambda_3)$ is not an ideal, namely:

$$\ell_3(X_2^2) = 1, X_2^2 \notin \mathfrak{P}(\Lambda_3), \text{ while } X_2 \in \mathfrak{P}(\Lambda_3).$$

∎

11 Reminds on Primary Decomposition

Every handbook in *Commutative Algebra* contains the so called *Lasker-Noether decomposition theorem*:

Theorem 11.1 (Lasker-Noether) *In a noetherian ring R, every ideal $\mathfrak{a} \subset R$ has an* irredundant primary decomposition $\mathfrak{a} = \cap_{i=1}^r \mathfrak{q}_i$ *such that:*

- $\mathfrak{q}_j$ *is a primary ideal, for all $j \in \{1, \ldots, r\}$, with $\mathfrak{p}_j = \sqrt{\mathfrak{q}_j}$,*
- $\mathfrak{q}_j \not\supset \bigcap_{\substack{i=1 \\ j \neq i}}^{r} \mathfrak{q}_i$, *for all $j \in \{1, \ldots, r\}$,*
- $\mathfrak{p}_j \neq \mathfrak{p}_i$, *for all $i \neq j \in \{1, \ldots, r\}$.*

If $\mathfrak{a} = \cap_{i=1}^r \mathfrak{q}_i = \cap_{j=1}^s \mathfrak{q}'_j$ are two irredundant primary representations of $\mathfrak{a}$ (where for each i, j we have $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ and $\mathfrak{p}'_j = \sqrt{\mathfrak{q}'_j}$), then:
– $r = s$,
– for all $i \in \{1, \ldots, r\}$, $\exists j \in \{1, \ldots, s\} : \mathfrak{p}_i = \mathfrak{p}'_j$;
– for all $j \in \{1, \ldots, s\}$, $\exists i \in \{1, \ldots, r\} : \mathfrak{p}'_j = \mathfrak{p}_i$.

For each $i \in \{1, \ldots, r\}$, the prime ideal $\mathfrak{p}_i$ is called associated prime ideal of $\mathfrak{a}$ *and the primary ideal $\mathfrak{q}_i$ is called* primary component of $\mathfrak{a}$; *each minimal element in $\{\mathfrak{p}_i : 1 \leq i \leq r\}$ is called* isolated prime of $\mathfrak{a}$, *while associated prime ideal which are not isolated are called* embedded, *a primary component $\mathfrak{q}_i$ is called* isolated *or* embedded *according to what is its radical.*

The isolated primary components of $\mathfrak{a}$ are uniquely determined. ∎

Remark 11.2 We briefly recall here the iter followed by E. Noether in [20] for proving the above facts, in order to emphasize a result of Macaulay/Gröbner which will be proved in the next section.

I In a commutative ring R, an ideal which is not a finite intersection of ideals strictly containing it is called *irreducible.*

II In a commutative ring R, every prime ideal $\mathfrak{p} \subset R$ is irreducible.

III In a noetherian ring R, every irreducible ideal is primary (but not conversely).

IV (Lasker-Noether) In a noetherian ring R, every ideal $\mathfrak{a} \subset R$ is a finite intersection of irreducible ideals.

V (Noether) In a noetherian ring R, *reduced representation* of an ideal $\mathfrak{a}$ is a representation of $\mathfrak{a}$ as intersection $\mathfrak{a} = \cap_{j=1}^r \mathfrak{i}_j$ of finitely many irreducible ideals s.t.

– for all $j \in \{1, \ldots, r\}$, $\mathfrak{i}_j \not\supset \bigcap_{\substack{h=1 \\ j \neq h}}^{r} \mathfrak{i}_h$,

– there is no irreducible ideal ${\mathfrak{i}_j}' \supset \mathfrak{i}_j$ such that $\mathfrak{a} = \left(\bigcap_{\substack{h=1\\ j\neq h}}^{r} \mathfrak{i}_h\right) \cap {\mathfrak{i}_j}'$.

VI (Noether) In a noetherian ring R, each ideal $\mathfrak{a} \subset R$ has a reduced representation as intersection of finitely many irreducible ideals.

VII A primary component $\mathfrak{q}_j$ of an ideal $\mathfrak{a}$ contained in a noetherian ring R, is called *reduced* if there is no primary ideal ${\mathfrak{q}_j}' \supset \mathfrak{q}_j$ such that $\mathfrak{a} = \left(\bigcap_{\substack{i=1\\ j\neq i}}^{r} \mathfrak{q}_i\right) \cap {\mathfrak{q}_j}'$.

VIII In an irredundant primary decomposition of an ideal of a noetherian ring, each primary component can be chosen to be reduced.

We recall here some well-known examples which show that the statements about uniqueness of representation cannot be improved.

Example 11.3 (Hentzelt) *All the examples live in the polynomial ring* $\mathbb{Q}[X,Y]$.

1. *The decomposition* $(X^2, XY) = (X) \cap (X^2, XY, Y^\lambda)$, *for all* $\lambda \in \mathbb{N}, \lambda \geq 1$, *where* $\sqrt{(X^2, XY, Y^\lambda)} = (X,Y) \supset (X)$, *shows that embedded components are not unique; however, since*
$$(X^2, XY, Y) = (X^2, Y) \supsetneq (X^2, XY, Y^\lambda), \text{ for each } \lambda > 1,$$
(X^2, Y) *is a reduced embedded irreducible component and* $(X^2, XY) = (X) \cap (X^2, Y)$ *is a reduced representation.*
2. *The decompositions* $(X^2, XY) = (X) \cap (X^2, Y + aX)$, *as* $a \in \mathbb{Q}$ *(where* $\sqrt{(X^2, Y + aX)} = (X,Y) \supset (X)$, *and* $(X^2, Y + aX)$ *is reduced) show that also reduced representations are not unique; note that, setting* $a = 0$, *we find again* $(X^2, XY) = (X) \cap (X^2, Y)$.

Example 11.4 *We also recall the reduced representation*
$$(X^2, XY, Y^\lambda) = (X^2, Y) \cap (X, Y^\lambda)$$
of the primary ideal $\mathfrak{a}_\lambda := (X^2, XY, Y^\lambda)$ *into reduced irreducible components.*
Neither such decomposition is unique since we also have
$$\mathfrak{a}_\lambda = (X^2, Y + aX) \cap (X, Y^\lambda) \text{ as } a \in \mathbb{Q}.$$

Let us also remark that these reduced irreducible components give the irredundant primary representations
$$\begin{aligned}(X^2, XY) &= (X) \cap \mathfrak{a}_\lambda \\ &= (X) \cap (X^2, Y + aX) \cap (X, Y^\lambda) \\ &= (X) \cap (X^2, Y + aX)\end{aligned}$$
in terms of reduced primary components. ■

Example 11.5 (Noether) *In the same context it is worthwhile to recall the decompositions in* $\mathbb{Q}[X,Y,Z]$
$$\begin{aligned}\mathfrak{a}_\lambda &= (X^2, XY, Y^2, YZ) \cap (X, Y^\lambda), \\ (X^2, XY, Y^2, YZ) &= (X^2, Y) \cap (X, Y^2, Z),\end{aligned}$$
whence
$$\begin{aligned}\mathfrak{a}_\lambda &= (X^2, XY, Y^2, YZ) \cap (X, Y^\lambda) \\ &= (X^2, Y) \cap (X, Y^\lambda)\end{aligned}$$
because $(X, Y^2, Z) \supset (X, Y^\lambda)$ *for all* $\lambda \geq 2$. ■

We will show in Section 12 that in an irredundant primary decomposition of an ideal, for each embedded associated prime $\mathfrak{p}$ it is possible to determine a reduced primary component $\mathfrak{q}$ associated to it, together with a reduced decomposition of $\mathfrak{q}$ into irreducible components associated to $\mathfrak{p}$.

12 Macaulay Bases and Primary Decompositions

Consider:
a noetherian inf-limited ordering $<$ on $\mathcal{T}$,
an $\mathfrak{m}$-closed ideal $\mathfrak{a} \subset \mathcal{P} \subset \mathcal{S}$,
the (finite) corner set $\mathbf{C}(\mathfrak{a}) := \{\omega_1, \dots, \omega_s\}$,
the (not-necessarily finite) set $\mathbf{N}(\mathfrak{a})$,
the Macaulay basis $\{\ell(\tau) : \tau \in \mathbf{N}(\mathfrak{a})\}$ and
the vector subspace $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ it generates.
For each $j, 1 \le j \le s$, denote

$$\Lambda_j := \mathrm{Span}_{\mathbf{k}}\{\upsilon\ell(\omega_j) : \upsilon \in \mathcal{T}, \omega_j \in \mathbf{C}(\mathfrak{a})\} \text{ and } \mathfrak{q}_j := \mathfrak{P}(\Lambda_j).$$

Note that

$$\mathfrak{q}_i \subset \mathfrak{q}_{i'} \iff \Lambda_i \supset \Lambda_{i'}.$$

Moreover, let $J \subset \{1, \dots, s\}$ be the subset of indices corresponding to the minimal elements of $\{\mathfrak{q}_j : 1 \le j \le s\}$

Lemma 12.1 (Macaulay) *With the above notation, for each j, denoting*

$$\Lambda'_j := \mathrm{Span}_{\mathbf{k}}\{\upsilon\ell(\omega_j) : \upsilon \in \mathcal{T} \cap \mathfrak{m}\}$$

we have:
1. $\dim_{\mathbf{k}}(\Lambda'_j) = \dim_{\mathbf{k}}(\Lambda_j) - 1$,
2. $\ell(\omega_j) \notin \Lambda'_j = \mathfrak{M}(\mathfrak{q}_j : \mathfrak{m})$,
3. *for each $\mathfrak{m}$-primary ideal $\mathfrak{q}' \supset \mathfrak{q}_j \Longrightarrow \mathfrak{M}(\mathfrak{q}') \subseteq \Lambda'_j$.*

Proof. For each $h, 1 \le h \le n$, denoting $l_h := X_h\ell(\omega_j)$, we have

$$\begin{aligned}\Lambda'_j \subset \sum_h \mathrm{Span}_{\mathbf{k}}\{\upsilon l_h : \upsilon \in \mathcal{T}\} &= \sum_h \mathfrak{M}(\mathfrak{q}_j : X_h)\\ &= \mathfrak{M}\left(\cap_h(\mathfrak{q}_j : X_h)\right)\\ &= \mathfrak{M}(\mathfrak{q}_j : \mathfrak{m}).\end{aligned}$$

As $\mathfrak{q}_j : \mathfrak{m} \ne \mathfrak{q}_j$ we have $\dim_{\mathbf{k}}(\Lambda_j) > \dim_{\mathbf{k}}(\mathfrak{M}(\mathfrak{q}_j : \mathfrak{m})) \ge \dim_{\mathbf{k}}(\Lambda'_j) \ge \dim_{\mathbf{k}}(\Lambda_j) - 1$, whence the first claim. ∎

Corollary 12.2 *With the notation above, if $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, then it is possible to enumerate the set $\mathbf{N}(\mathfrak{a}) := \{\tau_1, \dots, \tau_s\}$ so that, for all σ:*
$L_\sigma := \mathrm{Span}_{\mathbf{k}}(\{\ell(\tau_1), \dots, \ell(\tau_\sigma)\})$ is a $\mathcal{P}$-module, $\mathfrak{a}_\sigma = \mathfrak{P}(L_\sigma)$ is a zero-dimensional ideal and there is a chain $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_s = \mathfrak{a}$.

Proof. The proof can be done by induction on $s := \#\mathbf{N}(\mathfrak{a})$, being trivial if $\#\mathbf{N}(\mathfrak{a}) = 1$ (i.e. $\mathbf{N}(\mathfrak{a}) = \{1\}$). Choose any element $\omega_j \in \mathbf{C}(\mathfrak{a})$, $j \in J$, and set

$$\tau_s := \omega_j, \quad L_{s-1} := \mathrm{Span}_{\mathbf{k}}(\{\ell(\omega), \omega \in \mathbf{N}(\mathfrak{a}), \omega \ne \tau_s\}.$$

Then $\ell(\omega_j) \notin L_{s-1}$,
$\dim_{\mathbf{k}}(L_{s-1}) = s - 1$,
$\#\mathbf{N}(\mathfrak{a}_{s-1}) = s - 1$, so that
$\mathbf{N}(\mathfrak{a}_{s-1}) = \{\omega \in \mathbf{N}(\mathfrak{a}), \omega \ne \tau_s\}$
and the claim follows by induction. ∎

Corollary 12.3 *For a zero-dimensional ideal $\mathfrak{a}$, with $\deg(\mathfrak{a}) = s$, there is a finite ordered set of l.i. $\mathbf{k}$-linear functionals $\mathbb{L} = \{\ell_1, \dots, \ell_s\}$ such that: $L := \mathrm{Span}_{\mathbf{k}}(\mathbb{L}) = \mathfrak{L}(\mathfrak{a})$, each vector subspace $L_\sigma := \mathrm{Span}_{\mathbf{k}}(\{\ell_1, \dots, \ell_\sigma\})$, $1 \le \sigma \le s$, is a $\mathcal{P}$-module each $\mathfrak{a}_\sigma = \mathfrak{P}(L_\sigma)$ is a zero-dimensional ideal, and there is a chain $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \cdots \supset \mathfrak{a}_s = \mathfrak{a}$.*

Proof. Fix any term-ordering $<$ and consider the irredundant primary decomposition $\mathfrak{a} = \cap_{h=1}^{r} \mathfrak{q}_h$. For each i, let us denote $\mathsf{a}_i := (a_{i1}, \ldots, a_{in}) \in k^n$ and $\mathfrak{m}_i := \sqrt{\mathfrak{q}_i} = (X_1 - a_{i1}, \ldots, X_n - a_{in})$. Let $\lambda_i : \mathcal{P} \to \mathcal{P}$, be the translation $\lambda_i(X_j) = X_j + a_{ij}$, for all j, and let $\{\tau_{i1}, \ldots, \tau_{i\mu_i}\} = \mathbf{N}(\lambda_i(\mathfrak{q}_i))$ be enumerated so that Corollary 12.2 holds.
Setting

$$\mathbb{L} := \{\ell(\tau_{ij})\lambda_i(\cdot), 1 \le i \le t, 1 \le j \le \mu_i\} = \{\ell_1, \ldots, \ell_s\},$$

we have $\deg(\mathfrak{a}) = \sum_{i=1}^{r} \mu_i = \sum_{i=1}^{r} \deg(\mathfrak{q}_i)$ and $L := \mathrm{Span}_{\mathbf{k}}(\mathbb{L}) = \mathfrak{L}(\mathfrak{a})$. The claim is obtained, by Corollary 12.2, enumerating the set $\mathbb{L}$ so that for each α, β, $\ell_\alpha = \ell(\tau_{i_\alpha j_\alpha})\lambda_{i_\alpha}(\cdot)$, $\ell_\beta = \ell(\tau_{i_\beta j_\beta})\lambda_{i_\beta}(\cdot)$ we have $i_\alpha = i_\beta, j_\alpha < j_\beta \Longrightarrow \alpha < \beta$. ■

Theorem 12.4 (Gröbner) *With the above notation, for an $\mathfrak{m}$-primary ideal $\mathfrak{a}$, it holds:*
1. *each Λ_j is a finite-dimensional stable vectorspace,*
2. *each $\mathfrak{q}_j$ is an $\mathfrak{m}$-primary ideal,*
3. *each $\mathfrak{m}$-primary ideal $\mathfrak{q}_j$ is reduced,*
4. *each $\mathfrak{m}$-primary ideal $\mathfrak{q}_j$ is irreducible,*
5. *$\mathfrak{a} := \cap_{j \in J} \mathfrak{q}_j$ is a reduced representation of $\mathfrak{q}$.*

Proof. 1. is trivial by construction; 2. is a direct consequence of 1.; 3. if $\mathfrak{q}_j$ is not reduced, then exists $\mathfrak{q}' \supset \mathfrak{q}_j$ such that $\mathfrak{a} = \cap_{i \ne j} \mathfrak{q}_i \bigcap \mathfrak{q}'$ and Lemma 12.1 implies $\ell(\omega_j) \notin \Lambda'_j \supseteq \mathfrak{M}(\mathfrak{q}')$. Then, looking to the leading terms of the $\ell(\omega_i)s$, it is easy to see that

$$\ell(\omega_j) \notin \sum_{i \ne j} \Lambda_i + \mathfrak{M}(\mathfrak{q}') = \mathfrak{M}(\mathfrak{a}) = \Lambda;$$

4. if $\mathfrak{q}_j = \mathfrak{q}' \cap \mathfrak{q}''$ is reducible, Lemma 12.1 implies $\ell(\omega_j) \notin \mathfrak{M}(\mathfrak{q}') + \mathfrak{M}(\mathfrak{q}'')$, i.e., again, the contradiction $\ell(\omega_j) \notin \Lambda$; 5. since $\mathfrak{M}(\mathfrak{a}) = \Lambda = \sum_j \Lambda_j = \sum_{j \in J} \Lambda_j = \sum_{j \in J} \mathfrak{M}(\mathfrak{q}_j)$, the representation $\mathfrak{a} := \cap_{j \in J} \mathfrak{q}_j$ is reduced being the components reduced by 3., moreover, redundant components have been removed by restricting the indices to J. ■

Example 12.5 *If* $\mathfrak{a} = \mathfrak{m}^2 = (X^2, XY, Y^2) \subset \mathbf{k}[X, Y]$, *then*
$\mathbf{C}(\mathfrak{a}) = \{X, Y\}$, $\Lambda = \mathrm{Span}_{\mathbf{k}}\{M(1), M(X), M(Y)\}$, *and*
$\omega_1 := X$, $\Lambda_1 = \mathrm{Span}_{\mathbf{k}}\{M(1), M(X)\}$, $\mathfrak{q}_1 = (X^2, Y)$*;*
$\omega_2 := Y$, $\Lambda_2 = \mathrm{Span}_{\mathbf{k}}\{M(1), M(Y)\}$, $\mathfrak{q}_1 = (X, Y^2)$*;*
whence $(X^2, XY, Y^2) = (X^2, Y) \cap (X, Y^2)$. ■

Example 12.6 *In Ex. 10.4* $\mathfrak{a} = (X_2^2 - X_1^2, X_1X_2, X_1^3)$, $\mathbf{C}(\mathfrak{a}) = \{X_1^2, X_2\}$
$\Lambda = \mathrm{Span}_{\mathbf{k}}\{M(1), M(X_1), M(X_1^2) + M(X_2^2), M(X_2)\} = \mathrm{Span}_{\mathbf{k}}\{\ell_1, \ell_2, \ell_3, \ell_4\}$,
and $\omega_1 := X_2$, $\Lambda_2 = \mathrm{Span}_{\mathbf{k}}\{M(1), M(X_2)\}$, $\mathfrak{q}_1 = (X_1, X_2^2)$*,*
$\omega_2 := X_1^2$, $\Lambda_2 = \Lambda$, $\mathfrak{q}_2 = \mathfrak{a}$,
namely, $X_2\ell_3 = M(X_2)$, $X_1\ell_3 = M(X_1)$, $X_1^2\ell_3 = X_2^2\ell_3 = M(1)$,
thus $\mathfrak{a}$ *is irreducible.*
In connection with Corollary 12.2 we have to set

$$\tau_4 := X_1^2,\ L_3 := \mathrm{Span}_{\mathbf{k}}\{M(1), M(X_1), M(X_2)\},$$

obtaining $\mathfrak{a}_3 = (X_1^2, X_1X_2, X_2^2) = (X_1, X_2^2) \cap (X_1^2, X_2)$.
There are therefore two possible orderings of $\mathbf{N}(\mathfrak{a})$*, satisfying Corollary 12.2:*
$\mathbf{N}(\mathfrak{a}) = \{1, X_1, X_2, X_1^2\}$, *which returns the chain*

$$(X_1, X_2) \supset (X_1^2, X_2) \supset \mathfrak{a}_3 \supset \mathfrak{a},\ \textit{and}$$

$\mathbf{N}(\mathfrak{a}) = \{1, X_2, X_1, X_1^2\}$ *which returns the chain*

$$(X_1, X_2) \supset (X_1, X_2^2) \supset \mathfrak{a}_3 \supset \mathfrak{a}$$

■

If $\mathfrak{a}$ is not $\mathfrak{m}$-primary, let

$\mathbf{C}(\mathfrak{a}) = \{\omega_1, \dots, \omega_t\}$
$\rho := \max\{\deg(\omega_j) + 1 : \omega_j \in \mathbf{C}(\mathfrak{a})\} + 1$ so that $\mathfrak{q}' := \mathfrak{a} + \mathfrak{m}^\rho$ is an $\mathfrak{m}$-primary component of $\mathfrak{a}$, and $\Lambda \cap \nabla_\rho = \mathfrak{M}(\mathfrak{q}')$,
$\mathfrak{a} = \cap_{i=1}^r \mathfrak{q}_i$ an irredundant primary representation of $\mathfrak{a}$, with $\sqrt{\mathfrak{q}_1} = \mathfrak{m}$,
$\mathfrak{b} := \mathfrak{a} : \mathfrak{m}^\infty = \cap_{i=2}^r \mathfrak{q}_i$ and $\mathfrak{b} = \cap_{i=1}^u \mathfrak{Q}_i$ a reduced representation of it,
$\mathbf{C}(\mathfrak{q}') = \{\omega_1, \dots, \omega_t, \omega_{t+1}, \dots, \omega_s\} \supset \mathbf{C}(\mathfrak{a})$
for each $j, 1 \le j \le s$, $\Lambda_j := Span_{\mathbf{k}}\{\upsilon\ell(\omega_j) : \upsilon \in \mathcal{T}\}$, and $\mathfrak{q}_j := \mathfrak{P}(\Lambda_j)$
$\mathfrak{q} := \cap_{j=1}^s \mathfrak{q}_j$

Then

Corollary 12.7 *With the notation above, it holds:*

1. $\mathfrak{q} \subset \mathfrak{q}'$ *is a reduced* $\mathfrak{m}$*-primary component of* $\mathfrak{a}$,
2. $\mathfrak{q}' := \cap_{j=1}^t \mathfrak{q}_j$ *is a reduced representation of* $\mathfrak{q}'$,
3. $\mathfrak{q} = \cap_{j=1}^s \mathfrak{q}_j$ *is a reduced representation of* $\mathfrak{q}$,
4. $\mathfrak{q}_i \supset \mathfrak{b} \iff i > t$,
5. $\mathfrak{a} = \cap_{i=1}^u \mathfrak{Q}_i \bigcap \cap_{j=1}^t \mathfrak{q}_j$ *is a reduced representation of* $\mathfrak{a}$. ∎

Example 12.8 *In Ex. 11.3.1. we have* $\mathfrak{a} := (X^2, XY)$, $\mathbf{C}(\mathfrak{a}) = \{X\}$,
$\Lambda = \mathrm{Span}_{\mathbf{k}}\{M(1), M(X)\} \cup \{M(Y^i), i \in \mathbb{N}\}$, *then*
$\mathfrak{b} = \mathfrak{a} : \mathfrak{m}^\infty = (X)$
$\rho = 3$, $\mathfrak{q}' = \mathfrak{a} + \mathfrak{m}^3 = (X^2, XY, Y^3)$, $\mathbf{C}(\mathfrak{q}') = \{X, Y^2\}$
$\omega_1 = X$, $\Lambda_1 = \mathrm{Span}_{\mathbf{k}}\{M(1), M(X)\}$, $\mathfrak{q}_1 = (X^2, Y)$;
$\omega_2 = Y^2$, $\Lambda_2 = \mathrm{Span}_{\mathbf{k}}\{M(1), M(Y), M(Y^2)\}$, $\mathfrak{q}_1 = (X, Y^3) \supset (X)$;
whence $(X^2, XY) = (X) \cap (X^2, Y)$.

Moreover if Ex. 11.3.2. shows that reduced representation (and even the notion of Macaulay basis) strongly depend on the choice of a frame of coordinates. In fact, choosing, for each $a \in \mathbb{Q}, a \neq 0$,
$\Lambda = \mathrm{Span}_{\mathbf{k}}\{M(1), M(X) - aM(Y)\} \cup \{M(Y^i), i \in \mathbb{N}\}$, *we obtain*
$\rho = 3$, $\Lambda \cap \nabla_\rho = \{M(1), M(X) - aM(Y), M(Y), M(Y^2)\}$,
$\omega_1 := X$, $\Lambda_1 = \{M(1), M(X) - aM(Y)\}$, $\mathfrak{q}_1 = (X^2, Y + aX)$,
$\omega_2 := Y^2$, $\Lambda_2 = \{M(1), M(Y)\}$, $\mathfrak{q}_2 = (X, Y^3) \supset (X)$,
whence $(X^2, XY) = (X) \cap (X^2, Y + aX)$.

Let us now perform a generic change of coordinates in Ex. 11.3.1.
$\Phi : \mathbb{Q}[X, Y] \to \mathbb{Q}[X, Y]$, $\Phi(X) = aX + bY, \Phi(Y) = cX + dY$, $ad - bc \neq 0 \neq a$,
we obtain:
$\mathfrak{a} = (aXY + bY^2, a^2X^2 - bY^2)$,
$\Lambda = \mathrm{Span}_{\mathbf{k}}\{M(1), M(X), M(Y), a^2M(Y^2) - abM(XY) + b^2M(X^2), \dots\}$,
$\mathfrak{b} := (aX + bY)$
$\rho = 3$, $\mathfrak{q}' = \mathfrak{a} + \mathfrak{m}^3$, $\mathbf{C}(\mathfrak{q}') = \{X, Y^2\}$,
$\Lambda \cap \nabla_\rho = \mathrm{Span}_{\mathbf{k}}\{M(1), M(X), M(Y), a^2M(Y^2) - abM(XY) + b^2M(X^2)\}$,
$\omega_1 := X$, $\Lambda_1 = \{M(1), M(X)\}$, $\mathfrak{q}_1 = (X^2, Y)$,
$\omega_2 := Y^2$, $\Lambda_2 = \{M(1), aM(Y) - bM(X), a^2M(Y^2) - abM(XY) + b^2M(X^2)\}$,
$\mathfrak{q}_2 = (aX + bY, Y^3) \supset (aX + bY)$,
whence $(aXY + bY^2, a^2X^2 - bY^2) = (aX + bY) \cap (X^2, Y)$. *So far we have chosen* $\{M(1), M(X), M(Y)\}$ *as basis of* ∇_3 *however what we need to do is to extend the basis* $\{M(1), aM(X) - bM(Y)\}$ *of* $\mathfrak{M}(\mathfrak{b}) \cap \nabla_3$ *to a basis of* ∇_3, *of course any choice* $eM(Y) + fM(X), ae - fb \neq 0$ *is acceptable giving the reduced primary ideal* $\mathfrak{P}(\{M(1), eM(Y) + fM(X)\}) = (X^2, eX - fY)$ *and the irredundant reduced primary decomposition* $\mathfrak{a} = (aX + bY) \cap (X^2, eX - fY)$. ∎

Example 12.9 *One naturally should expect that irredundance should be preserved by change of coordinates; it* does *as shown by this example.*

If we consider the irreducible $\mathfrak{m}$*-primary ideal* $\mathfrak{q} := (X^2, Y^2)$ *we know that the generic initial ideal has the shape* $\mathrm{gin}(\mathfrak{q}) = (X^3, XY, Y^2)$; *thus we could fear to obtain a decomposition* $(X^3, XY, Y^2) = (X^3, Y) \cap (X, Y^2)$; *this is* not *what happens; in fact if we perform a generic change of coordinates*

$$\Phi : \mathbb{Q}[X, Y] \to \mathbb{Q}[X, Y],\ \Phi(X) = aX + bY, \Phi(Y) = cX + dY,\ ad - bc \neq 0 \neq a,$$

we obtain
$\mathfrak{a} = \big((aX+bY)^2, 2bdXY + bcX^2 + adX^2, X^3\big)$
$\Lambda = \mathrm{Span}_{\mathbf{k}}\{M(1), \alpha M(X) + \beta M(Y), \gamma M(X) + \delta M(Y), \ell\},\ \ \Delta := \alpha\delta - \beta\gamma \neq 0$
where $\ell := -(ad+bc)M(XY) + 2bdM(X^2) + 2acM(Y^2)$*; we thus obtain*
$\ell X = -(ad+bc)M(Y) + 2bdM(X), \ell Y := -(ad+bc)M(X) + 2acM(Y), \ell XY = M(1)$
with $\Delta = (ad+bc)^2 - 4abcd = (ad-bc)^2$*, thus proving irreducibility.* ∎

13 Horner representation of Macaulay Bases

The description of the Noether equations necessarily requires a compact and less-consuming form, as Example 9.6 shows.

If we denote, for each $j, 1 \leq j \leq n$,

$$\mathbb{M}[j,n] := \{M(\tau) : \tau = X_1^{a_1} \cdots X_n^{a_n} \in \mathcal{T}, a_1 = \cdots = a_{j-1} = 0 \neq a_j\} \subset \mathbb{M},$$

then each element $\ell \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M} \setminus \{\mathrm{Id}\})$ can be uniquely expressed, (see [15]), as

$$\ell = \ell^{(1)} + \cdots + \ell^{(j)} + \cdots + \ell^{(n)},\ \ \ell^{(j)} \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M}[j,n])\ \forall\, j,$$

we will also introduce the notation

$$\ell^{(\geq j)} := \sum_{i=j}^{n} \ell^{(i)}.$$

Lemma 13.1 *[15] Let* $\ell = \ell^{(1)} + \cdots + \ell^{(n)} \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M} \setminus \{\mathrm{Id}\})$*. The following hold:*

1. $\lambda_i(\ell) = \lambda_i(\ell^{(1)}) + \cdots + \lambda_i(\ell^{(i-1)}) + \ell^{(i)}$;
2. $(\lambda_i(\ell))^{(j)} = \begin{cases} \lambda_i(\ell^{(j)}) & \text{if } j < i, \\ \ell^{(j)} & \text{if } j = i, \\ 0 & \text{if } j > i; \end{cases}$
3. $\ell^{(i)} = (\lambda_i(\ell))^{(\geq i)} = \lambda_i(\ell^{(\geq i)})$.

∎

We can thus formulate

Corollary 13.2 (Macaulay) *Given a finite dimensional stable vector subspace* $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ *with* $\mathbf{k}$*-basis* $B := \{\ell_1, \ldots, \ell_s\}$*,* $\ell_1 = \mathrm{Id}$*, let* $\ell \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ *be such that the vector subspace generated by* $B \cup \{\ell\}$ *is stable.*

Then there are $c_{ij} \in k, 1 \leq j \leq r, 1 \leq i \leq s$ *such that*

$$\ell^{(j)} = \sum_{i=1}^{s} c_{ij} \rho_j(\ell_i^{(\geq j)}).$$

∎

Corollary 13.3 *If* $\Lambda \subset \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ *is a finite dimensional stable vector subspace with* $\dim_{\mathbf{k}}(\Lambda) = s$*, then there are* $\frac{ns(s+1)}{2}$ *elements* $c_{ijh} \in \mathbf{k}, 1 \leq j \leq n, 1 \leq i < h \leq s$ *such that, setting*

$$\begin{aligned} \ell_1 &:= \mathrm{Id}, \\ \ell_h^{(j)} &:= \textstyle\sum_{i=1}^{h-1} c_{ijh} \rho_j(\ell_i^{(\geq j)}), && 1 < h \leq s, 1 \leq j \leq n \\ \ell_h &:= \textstyle\sum_{j=1}^{r} \ell_h^{(j)}, && 1 < h \leq s, \end{aligned}$$

it holds

$$\Lambda = \mathrm{Span}_{\mathbf{k}}\{\ell_h, 1 \leq h \leq s\}.$$

∎

Example 13.4 *In Example 9.6 we have*

$$\ell_0 := \mathrm{Id},$$
$$\ell_h := \begin{cases} \sum_{j=1}^{h} \rho_j(\ell_{h-j}^{(j)}), & 1 \le h \le n, \\ \sum_{j=1}^{r} \rho_j(\ell_{h-j}^{(j)}), & n \le h. \end{cases}$$

■

14 Four pointers

14.1 Polynomial Evaluation at Macaulay Basis

We recall that, via *recursive Horner representation,* each polynomial $f \in \mathcal{P}$ can be uniquely represented as

$$f(X_1, \dots, X_n) = \mathfrak{H}_0(f) + \sum_{j=1}^{r} X_j \mathfrak{H}_j(f),$$

where $\mathfrak{H}_0(f) = f(\mathbf{0}) \in k$, $\mathfrak{H}_j(f) \in \mathbf{k}[X_1, \dots, X_j]$, for all j and each $\mathfrak{H}_j(f)$ has recursively a similar Horner representation.

Assume we are given, via recursive Horner representation, a polynomial $f \in \mathcal{P}$ and the Macaulay basis of a primary ideal at the origin $\{\ell_1, \dots, \ell_s\}$ through the elements $c_{ijh} \in k, 1 \le j \le n, 1 \le i < h \le s$ such that, for each h and j

$$\ell_h^{(j)} = \sum_{i=1}^{h-1} c_{ijh} \rho_j(\ell_i^{(\ge j)}). \tag{9}$$

Proposition 14.1 *[15] For each $h, j, 1 \le j \le n, 1 \le h \le s$ there are polynomials $f_{hj} \in \mathbf{k}[X_1, \dots, X_j]$ such that*
$f_{hj} = \sum_{i=1}^{h-1} \sum_{\nu=j}^{n} c_{ijh} \mathfrak{H}_j(f_{i\nu})$;
$\ell_h^{(j)}(f) = f_{hj}(\mathbf{0}) = \sum_{i=1}^{h-1} \sum_{\nu=j}^{n} c_{ijh} (\mathfrak{H}_j(f_{i\nu}))(\mathbf{0})$ *or, equivalently,*
$\ell_h^{(j)}(f) = \mathfrak{H}_0(f_{hj}) = \sum_{i=1}^{h-1} \sum_{\nu=j}^{n} c_{ijh} \mathfrak{H}_0(\mathfrak{H}_j(f_{i\nu}))$.

Corollary 14.2 *[15] With the notation and assumptions above, it is possible to compute $\ell_h^{(j)}(f)$ for each $h, j, 1 \le j \le n, 1 \le h \le s$, with complexity $\mathcal{O}(n^2 s^2)$.*

Proof. We need to compute each $\mathfrak{H}_0(f_{hj})$ where each element f_{hj} is a Horner component of the recursive Horner representation of f, each f_{hj} is a combination of Horner components of $f_{i\nu}, i < h$ and

$$f_{1j} := \mathfrak{H}_0(f) + \sum_{i=1}^{j} X_i \mathfrak{H}_i(f)$$

for each j, because $\ell_1 = \mathrm{Id}$. ■

14.2 Computing a Macaulay basis

Let $<$ be an inf-limited ordering, $\mathfrak{a} \subset \mathcal{P}$ an $\mathfrak{m}$-primary ideal, $V := \mathfrak{M}(\mathfrak{a})$, $\Lambda := \{\ell_1, \dots, \ell_s\}$ a Macaulay basis of V. Then, by Corollary 13.3, the $\mathbf{k}$-basis

$$\Gamma := \{\rho_j(\ell_i^{(\ge j)}), 1 \le j \le r, 1 \le i \le s\}$$

satisfies the following:

Theorem 14.3 *For any $\ell \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M}) \setminus V$ such that $U := \{\lambda + a\ell : \lambda \in V, a \in k\}$ is stable, $\ell \in \mathrm{Span}_{\mathbf{k}}(\Gamma)$.* ■

We are now going to discuss the structure both of V and of each stable extension

$$U := \{\lambda + a\ell : \lambda \in V, a \in k\}$$

in view of Corollary 13.3 and Theorem 14.3; for that we will sistematically study the example introduced in Ex. 9.6 in the case $r = 3$ (varying the ordering).

Example 14.4 *Letting* $f_1 := X_2 - X_1^2, f_2 := X_3 - X_1^3,\ \mathfrak{a} := (f_1, f_2),$ *let us consider the refinement* $<$ *of* v_{w} *by the reverse lexicographical ordering induced by* $X_1 \succ X_2 \succ X_3$*. Then we have*

- *the Gröbner basis of* $\mathfrak{a}$ *w.r.t.* $<$ *is* $\{X_1^2 - X_2, X_1X_2 - X_3, X_2^2 - X_1X_3\}$;
- $\{X_1^2, X_1X_2, X_2^2\} = \mathbf{T}(\mathfrak{a})$;
- $\mathbf{N}(\mathfrak{a}) = \{1\} \cup \{X_1X_3^{i-1}, X_2X_3^{i-1}, X_3^i, i \in \mathbb{N}\} = \mathbf{T}\{\mathfrak{M}(\mathfrak{a})\}$.
- *For all* $i \in \mathbb{N}$, $\mathbf{T}(\ell_{3i-2}) = X_3^{i-1}$, $\mathbf{T}(\ell_{3i-1}) = X_1X_3^{i-1}$, $\mathbf{T}(\ell_{3i}) = X_2X_3^{i-1}$;
- *for all* $\rho \in \mathbb{N}$, $(X_1^2, X_1X_2, X_2^2, X_1X_3^{\rho-1}, X_2X_3^{\rho-1}, X_3^{\rho}) = \mathbf{T}(\mathfrak{a} + \mathfrak{m}^{\rho})$;
- *for all* $\rho \in \mathbb{N}$, $\{X_1^2 - X_2, X_1X_2 - X_3, X_2^2 - X_1X_3, X_1X_3^{\rho-1}, X_2X_3^{\rho-1}, X_3^{\rho}\}$

is the Gröbner basis of $\mathfrak{a} + \mathfrak{m}^{\rho}$ *w.r.t.* $<$*;*

- $\mathbf{N}(\mathfrak{a}) = \{1\} \cup \{X_1X_3^{i-1}, X_2X_3^{i-1}, X_3^i, i < \rho\} = \mathbf{T}\{\mathfrak{M}(\mathfrak{a})\}$.

In particular

$\ell_1 := M(1)$,
$\ell_2 := M(X_1)$,
$\ell_3 := M(X_2) + M(X_1^2)$,
$\ell_4 := M(X_3) + M(X_1X_2) + M(X_1^3)$,
$\ell_5 := M(X_1X_3) + M(X_2^2) + M(X_1^2X_2) + M(X_1^4)$,
$\ell_6 := M(X_2X_3) + M(X_1^2X_3) + M(X_1X_2^2) + M(X_1^3X_2) + M(X_1^5)$,
$\ell_7 := M(X_3^2) + M(X_1X_2X_3) + M(X_1^3X_3) + M(X_2^3) + M(X_1^2X_2^2)+$
$\quad +M(X_1^4X_2) + M(X_1^6)$;

as a consequence we have

$\rho_1(\ell_1) := M(X_1)$,
$\rho_2(\ell_1) := M(X_2)$,
$\rho_3(\ell_1) := M(X_3)$,
$\rho_1(\ell_2) := M(X_1^2)$,
$\rho_1(\ell_3) := M(X_1X_2) + M(X_1^3)$,
$\rho_2({\ell_3}^{(2)}) := M(X_2^2)$,
$\rho_1(\ell_4) := M(X_1X_3) + M(X_1^2X_2) + M(X_1^4)$,
$\rho_2({\ell_4}^{(\geq 2)}) := M(X_2X_3)$,
$\rho_3({\ell_4}^{(3)}) := M(X_3^2)$,
$\rho_1(\ell_5) := M(X_1^2X_3) + M(X_1X_2^2) + M(X_1^3X_2) + M(X_1^5)$,
$\rho_2({\ell_5}^{(2)}) := M(X_2^3)$,
$\rho_1(\ell_6) := M(X_1X_2X_3) + M(X_1^3X_3) + M(X_1^2X_2^2) + M(X_1^4X_2) + M(X_1^6)$,
$\rho_2({\ell_6}^{(2)}) := M(X_2^2X_3)$,
$\rho_1(\ell_7) := M(X_1X_3^2) + M(X_1^2X_2X_3) + M(X_1^4X_3) + M(X_1X_2^3) + M(X_1^3X_2^2)+$
$\quad +M(X_1^5X_2) + M(X_1^7)$,
$\rho_2({\ell_7}^{(\geq 2)}) := M(X_2X_3^2) + M(X_2^4)$,
$\rho_3({\ell_7}^{(3)}) := M(X_3^3)$.

This information (and others which will be deduced during the following discussion) can be submarized in the following tables:

	λ	$\lambda^{(1)}$		$\lambda^{(2)}$		$\lambda^{(3)}$	$\sigma_1(\lambda)$	$\sigma_2(\lambda)$	$\sigma_3(\lambda)$
1	ℓ_1					$M(1)$	0	0	0
X_1	ℓ_2	$\rho_1(\ell_1)$					ℓ_1	0	0
X_1^2		$\rho_1(\ell_2)$					ℓ_2	0	0
X_2				$\rho_2(\ell_1)$			0	ℓ_1	0
	ℓ_3	$\rho_1(\ell_2)$	+	$\rho_2(\ell_1)$			ℓ_2	ℓ_1	0
X_1X_2		$\rho_1(\ell_3)$					ℓ_3	ℓ_2	0
X_2^2				$\rho_2(\ell_3^{(2)})$			0	$\ell_3^{(2)}$	0
X_3						$\rho_3(\ell_1)$	0	0	ℓ_1
	ℓ_4	$\rho_1(\ell_3)$			+	$\rho_3(\ell_1)$	ℓ_3	ℓ_2	ℓ_1
X_1X_3		$\rho_1(\ell_4)$					ℓ_4	$\ell_3^{(1)}$	ℓ_2
	ℓ_5	$\rho_1(\ell_4)$	+	$\rho_2(\ell_3^{(2)})$			ℓ_4	ℓ_3	ℓ_2
X_2X_3				$\rho_2(\ell_4^{(3)})$			0	$\ell_4^{(3)}$	$\ell_3^{(2)}$
	ℓ_6	$\rho_1(\ell_5)$	+	$\rho_2(\ell_4^{(3)})$			ℓ_5	ℓ_4	ℓ_3
X_3^2						$\rho_3(\ell_4^{(3)})$	0	0	$\ell_4^{(3)}$
	ℓ_7	$\rho_1(\ell_6)$	+	$\rho_2(\ell_5^{(2)})$	+	$\rho_3(\ell_4^{(3)})$	ℓ_6	ℓ_5	ℓ_4
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$X_1X_3^i$		$\rho_1(\ell_{3i+1})$					ℓ_{3i+1}	$\ell_{3i}^{(1)}$	$\ell_{3i-1}^{(1)}$
	ℓ_{3i+2}	$\rho_1(\ell_{3i+1})$	+	$\rho_2(\ell_{3i}^{(\geq 2)})$			ℓ_{3i+1}	ℓ_{3i}	ℓ_{3i-1}
$X_2X_3^i$				$\rho_2(\ell_{3i+1}^{(\geq 2)})$			0	$\ell_{3i+1}^{(\geq 2)}$	$\ell_{3i}^{(\geq 2)}$
	ℓ_{3i+3}	$\rho_1(\ell_{3i+2})$	+	$\rho_2(\ell_{3i+1}^{(\geq 2)})$			ℓ_{3i+2}	ℓ_{3i+1}	ℓ_{3i}
X_3^{i+1}						$\rho_3(\ell_{3i+1}^{(3)})$	0	0	$\ell_{3i+1}^{(3)}$
	ℓ_{3i+4}	$\rho_1(\ell_{3i+3})$	+	$\rho_2(\ell_{3i+2}^{(\geq 2)})$	+	$\rho_3(\ell_{3i+1}^{(3)})$	ℓ_{3i+3}	ℓ_{3i+2}	ℓ_{3i+1}

λ			$\sigma_1(\lambda)$	$\sigma_2(\lambda)$	$\sigma_3(\lambda)$	$\lambda(f_1)(\mathbf{0})$	$\lambda(f_2)(\mathbf{0})$	$\mathbf{T}(\lambda)$
ℓ_1	=	$M(1)$	0	0	0	0	0	1
ℓ_2	=	$\rho_1(\ell_1)$	ℓ_1	0	0	0	0	X_1
$\ell_3^{(2)}$	=	$\rho_2(\ell_1)$	0	ℓ_1	0	1	0	X_2
$\ell_4^{(3)}$	=	$\rho_3(\ell_1)$	0	0	ℓ_1	0	1	X_3
$\ell_3^{(1)}$	=	$\rho_1(\ell_2)$	ℓ_2	0	0	-1	0	X_1^2
$\ell_4^{(1)}$	=	$\rho_1(\ell_3)$	ℓ_3	ℓ_2	0	0	-1	X_1X_2
$\ell_5^{(2)}$	=	$\rho_2(\ell_3{}^{(\geq 2)})$	0	$\ell_3{}^{(\geq 2)}$	0	0	0	X_2^2
$\ell_5^{(1)}$	=	$\rho_1(\ell_4)$	ℓ_4	$\ell_3{}^{(1)}$	ℓ_2	0	0	X_1X_3
$\ell_6^{(2)}$	=	$\rho_2(\ell_4{}^{(\geq 2)})$	0	$\ell_4{}^{(\geq 3)}$	$\ell_3{}^{(\geq 2)}$	0	0	X_2X_3
$\ell_7^{(3)}$	=	$\rho_3(\ell_4{}^{(\geq 3)})$	0	0	$\ell_4{}^{(\geq 3)}$	0	0	X_3^2
$\ell_6^{(1)}$	=	$\rho_1(\ell_5)$	ℓ_5	$\ell_4{}^{(1)}$	$\ell_3^{(1)}$	0	0	$X_1^2X_3$
$\ell_7^{(2)}$	=	$\rho_2(\ell_5{}^{(\geq 2)})$	0	$\ell_5{}^{(\geq 2)}$	0	0	0	X_2^3
$\ell_7^{(1)}$	=	$\rho_1(\ell_6)$	ℓ_6	$\ell_5{}^{(1)}$	$\ell_4^{(1)}$	0	0	$X_1^3X_3$
$\ell_8^{(2)}$	=	$\rho_2(\ell_6{}^{(\geq 2)})$	0	$\ell_6{}^{(\geq 2)}$	$\ell_5{}^{(\geq 2)}$	0	0	$X_2^2X_3$
$\ell_8^{(1)}$	=	$\rho_1(\ell_7)$	ℓ_7	$\ell_6{}^{(1)}$	$\ell_5^{(1)}$	0	0	$X_1X_3^2$
$\ell_9^{(2)}$	=	$\rho_2(\ell_7{}^{(\geq 2)})$	0	$\ell_7{}^{(\geq 2)}$	$\ell_6{}^{(\geq 2)}$	0	0	$X_2X_3^2$
$\ell_{10}^{(3)}$	=	$\rho_3(\ell_7{}^{(\geq 3)})$	0	0	$\ell_7{}^{(\geq 3)}$	0	0	X_3^3
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\ell_{3i+3}^{(1)}$	=	$\rho_1(\ell_{3i+2})$	ℓ_{3i+2}	$\ell_{3i+1}^{(1)}$	$\ell_{3i}^{(1)}$	0	0	$X_1X_3^i$
$\ell_{3i+4}^{(2)}$	=	$\rho_2(\ell_{3i+2}^{(\geq 2)})$	0	$\ell_{3i+2}^{(\geq 2)}$	$\ell_{3i+1}^{(\geq 2)}$	0	0	$X_2X_3^i$
$\ell_{3i+4}^{(1)}$	=	$\rho_1(\ell_{3i+3})$	ℓ_{3i+3}	$\ell_{3i+2}^{(1)}$	$\ell_{3i+1}^{(1)}$	0	0	$X_1X_2X_3^i$
$\ell_{3i+5}^{(2)}$	=	$\rho_2(\ell_{3i+3}^{(\geq 2)})$	0	$\ell_{3i+3}^{(\geq 2)}$	$\ell_{3i+2}^{(\geq 2)}$	0	0	$X_2^2X_3^i$
$\ell_{3i+5}^{(1)}$	=	$\rho_1(\ell_{3i+4})$	ℓ_{3i+4}	$\ell_{3i+3}^{(1)}$	$\ell_{3i+2}^{(1)}$	0	0	$X_1X_3^{i+1}$
$\ell_{3i+6}^{(2)}$	=	$\rho_2(\ell_{3i+4}^{(\geq 2)})$	0	$\ell_{3i+4}^{(\geq 2)}$	$\ell_{3i+3}^{(\geq 2)}$	0	0	$X_2X_3^{i+1}$
$\ell_{3i+7}^{(3)}$	=	$\rho_3(\ell_{3i+4}^{(\geq 3)})$	0	0	$\ell_{3i+4}^{(\geq 3)}$	0	0	X_3^{i+2}

$\blacksquare$

Definition 14.5 *The* corner set *of V is the set*

$$\begin{aligned}\mathbf{C}(V) &:= \{\tau \in \mathcal{T} : M(\tau) \in \mathbf{N}(V), \sigma_i(M(\tau)) \in \mathbf{T}\{V\}\ \forall\ i\} \\ &= \{\tau \in \mathbf{T}(\mathfrak{P}(V)) : \text{ all its predecessors } \omega \in \mathbf{N}(\mathfrak{P}(V))\} \\ &= \mathbf{G}(\mathfrak{P}(V))\end{aligned}$$

Any element

$$\ell := M(\mathbf{T}(\ell)) + \sum_{\omega \in \mathcal{T}} c_\omega M(\omega) \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M}) \setminus V \quad \textit{such that}$$

*c*1) $\mathbf{T}(\ell) \in \mathbf{C}(V)$,

*c*2) $\sigma_j(\ell) \in V$ *for each* j,

c3) $c_\omega \neq 0 \Longrightarrow \omega \notin \mathbf{T}\{V\}$;
is called a continuation *of* V *at* $\tau := \mathbf{T}(\ell)$.
An elementary *continuation of* V *at* $\tau \in \mathbf{C}(V)$ *is a continuation*

$$\ell := M(\mathbf{T}(\ell)) + \sum_{\omega \in \mathcal{T}} c_\omega M(\omega)$$

which, moreover, satisfies

c4) *if* $M(\omega) \in \mathbf{C}(V), c_\omega \neq 0$, *then there is no continuation of* V *at* ω. ∎

Lemma 14.6 *[15] The following conditions are equivalent:*
1. U *is stable and* $\Lambda \cup \{\ell\}$ *is its Macaulay basis,*
2. $\tau := \mathbf{T}(\ell) \in \mathbf{C}(V)$ *and* ℓ *is a continuation of* V *at* τ.

Example 14.7 *For instance, for* $\rho = 3$, *we have*

$$V := \mathfrak{M}(\mathfrak{a}_3) = \mathrm{Span}_{\mathbf{k}}(\ell_1, \ell_2, \ell_3), \text{ and } \quad \mathbf{C}(V) := \{X_1^2, X_1X_2, X_2^2, X_3\}.$$

The elementary continuation of V
at X_1^2 *is* $\lambda := a\rho_1(\ell_2), a \in k \setminus \{0\}$,
at X_1X_2 *is* $\lambda := a\rho_1(\ell_3), a \in k \setminus \{0\}$,
at X_3 *is* $\lambda := a\rho_3(\ell_1), a \in k \setminus \{0\}$,
and the continuation of V
at X_1^2 *is* $\lambda := a\rho_1(\ell_2), a \in k \setminus \{0\}$,
at X_1X_2 *is* $\lambda := a\rho_1(\ell_3) + b\rho_1(\ell_2), a, b \in k, a \neq 0$,
at X_3 *is* $\lambda := a\rho_3(\ell_1) + b\rho_1(\ell_3) + c\rho_1(\ell_2), a, b, c \in k, a \neq 0$.
On the other side there is no continuation at X_2^2 *as for each* $\lambda : \mathbf{T}(\lambda) = X_2^2$ *we necessarily have*

$$\sigma_2(\lambda) = a{\ell_3}^{(2)} + b\ell_2 + c\ell_1, a, b, c \in k, a \neq 0$$

so that $\sigma_2(\lambda) \notin V$.

∎

The relation between continuations and elementary ones is clarified (see [15]) by the following list of results

Lemma 14.8 *Let* ℓ' *and* ℓ'' *be two different continuations of* V *at* τ. *Then* $\ell' - \ell''$ *is a continuation of* V *at some* $\omega > \tau$, $\omega \in \mathbf{C}(V)$. ∎

Corollary 14.9 *If continuations of* V *at* t *exist, then there is exactly a single elementary continuation* ℓ *of* V *at* t, *which we will denote* $C_{V,t}$. ∎

Theorem 14.10 *The following conditions are equivalent:*
1. $U := \{\lambda + a\ell : \lambda \in V, a \in k\}$ *is stable and* $\Delta \cup \{\ell\}$ *is its Macaulay basis.*
2. $\exists\, t_0 < \cdots < t_v$, $M(t_i) \in \mathbf{C}(V)$ *and* $c_i \in k \setminus \{0\}, 1 \leq i \leq v$, *such that*

$$\ell = C_{V,t_0} + \sum_{i=1}^{v} c_i C_{V,t_i}.$$

∎

Fig. 1 Macaulay basis from any basis

$(\Lambda, \mathcal{M}) :=$ **MacaulayBasis**$(F, <)$
where
$F := \{f_1, \dots, f_t\} \subset \mathcal{P}$,
$\mathfrak{a} := (F)$ an $\mathfrak{m}$-primary ideal,
$<$ an inf-limited ordering,
$\Lambda := \{\ell_1, \dots, \ell_s\}$ the Macaulay basis of $\mathfrak{M}(\mathfrak{a})$
$\mathcal{M} = \{\left(b_{ij}^{(h)}\right) \in k^{s^2}, 1 \le h \le n\}$ the set of the square matrices $\left(b_{ij}^{(h)}\right)$ defined by $\sigma_h(\ell_i) = \sum_{j=1}^{s} b_{ij}^{(h)} \ell_j$.
$i := 1$, $\ell_1 :=$ Id, $\Lambda := \{\text{Id}\}$ $V := \text{Span}_{\mathbf{k}}(\Lambda)$, $\mathcal{C} := \mathbf{C} := \emptyset$,
$\mathbf{B} := \mathbf{G} := \{X_j, 1 \le j \le r\}$,
For $1 \le j, h \le n$ **compute** $\sigma_h(M(X_j))$.
Repeat
$t := \max_<(\mathbf{G} \setminus \mathbf{C}), \mathbf{B} := \mathbf{B} \setminus \{t\}$
Compute (if it exists) $C_{U,t}$
If $C_{U,t}$ exists **then**
If exist c_τ such that $\text{ev}(C_{U,t}) = \sum_{\tau \in \mathcal{C}} c_\tau \, \text{ev}(C_{U,\tau})$ **then**
$i := i + 1, \ell_i := C_{U,t} - \sum_{\tau \in \mathbf{C}} c_\tau C_{U,\tau}$
For $1 \le h \le n$, $1 \le j < i$ **do**
Compute $b_{ij}^{(h)} : \sigma_h(\ell_i) = \sum_{j=1}^{s} b_{ij}^{(h)} \ell_j$;
$\mathbf{B} := \mathbf{B} \cup \{\mathbf{T}(\rho_j(\ell_i^{(\ge j)})), 1 \le j \le r\}$
$\mathbf{G}$ be the minimal basis of the monomial ideal generated by $\mathbf{B} \cup \mathbf{C}$
For $1 \le j, h \le n$ **compute** $\sigma_h \rho_j(\ell_i^{(\ge j)})$
else
$\mathcal{C} := \mathcal{C} \cup \{C_{U,t}\}$ $\mathbf{C} := \mathbf{C} \cup \{t\}$
until $\mathbf{G} \setminus \mathbf{C} := \emptyset$

Lemma 14.11 *Let $M(t) \in \mathbf{C}(V) \cap \mathbb{M}[\kappa, n]$ and let ${\ell_\iota}^{(\kappa)}$ be such that*

$$\rho_\kappa(\mathbf{T}(\ell_\iota^{(\kappa)}) = M(t).$$

For $\kappa \le j \le n$ let $J(j)$ denote the set of indices i such that
a) $\mathbf{T}(\rho_j(\ell_i^{(j)})) \notin \mathbf{T}\{V\}$,
b) $\mathbf{T}(\rho_j(\ell_i^{(j)})) > M(t)$,
c) if $\mathbf{T}(\rho_j(\ell_i^{(j)})) \in \mathbf{C}(V)$ *then there is no elementary continuation of V at* $\mathbf{T}(\rho_j(\ell_i^{(j)}))$.
The following conditions are equivalent:
1. *the elementary continuation $C_{V,t}$ exists;*
2. *there are values $a_{ji} \in k$, such that, for each μ*

$$\sigma_\mu \rho_\kappa(\ell_\iota^{(\kappa)}) + \sum_{j=1}^{n} \sum_{i \in J(j)} a_{ji} \sigma_\mu \rho_j(\ell_i^{(j)}) \in V.$$

Finally, if the above conditions are satisfied,

$$C_{V,t} = \rho_\kappa(\ell_\iota^{(\kappa)}) + \sum_{j=1}^{n} \sum_{i \in J(j)} a_{ji} \rho_j(\ell_i^{(j)}).$$

■

We can now present, in Figure 1, the algorithm proposed in [15] which uses the structure of the continuations of $\mathfrak{m}$-primary ideals in order to compute the Macaulay basis (w.r.t. an inf-limited ordering $<$) of any $\mathfrak{m}$-primary ideal given by means of a set of generators $F := \{f_1, \dots, f_t\} \subset \mathfrak{m}$; as it essentially consists of linear algebra reduction of sn vectors in $\mathbf{k}^{sn+t}$, its complexity is $\mathcal{O}(s^3 n^3)$.

The auxiliary tools needed by the algorithm are the following:

– The structure described in Corollary 13.3 and Theorem 14.3 implies that one can easily iteratively compute, for each $1 \le j, h \le n, 1 \le i \le s$, $\sigma_h(\ell_i)$ and $\sigma_h(\rho_j(\ell_i^{(\ge j)})) = \rho_j \sigma_h(\ell_i^{(\ge j)})$, since

$$\ell_i = \sum_{j=1}^{n} \sum_{\iota=1}^{i-1} c_{\iota j i} \rho_j(\ell_\iota^{(\ge j)}) \Longrightarrow \sigma_h(\ell_i) = \sum_{j=1}^{n} \sum_{\iota=1}^{i-1} c_{\iota j i} \sigma_h \rho_j(\ell_\iota^{(\ge j)}),$$

and

$$\sigma_h(\ell_i) = \sum_{\iota=1}^{i-1} c_\iota \ell_\iota \Longrightarrow \sigma_h(\rho_j(\ell_i^{(\ge j)})) = \begin{cases} 0 & \text{if } h < j \\ \ell_i^{(\ge j)}) & \text{if } h = j \\ \sum_{\iota=1}^{i-1} c_\iota \rho_j(\ell_\iota^{(\ge j)}) & \text{if } h > j. \end{cases}$$

For instance, in the Ex. 14.7 we have

$$\begin{aligned} \sigma_1(\ell_7) &= \sigma_1\rho_1(\ell_6) + \sigma_1\rho_2(\ell_5^{(2)}) + \sigma_1\rho_3(\ell_4^{(3)}) = \ell_6 + 0 + 0 = \ell_6, \\ \sigma_2(\ell_7) &= \sigma_2\rho_1(\ell_6) + \sigma_2\rho_2(\ell_5^{(2)}) + \sigma_2\rho_3(\ell_4^{(3)}) = \ell_5^{(1)} + \ell_5^{(2)} + 0 = \ell_5, \\ \sigma_3(\ell_7) &= \sigma_3\rho_1(\ell_6) + \sigma_3\rho_2(\ell_5^{(2)}) + \sigma_3\rho_3(\ell_4^{(3)}) = \ell_4^{(1)} + 0 + \ell_4^{(3)} = \ell_4, \\ \sigma_2\rho_1(\ell_7) &= \rho_1(\ell_5) = \ell_6^{(1)}, \\ \sigma_3\rho_1(\ell_7) &= \rho_1(\ell_4) = \ell_5^{(1)}, \\ \sigma_3\rho_2(\ell_7^{(\ge 2)}) &= \rho_2(\ell_4^{(\ge 2)}) = \rho_2(\ell_4^{(3)}) = \ell_6^{(2)}. \end{aligned}$$

In fact the complete table is obtained by means of this recursive evaluation.

– Since we can compute the values of $\sigma_h(\rho_j(\ell_i^{(\ge j)}))$, for each $1 \le j, h \le n, 1 \le i \le s$, to determine all the continuations of V at each element t in the corner set of V requires nothing more than efficient book-keeping.
For instance in the cases we discussed in Ex. 14.7 we have
at $X_1^2 \Longrightarrow \lambda := \rho_1(\ell_2)$ is a contination since $\sigma_h(\lambda) \in V$, for each h;
at $X_1X_2 \Longrightarrow \lambda := \rho_1(\ell_3)$ is a contination for the same reason;
at $X_2^2 \Longrightarrow \lambda := \rho_2({\ell_3}^{(2)})$ is not a continuation since, for each $a, b \in k$

$$\sigma_2(\lambda) = \sigma_2(\lambda + a\rho_1(\ell_2) + b\rho_1(\ell_3)) = {\ell_3}^{(2)} \notin V;$$

at $X_3 \Longrightarrow \lambda := \rho_3(\ell_1)$ is a contination since $\sigma_h(\lambda) \in V$ for each h;

– Finally, if for each $\ell \in \mathrm{Span}_{\mathbf{k}}(\mathbb{M})$ we denote

$$\mathrm{ev}(\ell) := (\ell(f_1), \cdots, \ell(f_t)) \in k^t,$$

and if $\{\ell_1, \ell_2, \ldots, \ell_s\}$ is the ordered Macaulay basis of $\mathfrak{a}$ (wrt $<$), which we aim to compute, setting for any $i < s$
• $V_i := \{\ell_1, \ell_2, \ldots, \ell_i\}$
• $C_i := \{\tau \in \mathbf{C}(V_i) :$ there is an elementary continuation of V_i at $\tau\}$,
we know that, for all i, $\exists\ c_\tau \in k$ such that $\ell_{i+1} = \sum_{\tau \in C_i} c_\tau C_{V_i,\tau}$.
Moreover, as

$$\ell_{i+1} \in \mathfrak{M}(\mathfrak{a}) \iff \mathrm{ev}(\ell_{i+1}) = \sum_{\tau \in C_i} c_\tau\, \mathrm{ev}(C_{V_i,\tau}) = 0,$$

ℓ_{i+1} can be obtained by solving this linear equation, as each $\mathrm{ev}(C_{V_i,\tau})$ can be computed by the scheme described in Section 14.1.

We end remarking that, given any finite set of polynomials $F := \{f_1, \ldots, f_t\}$ and the ideal $\mathfrak{a} \subseteq \mathcal{P}$ generated by F, in order to obtain the Macaulay basis (w.r.t. $<$) of the $\mathfrak{m}$-primary ideal $\mathfrak{a} + \mathfrak{m}^\rho$ for any $\rho \in \mathbb{N}$, we have to enlarge F by adding all monomials of degree ρ and to apply the algorithm presented in Figure 1, thus producing, "at least in imagination" (as Macaulay put it), the infinite Macaulay basis of the $\mathfrak{m}$-closed ideal $\cap_\rho \mathfrak{a} + \mathfrak{m}^\rho$.

Example 14.12 *We may verify the structure of Example 14.4 mainly checking its presentation in the included table and deducing that the algorithm performs the following computations:*
$t := X_1X_3^i$

$$\begin{aligned}
\ell^{(1)}_{3i+2} &= \rho_1(\ell_{3i+1}),\\
\sigma_2(\ell^{(1)}_{3i+2}) &= \sigma_2\rho_1(\ell^{(1)}_{3i+1}) = \ell^{(1)}_{3i},\\
\ell^{(2)}_{3i+2} &= \rho_2(\ell^{(\geq 2)}_{3i}),\\
\sigma_2(\ell_{3i+2}) &= \ell_{3i},\\
\sigma_3(\ell^{(1)}_{3i+2} + \ell^{(2)}_{3i+2}) &= \sigma_3\rho_1(\ell^{(1)}_{3i+1}) + \sigma_3\rho_2(\ell^{(\geq 2)}_{3i}) = \ell^{(1)}_{3i} + \ell^{(\geq 2)}_{3i} = \ell_{3i},\\
\ell_{3i+2} &= \rho_1(\ell^{(\geq 1)}_{3i+1}) + \rho_2(\ell^{(\geq 2)}_{3i});
\end{aligned}$$

$t := X_2X_3^i$

$$\begin{aligned}
\ell^{(1)}_{3i+3} &= \rho_1(\ell_{3i+2}),\\
\sigma_2(\ell^{(1)}_{3i+3}) &= \sigma_2\rho_1(\ell^{(1)}_{3i+2}) = \ell^{(1)}_{3i+1},\\
\ell^{(2)}_{3i+3} &= \rho_2(\ell^{(\geq 2)}_{3i+1}),\\
\sigma_2(\ell_{3i+3}) &= \ell_{3i+1},\\
\sigma_3(\ell^{(1)}_{3i+3} + \ell^{(2)}_{3i+3}) &= \sigma_3\rho_1(\ell^{(1)}_{3i+2}) + \sigma_3\rho_2(\ell^{(\geq 2)}_{3i+1}) = \ell^{(1)}_{3i} + \ell^{(\geq 2)}_{3i} = \ell_{3i},\\
\ell_{3i+3} &= \rho_1(\ell^{(\geq 1)}_{3i+2}) + \rho_2(\ell^{(\geq 2)}_{3i+1});
\end{aligned}$$

$t := X_3^{i+1}$

$$\begin{aligned}
\ell^{(1)}_{3i+4} &= \rho_1(\ell_{3i+3}),\\
\sigma_2(\ell^{(1)}_{3i+4}) &= \sigma_2\rho_1(\ell^{(1)}_{3i+3}) = \ell^{(1)}_{3i+2},\\
\ell^{(2)}_{3i+4} &= \rho_2(\ell^{(\geq 2)}_{3i+2}),\\
\sigma_2(\ell_{3i+4}) &= \ell_{3i+2},\\
\sigma_3(\ell^{(1)}_{3i+4} + \ell^{(2)}_{3i+4}) &= \sigma_3\rho_1(\ell^{(1)}_{3i+3}) + \sigma_3\rho_2(\ell^{(\geq 2)}_{3i+2}) = \ell^{(1)}_{3i+1} + \ell^{(2)}_{3i+1},\\
\ell^{(3)}_{3i+4} &= \rho_3(\ell^{(3)}_{3i+1}),\\
\sigma_3(\ell_{3i+4}) &= \ell_{3i+1},\\
\ell_{3i+4} &= \rho_1(\ell^{(\geq 1)}_{3i+2}) + \rho_2(\ell^{(\geq 2)}_{3i+1}) + \rho_3(\ell^{(3)}_{3i+1}).
\end{aligned}$$

■

14.3 Cerlienco–Mureddu Correspondence

Each zero-dimensional ideal $\mathsf{I} \subset \mathcal{P}$ can be considered as *given* if we know the set $\mathcal{Z}(\mathsf{I})$ of its roots and, for each $\mathsf{a} \in \mathcal{Z}(\mathsf{I})$, the Macaulay basis of the corresponding primary component of I.

For each $\mathsf{a} \in \mathcal{Z}(\mathsf{I})$, $\mathsf{a} := (a_1, \ldots, a_n)$, let us therefore denote:

- $\lambda_\mathsf{a} : \mathcal{P} \mapsto \mathcal{P}$ the translation $\lambda_\mathsf{a}(X_i) = X_i + a_i$, for all i,
- $\mathfrak{m}_\mathsf{a} = (X_1 - a_1, \ldots, X_n - a_n)$,
- $\mathfrak{q}_\mathsf{a}$ the $\mathfrak{m}_\mathsf{a}$-primary component of I,
- $\Lambda_\mathsf{a} := \mathfrak{M}(\lambda_\mathsf{a}(\mathfrak{q}_\mathsf{a})) \subset \mathrm{Span}_\mathsf{k}(\mathbb{M})$,
- $\ell_{\upsilon\mathsf{a}}$, for each $\upsilon \in \mathbf{N}_<(\lambda_\mathsf{a}(\mathfrak{q}_\mathsf{a}))$, the Macaulay equation $\ell_{\upsilon\mathsf{a}} := \ell(\upsilon)$ so that
- $\{\ell_{\upsilon\mathsf{a}} : \upsilon \in \mathbf{N}_<(\lambda_\mathsf{a}(\mathfrak{q}_\mathsf{a}))\}$ is the Macaulay basis of Λ_a.

Setting $s := \sum_{\mathsf{a}\in\mathcal{Z}(\mathsf{I})} \deg(\mathsf{q}_\mathsf{a})$ and

$$\mathbb{L} := \{\lambda_1,\ldots,\lambda_s\} := \{\ell_{\upsilon\mathsf{a}}\lambda_\mathsf{a} : \upsilon \in \mathbf{N}_<(\lambda_\mathsf{a}(\mathsf{q}_\mathsf{a})), \mathsf{a} \in \mathcal{Z}(\mathsf{I})\},$$

we know that $\mathrm{Span}_\mathbf{k}(\mathbb{L}) = \mathfrak{L}(\mathsf{I})$ and $\mathsf{I} = \mathfrak{P}(\mathrm{Span}_\mathbf{k}(\mathbb{L}))$; moreover (Corollary 12.2) we can assume $\mathbb{L}$ to be ordered so that, for each σ,

$$\mathsf{I}_\sigma = \mathfrak{P}(\mathrm{Span}_\mathbf{k}\{\{\lambda_1,\ldots,\lambda_\sigma\})$$

is an ideal.

We also set

$$\mathsf{X} := \{\mathsf{x}_1,\ldots,\mathsf{x}_s\} := \{(\mathsf{a},\upsilon) : \upsilon \in \mathbf{N}_<(\mathsf{q}_\mathsf{a}), \mathsf{a} \in \mathsf{Z}\}$$

enumerated so that $\mathsf{x}_j = (\mathsf{a}_j,\upsilon_j) \iff \lambda_j = \ell_{\upsilon_j\mathsf{a}_j}\lambda_{\mathsf{a}_j}$ and we set, for each $j, 1 \le j \le s$, $M(\lambda_j) := M(\upsilon_j)\lambda_{\mathsf{a}_j}$ where $\lambda_j = \ell_{\upsilon_j\mathsf{a}_j}\lambda_{\mathsf{a}_j}$.

Under the following equivalent assumptions:

- $\lambda = M(\lambda)$ for each $\lambda \in \mathbb{L}$,
- $\ell_{\upsilon\mathsf{a}} = M(\upsilon)$, for each $\lambda = \ell_{\upsilon\mathsf{a}}\lambda_\mathsf{a} \in \mathbb{L}$,
- each $\lambda_\mathsf{a}(\mathsf{q}_\mathsf{a})$ is a monomial ideal,

Cerlienco–Mureddu[8] Algorithm [5,6] associated to each such sets $\mathbb{L}$ and $\mathsf{X}(\Lambda)$, an order ideal $\mathbf{N} := \mathbf{N}(\mathbb{L})$ and a bijection $\Phi := \Phi(\mathbb{L}) : \mathbb{L} \mapsto \mathbf{N}$, which satisfies

$$\mathbf{N}_<(\mathbb{L}) = \mathbf{N}(\mathfrak{P}(\mathrm{Span}_\mathbf{k}(\mathbb{L})))$$

for the lexicographical ordering induced by $X_1 < \cdots < X_n$.

Cerlienco–Mureddu result has been generalized to each zero-dimensional ideal in [18] where it is proved:

For each $m < n$, denote

- π_m the projection
$$\pi_m : k^n \mapsto k^m, \pi_m(x_1,\ldots,x_n) = (x_1,\ldots,x_m);$$
- for each Noetherian equation
$$\ell(\tau) := M(\tau) + \sum_{t\in\mathbf{T}(\mathsf{I})} \gamma(t,\tau,<)M(t) = \tau^{-1} + \sum_{t\in\mathbf{T}(\mathsf{I})} \gamma(t,\tau,<)t^{-1},$$
$\tau = X_1^{d_1}\cdots X_n^{d_n}$, denote
$$\pi_m(\ell(\tau)) := (\sigma_{X_m^{d_m}\cdots X_n^{d_n}}(\ell(\tau)))(X_1^{-1},\ldots,X_m^{-1},0,\ldots,0) \in \mathbf{k}[X_1^{-1},\ldots,X_m^{-1}];$$
- finally for each $\lambda = \ell_{\upsilon\mathsf{a}}\lambda_\mathsf{a}$ set
$$\pi_m(\lambda) := \pi_m(\ell_{\upsilon\mathsf{a}}\lambda_\mathsf{a}) := \pi_m(\ell_{\upsilon\mathsf{a}})\lambda_{\pi_m(\mathsf{a})}.$$

Let

$$\mathbb{L} := \{\lambda_1,\ldots,\lambda_s\}, \quad \mathsf{X} := \{\mathsf{x}_1,\ldots,\mathsf{x}_s\} \subset k^n \times \mathcal{T},$$
$$\mathsf{x}_i = (\mathsf{a}_i,\upsilon_i), \mathsf{a}_i := (a_{i1},\ldots,a_{in}), \upsilon_i = \prod_{l=1}^n X_l^{\alpha_{il}}$$

be the Macaulay representation of a zero-dimensional ideal $\mathsf{I} \subset \mathcal{P}$.

By induction on $s = \#(\mathsf{X})$, consider $\mathbb{L}' := \{\lambda_1,\ldots,\lambda_{s-1}\}$ and the corresponding order ideal $\mathbf{N}' := \mathbf{N}(\mathbb{L}')$ and bijection $\Phi' := \Phi(\mathbb{L}')$.

Denote, for each $\nu, 1 \le \nu < n$, and each $\delta \in \mathbb{N}$,

$$\mathbb{Y}_{\nu\delta} := \mathrm{Span}_\mathbf{k}\{\pi_\nu(\lambda) : \lambda \in \mathbb{L}', \exists\omega \in \mathcal{T} \cap \mathbf{k}[X_1,\ldots,X_\nu] : \Phi'(\lambda) = \omega X_{\nu+1}^\delta\}.$$

With this notation, let us set

[8] Note that in [7] another nice algorithm is given for solving the problem, apparently adding a new point to the given set of points one has to repeat this algorithm while Cerlienco-Mureddu's one works automatically

$m := \max(j : \pi_j(\lambda_s) \in \mathrm{Span}_{\mathbf{k}}(\pi_j(\mathbb{L}'))$,
$d := \min\{\delta : \pi_m(\lambda_s) \notin \mathbb{Y}_{m\delta}\}$,
$\mathbb{W} := \{\pi_m(\lambda) : \Phi'(\lambda) = \omega X_{m+1}^d, \omega \in\in \mathcal{T} \cap \mathbf{k}[X_1, \dots, X_m]\} \cup \{\pi_m(\lambda_s)\}$
$\omega := \Phi(\mathbb{W})(\pi_m(\lambda_s))$,
$t_s := \omega X_{m+1}^d$

where $\mathbf{N}(\mathbb{W})$ and $\Phi(\mathbb{W})$ are the result of the application of the present algorithm to $\mathbb{W}$, which can be inductively applied since $\#(\mathbb{W}) \leq s-1$. We then define

$$\mathbf{N} := \mathbf{N}' \cup \{t_s\} \text{ and } \Phi(\lambda_i) := \begin{cases} \Phi'(\lambda_i) & i < s \\ t_s & i = s. \end{cases}$$

Proposition 14.13 *It holds* $\mathbf{N} := \mathbf{N}_<(\mathsf{I})$. ∎

14.4 The Axis-of-Evil Theorem

A series of three papers [16–18] merged Lazard Theorem [11], Möller's Algorithm [4,1] Gianni–Kalkbrener Theorem [8,10] and Cerlienco–Mureddu Correspondence giving a strong description of the structure of the Gröbner basis and of the dual basis of a zero-dimensional ideal.

We quote here just

Theorem 14.14 *Let*

$\mathsf{X} := \{\mathsf{x}_1, \dots, \mathsf{x}_s\} \subset k^n$ *be a finite set of points*
$\mathsf{I} \subset \mathcal{P}$ *the radical ideal whose roots are the elements in* X
$\mathbf{N} := \mathbf{N}_<(\mathsf{I})$ *the result of Cerlienco–Mureddu Correspondence*
$\mathbf{G}_<(\mathsf{I}) := \{\mathsf{t}_1, \dots, \mathsf{t}_r\}$, *the minimal basis of* $\mathbf{T}_<(\mathsf{I}) := \mathcal{T} \setminus \mathbf{N}$,
$\mathsf{t}_1 =: X_1^{d_{1i}} \cdots X_\nu^{d_{\nu i}}$ *for each* i

Then there is a combinatorial algorithm which for each $i, m, \delta, 1 \leq i \leq r$, $1 \leq m \leq \nu$, $1 \leq \delta \leq d_{mi}$ *returns a partition* $\mathsf{X} = \sqcup_{m\delta i} \mathsf{X}_{m\delta i}$ *such that denoting, for each* i, m, δ,

$\mathsf{N}_{m\delta i} := \mathbf{N}(\mathsf{X}_{m\delta i})$ *the result of Cerlienco–Mureddu Correspondence*
$\gamma_{m\delta i} := X_m + \sum_{\omega \in \mathsf{N}_{m\delta i}} c(\gamma_{m\tau}, \omega)\omega$ *the unique polynomial (computable by interpolation) s.t.* $\gamma_{m\delta i}(\mathsf{x}) = 0$ *for all* $\mathsf{x} \in \mathsf{X}_{m\delta i}$

and

$\gamma_{mi} := \prod_\delta \gamma_{m\delta i}$ *for each* m, i
$P_i := \gamma_{\nu i}$ *for each* i
$L_i := \prod_{j=1}^{\nu-1} \gamma_{ji} \in \mathbf{k}[X_1, \dots, X_{\nu-1}]$ *for each* i
$H_i := L_1 P_i$ *for each* i

it holds:

1. $\{H_1, \dots, H_r\}$ *is a (not-reduced) minimal Gröbner basis of* I
2. *let* j_ν *be the value such that* $\mathsf{t}_{j_\nu} < X_{\nu+1} \leq \mathsf{t}_{j_\nu+1}$; *then* $\{H_{\mathsf{t}_1}, \dots, H_{\mathsf{t}_{j_\nu}}\}$ *is a minimal Gröbner basis of* $\mathsf{I} \cap \mathbf{k}[X_1, \dots, X_\nu]$;
3. *for each* $\delta \in \mathbb{N}$, *let* $j(\nu\delta)$ *be the value such that* $\mathsf{t}_{j(\nu\delta)} < X_{\nu+1}^\delta \leq \mathsf{t}_{j(\nu\delta)+1}$; *then* $\{L_1, \dots, L_{j_{\nu\delta}}\}$ *is a Gröbner basis of* $\mathfrak{I}(\mathsf{Y}_{\nu\delta})$.
4. *for each* $i, 2 \leq i \leq r$, $P_i \in (H_j, j < i) : L_i$.

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