33. Möller II

In connection with his solution of Problem 23.3.1, Macaulay gave an algorithm, which, given an order ideal

$$\mathbf{N} \subset \mathcal{T} := \{X_1^{a_1} \cdots X_n^{a_n} : (a_1, \dots, a_n) \in \mathbb{N}^n\}$$

prodices

- a finite set of points,

$$\mathsf{X} := \{\mathsf{a}_1, \dots, \mathsf{a}_s\} \subset k^n, \quad \mathsf{a}_i := (a_{i1}, \dots, a_{in}),$$

  $\#(\mathbf{N}) = \#(\mathsf{X})$ and
- a bijection $\Phi : \mathsf{X} \mapsto \mathbf{N}$,
- a set of polynomials

$$g_\tau \in \mathcal{P} := k[X_1, \dots, X_n], \quad \tau \in \{X_i\omega : \omega \in \mathbf{N}, 1 \leq i \leq n\}$$

such that, denoting

$$\{\mathsf{I}\} := \{\mathsf{f} : \mathsf{f}(\mathsf{a}_{\mathsf{i}1}, \dots, \mathsf{a}_{\mathsf{in}}) = 0, 1 \leq \mathsf{i} \leq \mathsf{s}\}$$

and, for each $\tau \in \mathbf{N}$ ℓ_τ the functional defined by

$$\ell_\tau(f) = f(a_{i1}, \dots, a_{in}), \quad f \in \mathcal{P}, \mathsf{a}_i := \Phi^{-1}(\tau)$$

it holds

- $\mathbf{N} = \mathbf{N}(\mathsf{I})$,
- $\{g_\tau : \tau \in \mathbf{G}(\mathsf{I})\}$ is the reduced Gröbner basis of I w.r.t. the lexicographical ordering induced by $X_1 < \cdots < X_n$,
- $\{g_\tau : \tau \in \mathbf{G}(\mathsf{I})\}$ and $\{\ell_\tau : \tau \in \mathbf{N}\}$ are inverse.

After presenting a slight generalization of this construction by Macaulay (Section 33.1) I present some recent and interesting converse results

- Lazard descriptio of the structure of the lexicographical Gröbner basis of an ideal in 2 variables (Theorem 33.1.1),
- an algorithm by Cerlienco and Mureddu which, given a finite set $\mathsf{X} \subset k^n$ of points computes, with the notionation above, the order ideal $\mathbf{N}(\mathsf{I})$ and a bijection $\Phi : \mathsf{X} \mapsto \mathbf{N}$ satisfying the properties granted by Macaulay's result (Section 33.2)

I merge them into a description of both the Gröbner structure and the inverse system of any ideal of points (Section 33.3); the tool to prove this Structural Theorem is a direct application of Möller Algorithm (Section 33.5).

33.1 Macaulay's Trick

In connection with his solution of Problem 23.3.1, Macaulay needed to show, for any function $H(T) : \mathbb{N} \mapsto \mathbb{N}$ satisfying the formula of Lemma 23.3.2, the existence of an ideal $\mathsf{I} \subset \mathcal{P}$ satisfying $H(T;\mathsf{I}) = H(T)$, at least in the case of a zero-dimensional ideal; if the ideal is assumed to be homogeneous, the extremal monomial ideal L, for which ${}^hH(T;\mathsf{L}) = {}^hH(T)$, is the required solution; but for the non-homogeneous case, Macaulay needed to produce an ideal I such that $H(\mathsf{I}) = H(\mathsf{L})$ and therefore also the relation $\mathbf{T}_<(\mathsf{I}) = \mathsf{L}$ for any degree-compatible term-ordering $<$.

We discuss here a slightly extension of his trick, which allows to solve the following

Problem 33.1.1. Given a finite set of terms $m_1, \dots, m_r \in \mathcal{T}$ and a term-ordering $<$ on $\mathcal{T}$, produce a set of elements $g_1, \dots, g_r \in \mathcal{P}$ such that

- $\mathbf{T}(g_i) = m_i$, for each i,
- $G := \{g_1, \dots, g_r\}$ is a Gröbner basis;

so that, denoting I the ideal generated by G, it holds

- $\mathbf{T}(\mathsf{I}) = \mathbf{T}(G) = (m_1, \dots, m_r)$.

Let
$$M := \{n_1, \dots, n_s\} \subset \mathcal{T}$$
be a finite sequence[1] such that

for each $i, 1 \le i \le r$, exists $J_i \subset \{1, \dots, s\}$ such that $m_i = \prod_{l \in J_i} n_l$;
for each $i, j, 1 \le i < j \le r$, $\mathrm{lcm}(m_i, m_j) = \prod_{l \in J_i \cup J_j} n_l$.

Clearly such a list M can be easily obtained, by repeated gcds. Now let us choose, for each $l, 1 \le l \le s$, an element $h_l \in \mathcal{P}$ such that $\mathbf{T}(h_l) < n_l$ and let us define

$\gamma_l := n_l - h_l$, for each $l, 1 \le l \le s$,
$g_i := \prod_{l \in J_i} \gamma_l$, for each $i, 1 \le i \le r$.

[1] *Caveat lector!* A *sequence* and not just a *set*. If we have $m_1 := X^2, m_2 := XY$, we must return $n_1 := n_2 := X, n_3 := Y$ and $J_1 := \{1, 2\}, J_2 := \{1, 3\}$.

With this notation, for each pair $i, j, 1 \leq i < j \leq r$, it holds by construction $t_{ij} = \prod_{l \in J_j \setminus J_i} n_l$, and $t_{ji} = \prod_{l \in J_i \setminus J_j} n_l$, where t_{ij}, t_{ji} are the elements satifying

$$t_{ij}\mathbf{T}(g_i) = \mathbf{T}(i,j) = \mathrm{lcm}(\mathbf{T}(g_i), \mathbf{T}(g_j)) = t_{ji}\mathbf{T}(g_j).$$

Proposition 33.1.1. *$G := \{g_1, \dots, g_r\}$ is a Gröbner basis.*

Proof. We have to prove, for each pair $i, j, 1 \leq i < j \leq r$, that the S-pair $S(i,j)$ has a Gröbner representation. To do so, let us define

$$\phi_{ij} := \left(\prod_{l \in J_j \setminus J_i} \gamma_l \right) - t_{ij} \text{ and } \phi_{ji} := \left(\prod_{l \in J_i \setminus J_j} \gamma_l \right) - t_{ji}.$$

Clearly, since

$$t_{ij} = \mathbf{T}\left(\prod_{l \in J_j \setminus J_i} \gamma_l \right) \text{ and } t_{ji} = \mathbf{T}\left(\prod_{l \in J_i \setminus J_j} \gamma_l \right),$$

it holds $\mathbf{T}(\phi_{ij}) < t_{ij}$ and $\mathbf{T}(\phi_{ji}) < t_{ji}$. Therefore we can claim that

$$S(i,j) = -\phi_{ij}g_i + \phi_{ji}g_j$$

is the required standard representation. In fact we have

$$\begin{aligned}
0 &= -\prod_{l \in J_i \cup J_j} \gamma_l + \prod_{l \in J_j \cup J_i} \gamma_l \\
&= -\left(\prod_{l \in J_j \setminus J_i} \gamma_l \right) g_i + \left(\prod_{l \in J_i \setminus J_j} \gamma_l \right) g_j \\
&= -(\phi_{ij} + t_{ij})g_i + (\phi_{ji} + t_{ji})g_j \\
&= -\phi_{ij}g_i + \phi_{ji}g_j - (t_{ij}g_i - t_{ji}g_j) \\
&= -\phi_{ij}g_i + \phi_{ji}g_j - S(i,j),
\end{aligned}$$

so that, the claim holds, since

$$\mathbf{T}(\phi_{ij}g_i) < t_{ij}\mathbf{T}(g_i) = \mathbf{T}(i,j) = t_{ji}\mathbf{T}(g_j) > \mathbf{T}(\phi_{ji}g_j).$$

⊙

For any finite set X of points

$$\mathsf{X} := \{\mathsf{a}_1, \dots, \mathsf{a}_s\} \subset k^n, \quad \mathsf{a}_i := (a_{i1}, \dots, a_{in})$$

let us denote

for each i, ℓ_i the linear functional $\ell_i \in \mathcal{P}^*$ defined by

$$\ell_i(f) = f(a_{i1}, \ldots, a_{in}) \text{ for each } f(X_1, \ldots, X_n) \in \mathcal{P};$$

$\mathbb{L}(\mathsf{X}) := \mathrm{Span}_k(\{\ell_i, 1 \le i \le s\}) \subset \mathcal{P}^*$,
$\mathsf{I}(\mathsf{X}) := \{f \in \mathcal{P} : f(\mathsf{a}_i) = 0, \text{ for each } i\} = \mathfrak{P}(\mathbb{L}(\mathsf{X}))$.

With this notation we can now present Macaulay's result: let $\mathbf{N} \subset \mathcal{T}$ be a finite order ideal of $\mathcal{T}$, and let

$$\mathbf{G} := \{m_1, \ldots, m_r\}, \quad m_l = X_1^{e_{1l}} \cdots X_n^{e_{nl}}, \text{ for each } l,$$

be the minimal basis of the monomial ideal $\mathcal{T} \setminus \mathbf{N}$.

Since $\mathbf{N}$ is finite, for each i exists $d_i \in \mathbb{N}$ such that

$$X_i^{d_i} \in \mathbf{G} \text{ and } e_{il} \le d_i, \text{ for each } l.$$

Let us then take, for each $i, j, k, j \ne k$, elements

$$a_{ij} \in k, 1 \le i \le n, 0 \le j < d_i : a_{ij} \ne a_{ik},$$

and let us define, for each $l, 1 \le l \le r$,

$$g_l := \prod_{i=1}^{n} \prod_{j=0}^{e_{il}-1} (X_i - a_{ij}),$$

which is such that $\mathbf{T}(g_l) = m_l$.

Moreover, to each term $t = X_1^{e_1} \cdots X_n^{e_n} \in \mathbf{N}$ let us associate the affine point

$$\mathsf{a}(t) := (a_{1e_1}, \ldots a_{ne_n}) \in k^n,$$

and let $\mathsf{X} := \{\mathsf{a}(t) : t \in \mathbf{N}\}$. Then:

Corollary 33.1.1 (Macaulay).

Under this notation, for any degree-compatible term-ordering, it holds

(1) $\mathbf{N} = \mathbf{N}(\mathsf{I}(\mathsf{X}))$,
(2) $\mathcal{G}(\mathsf{I}(\mathsf{X})) := \{g_1, \ldots, g_r\}$ *is the reduced Gröbner basis of* $\mathsf{I}(\mathsf{X})$.

$\boxed{\odot}$

Since $e_i \le d_i$, for each $t = X_1^{e_1} \cdots X_n^{e_n} \in \{X_j\tau : 1 \le j \le n, \tau \in \mathbf{N}\}$ and each i, it is natural to consider also the polynomials

$$g_t := \prod_{i=1}^{n} \prod_{j=0}^{e_i-1} (X_i - a_{ij}), \quad t = X_1^{e_1} \cdots X_n^{e_n} \in \{X_j\tau : 1 \le j \le n, \tau \in \mathbf{N}\}$$

and investigate their relation with Lagrange Interpolation Formula (Corollary 28.2.1).

Let us order $\mathbf{N} := \{t_1, \ldots, t_s\}$ in such a way that $t_1 < t_2 < \cdots < t_s$, where $<$ is the lexicographical ordering induced by $X_1 < \cdots < X_n$, and let us write $\mathsf{a}_i := \mathsf{a}(t_i)$ in order to fix a suitable enumeration of X and $\mathbb{L}(\mathsf{X})$. Moreover let us define $q_i := g_{t_i}$, for each $i, 1 \le i \le s$. Then

Lemma 33.1.1. *For any degree-compatible term-ordering, it holds*

(1) $\{g_t : t \in \mathbf{B}(\mathsf{I}(\mathsf{X}))\}$, *is the border basis of* $\mathsf{I}(\mathsf{X})$*;*
(2) $\{g_t : t \in \mathbf{G}(\mathsf{I}(\mathsf{X}))\}$, *is the reduced Gröbner basis of* $\mathsf{I}(\mathsf{X})$*;*
(3) $\mathbf{q} := \{q_i : 1 \le i \le s\}$ *is a triangular set of* $\mathbb{L}(\mathsf{X})$.

$\boxed{\odot}$

For $n = 2$, the structure of the Gröbner basis constructed by Macaulay for the ideal $\mathsf{I}(\mathsf{X})$ is an illustrating example of Lazard Theorem which describes the structure of the lexicographical Gröbner basis for any ideal $\mathsf{I} \subset k[X_1, X_2]$:

Theorem 33.1.1 (Lazard). *Let* $\mathcal{P} := k[X_1, X_2]$ *and let* $<$ *be the lexicographical ordering induced by* $X_1 < X_2$.

Let $\mathsf{I} \subset \mathcal{P}$ *be an ideal and let* $\{f_0, f_1, \dots, f_k\}$ *be a Gröbner basis of* I *ordered so that*

$$\mathbf{T}(f_0) < \mathbf{T}(f_1) < \cdots < \mathbf{T}(f_k).$$

Then

- $f_0 = PG_1 \cdots G_{k+1}$,
- $f_j = PH_jG_{j+1} \cdots G_{k+1}, 1 \le j < k$,
- $f_k = PH_kG_{k+1}$,

where

P *is the primitive part of* $f_0 \in k[X_1][X_2]$*;*
$G_i \in k[X_1], 1 \le i \le k+1$*;*
$H_i \in k[X_1][X_2]$ *is a monic polynomial of degree* $d(i)$*, for each* i*;*
$d(1) < d(2) < \cdots < d(k)$*;*
$H_{i+1} \in (G_1 \cdots G_i, H_1G_2 \cdots G_i, \dots, H_jG_{j+1} \cdots G_i, \dots, H_{i-1}G_i, H_i)$ *for all* i.

$\boxed{\odot}$

Proof. Let P and G_{k+1} be, respectively, the primitive part and the content of $\gcd(f_0, \dots, f_h)$ in $k[X_1][X_2]$; since a set $\{g_0, \dots, g_h\}$ is a manimal Gröbner basis if and only if the same is true for $\{gg_0, \dots, gg_h\}$ we can divide by PG_{k+1} and assume wlog that $P = G_{k+1} = 1$ and $\gcd(f_0, \dots, f_h) = 1$.

Since, for each i, $\mathbf{T}(f_i) < \mathbf{T}(f_{i+1})$ necessarily we have $d(i) \le d(i+1)$ but $d(i) = d(i+1)$ would imply $\mathbf{T}(f_i) \mid \mathbf{T}(f_{i+1})$ so that we have $d(i) < d(i+1)$.

Setting $g_i := \mathrm{Lp}(f_i)$ for each i, both $X_2^{d(i+1)-d(i)} f_i$ and f_{i+1} are in the ideal and have degree $d(i+1)$ in X_2; therefore successive euclidean division of the leading polynsomial leads to a polynomial $f := \mathrm{Lp}(f)X_2^{d(i+1)} + \cdots$ in the ideal, where $\mathrm{Lp}(f) = \gcd(g_i, g_{i+1})$.

Therefore $\mathbf{T}(f)$ is multiple of some $\mathbf{T}(f_j)$. If $g_{i+1} \ne \gcd(g_i, g_{i+1})$, necessarily $j < i+1$ and $\mathbf{T}(f_j)$ divides $\mathbf{T}(f_{i+1})$ getting a contradiction. As a conclusion $g_{i+1} \mid g_i$ and we can set $G_{i+1} := \frac{g_i}{g_{i+1}}$.

Since $G_{i+1}f_{i+1} - X_2^{d(i+1)-d(i)}f_i$ is a polynomial of degree less than $d(i+1)$ in X_2 which reduces to zero by the Gröbner basis, it follows that $G_{i+1}f_{i+1} \in (f_0, \ldots, f_i)$; therefore, inductively

$$g_i \mid f_j \text{ for each } j \leq i \implies g_{i+1} \mid f_j \text{ for each } j \leq i+1.$$

Therefore, $\gcd(f_0, \ldots, f_h) = 1$ implies that $g_h = 1$ and each g_i divides f_i.

Setting $H_i := \frac{f_i}{g_i}$ for all i, from $G_{i+1}f_{i+1} \in (f_0, \ldots, f_i)$ diving by

$$G_{i+1}g_{i+1} = g_i = G_{i+1} \cdots G_h$$

we obtain the last claim. $\boxed{\odot}$

33.2 Cerlienco–Mureddu Correspondence

Cerlienco and Mureddu solved a partial converse of Macaulay's result:

Problem 33.2.1. Given a finite set of points,

$$\{\mathsf{a}_1, \ldots, \mathsf{a}_s\} \subset k^n, \quad \mathsf{a}_i := (a_{i1}, \ldots, a_{in}),$$

to compute $\mathbf{N}(\mathsf{I})$ w.r.t. the lexicographical ordering $<$ induced by $X_1 < \cdots < X_n$ where

$$\mathsf{I} := \{f \in \mathcal{P} : f(\mathsf{a}_i) = 0, 1 \leq i \leq s\}.$$

$\boxed{\odot}$

Remark that a zero-dimensional ideal $\mathsf{I} \subset \mathcal{P}$ can be considered as *given* if we know

- the set $\mathcal{Z}(\mathsf{I})$ and
- for each $\mathsf{a} \in \mathcal{Z}(\mathsf{I})$, a Macaulay basis of the corresponding primary component of I.

Let us set

- $<$ the lexicographical ordering $<$ induced by $X_1 < \cdots < X_n$;
- $\mathsf{I} \subset \mathcal{P}$ be a zero dimensional ideal;
- for each $\mathsf{a} \in \mathsf{Z} := \mathcal{Z}(\mathsf{I})$, $\mathsf{a} := (a_1, \ldots, a_n)$:
  - $\lambda_\mathsf{a} : \mathcal{P} \mapsto \mathcal{P}$ the translation $\lambda_\mathsf{a}(X_i) = X_i + a_i$, for each i,
  - $\mathfrak{m}_\mathsf{a} = (X_1 - a_1, \ldots, X_n - a_n)$,
  - $\mathfrak{q}_\mathsf{a}$ the $\mathfrak{m}_\mathsf{a}$-primary component of I,
  - $\Lambda_\mathsf{a} := \mathfrak{M}(\lambda_\mathsf{a}(\mathfrak{q}_\mathsf{a})) \subset \mathrm{Span}_K(\mathbb{M})$,
  - $\ell_{\upsilon\mathsf{a}}$, for each $\upsilon \in \mathbf{N}_<(\lambda_\mathsf{a}(\mathfrak{q}_\mathsf{a}))$, the Macaulay equation $\ell_{\upsilon\mathsf{a}} := \ell(\upsilon)$ so that
  - $\{\ell_{\upsilon\mathsf{a}} : \upsilon \in \mathbf{N}_<(\lambda_\mathsf{a}(\mathfrak{q}_\mathsf{a}))\}$ is the Macaulay basis of Λ_a, enumerated in order to satisfy the properties of Corollary 32.3.1[2],
- $s := \sum_{\mathsf{a} \in \mathsf{Z}} \deg(\mathfrak{q}_\mathsf{a})$;

[2] Remark that in particular $\upsilon = \mathbf{T}_<(\ell_{\upsilon\mathsf{a}})$.

- $\mathbb{L} := \{\lambda_1, \ldots, \lambda_s\} := \{\ell_{v\mathsf{a}}\lambda_{\mathsf{a}} : v \in \mathbf{N}_<(\lambda_{\mathsf{a}}(\mathfrak{q}_{\mathsf{a}})), \mathsf{a} \in \mathsf{Z}\}$ ordered as stated in Corollary 32.3.2;
- $\mathsf{X} := \{\mathsf{x}_1, \ldots, \mathsf{x}_s\} := \{(\mathsf{a}, v) \in \mathbf{N}_<(\lambda_{\mathsf{a}}(\mathfrak{q}_{\mathsf{a}})), \mathsf{a} \in \mathsf{Z}\}$ enumerated so that

$$\mathsf{x}_j = (\mathsf{a}, v) \iff \lambda_j = \ell_{v\mathsf{a}}\lambda_{\mathsf{a}};$$

- for each $j, 1 \leq j \leq s$, $M(\lambda_j) := M(v)\lambda_{\mathsf{a}}$ where $\lambda_j = \ell_{v\mathsf{a}}\lambda_{\mathsf{a}}$;

and let us remark that Cerlienco and Mureddu state their result under the further assumption that

$$\text{for each } j, 1 \leq j \leq s,\ \ell_{v\mathsf{a}}\lambda_{\mathsf{a}} =: \lambda_j = M(\lambda_j) = M(v)\lambda_{\mathsf{a}} \text{ i.e. } \ell_{v\mathsf{a}} = M(v).$$

Therefore, with the notation above:

Definition 33.2.1. *The ordered sets* $\mathbb{L}(\mathsf{I}) := \mathbb{L}$ *and* $\mathsf{X}(\mathsf{I}) := \mathsf{X}$ *are called, respectively, a* Macaulay representation *and a* CM-scheleton *of* I.

If, moreover, for each $\lambda = \ell_{v\mathsf{a}}\lambda_{\mathsf{a}} \in \mathbb{L}$, $\lambda = M(\lambda) = M(v)\lambda_{\mathsf{a}}$, *then* I *is called a* CM-ideal, $\mathsf{X}(\mathsf{I})$ *its* CM-scheme, *and each* $\mathsf{x} = (\mathsf{a}, v) \in \mathsf{X}(\mathsf{I})$ *a* CM-condition. ⊙

Lemma 33.2.1. *The following holds:*

(1) $\mathsf{I} = \bigcap_{\mathsf{a}\in\mathsf{Z}} \mathfrak{q}_{\mathsf{a}} = \mathfrak{P}(\mathrm{Span}_k(\mathbb{L}))$;
(2) *for each* $j, 1 \leq j \leq s$, $\mathsf{x}_j = (\mathsf{a}, v)$ *and each* $v' \mid v$ *there is* $i < j$ *such that* $\mathsf{x}_i = (\mathsf{a}, v')$;
(3) *for each* $j, 1 \leq j \leq s$, $\mathsf{x}_j = (\mathsf{a}, v) \in \mathsf{X}$, *and each* $f \in \mathcal{P}$

$$M(\lambda_j)(f) = M(v)(\lambda_{\mathsf{a}}(f)) = (D(v)(f))(\mathsf{a}) = c(v, \lambda_{\mathsf{a}}(f));$$

(4) *for each* $\sigma, 1 \leq \sigma \leq s$, $\mathbb{L}_\sigma := \{\lambda_1, \ldots, \lambda_\sigma\}$ *and* $\mathsf{X}_\sigma := \{\mathsf{x}_i, 1 \leq i \leq \sigma\}$ *are a Macaulay representation and a CM-scheleton of* $\mathsf{I}_\sigma = \mathfrak{P}(\mathrm{Span}_k(\mathbb{L}_\sigma))$;
(5) $\mathsf{I}_1 \subset \ldots \subset \mathsf{I}_\sigma \subset \mathsf{I}_{\sigma+1} \subset \ldots \mathsf{I}$;
(6) $\mathsf{I} = \sqrt{\mathsf{I}} \iff v = 1$ for each $(\mathsf{a}, v) \in \mathsf{X} \iff \#\mathsf{X} = \#\mathsf{Z}$. ⊙

Cerlienco–Mureddu result consists in proposing an algorithm which to each CM-scheme

$$\mathsf{X} := \{\mathsf{x}_1, \ldots, \mathsf{x}_s\} \subset k^n \times \mathcal{T}, \mathsf{x}_i = (\mathsf{a}_i, v_i), \mathsf{a}_i := (a_{i1}, \ldots, a_{in}), v_i = \prod_{l=1}^{n} X_l^{\alpha_{il}}$$

associates

- an order ideal $\mathbf{N} := \mathbf{N}(\mathsf{X})$ and
- a bijection $\Phi := \Phi(\mathsf{X}) : \mathsf{X} \mapsto \mathbf{N}$,

which, as we will proof later, satisfies

Fact 33.2.1. *For* $\mathsf{X} = \mathsf{X}(\mathsf{I}) \subset k^n \times \mathcal{T}$ *holds* $\mathbf{N}_<(\mathsf{I}) = \mathbf{N}(\mathsf{X})$ *for the lexicographical ordering induced by* $X_1 < \cdots < X_n$. ⊙

Since they do so by induction on $s = \#(\mathsf{X})$ let us consider the subset $\mathsf{X}' := \{\mathsf{x}_1, \ldots, \mathsf{x}_{s-1}\}$, and the corresponding[3] order ideal $\mathbf{N}' := \mathbf{N}(\mathsf{X}')$ and bijection $\Phi' := \Phi(\mathsf{X}')$.

We need also to consider, for each $m < n$, the set

$$\mathcal{T}[1,m] := \mathcal{T} \cap k[X_1, \ldots, X_m] = \{X_1^{a_1} \cdots X_m^{a_m} : (a_1, \ldots, a_m) \in \mathbb{N}^m\},$$

and the projection

$$\pi_m : k^n \mapsto k^m, \quad \pi_m(x_1, \ldots, x_n) = (x_1, \ldots, x_m),$$

which we freely use to denote also the projections

$$\pi_m : \mathcal{T} \simeq \mathbb{N}^n \mapsto \mathbb{N}^m \simeq \mathcal{T}[1,m], \quad \pi_m(X_1^{\alpha_1} \cdots X_n^{\alpha_n}) = X_1^{\alpha_1} \cdots X_m^{\alpha_m}$$

and

$$\pi_m : k^n \times \mathcal{T} \mapsto k^m \times \mathcal{T}[1,m], \quad \pi_m(\mathsf{a}, \tau) = (\pi_m(\mathsf{a}), \pi_m(\tau)).$$

Also, for a CM-condition $\mathsf{x} = (\mathsf{a}, \upsilon) \in k^m \times \mathcal{T}[1,m]$ we also set

$$\pi_m(\lambda) := \pi_m(M(\upsilon)\lambda_{\mathsf{a}}) := M(\pi_m(\upsilon))\lambda_{\pi_m(\mathsf{a})}.$$

With a slight abuse of notation, if $\mathsf{I}(\mathsf{X})$ is radical, we simply identify each $\mathsf{x}_i = (\mathsf{a}_i, 1)$ with a_i.

With this notation, let us set

$$\begin{aligned}
&m := \max(j : \text{ exists } i < s : \pi_j(\mathsf{x}_i) = \pi_j(\mathsf{x}_s));\\
&d := \#\{\mathsf{x}_i, i < s : \pi_m(\mathsf{x}_i) = \pi_m(\mathsf{x}_s), \Phi'(\mathsf{x}_i) \in \mathcal{T}[1, m+1]\};\\
&\mathsf{W} := \{\mathsf{x}_i : \Phi'(\mathsf{x}_i) = \omega_i X_{m+1}^d, \omega_i \in \mathcal{T}[1,m]\} \cup \{\mathsf{x}_s\};\\
&\mathsf{V} := \pi_m(\mathsf{W});\\
&\omega := \Phi(\mathsf{V})(\pi_m(\mathsf{x}_s));\\
&t_s := \omega X_{m+1}^d;
\end{aligned}$$

where $\mathbf{N}(\mathsf{V})$ and $\Phi(\mathsf{V})$ are the result of the application of the present algorithm to V, which can be inductively applied since $\#(\mathsf{V}) \leq s-1$. We then define

- $\mathbf{N} := \mathbf{N}' \cup \{t_s\}$,
- $\Phi(\mathsf{x}_i) := \begin{cases} \Phi'(\mathsf{x}_i) & i < s \\ t_s & i = s \end{cases}$

Example 33.2.1. Let us consider the set $\mathsf{Y} := \{\mathsf{a}_i, 1 \leq i \leq 6\}$ where

$$\begin{array}{lllllllll}
\mathsf{a}_1 & = & (0,0) & \mathsf{a}_2 & = & (0,1) & \mathsf{a}_3 & = & (2,0) \\
\mathsf{a}_4 & = & (0,2) & \mathsf{a}_5 & = & (1,0) & \mathsf{a}_6 & = & (1,1);
\end{array}$$

Cerlienco–Mureddu Algorithms returns:

(0,0) $\mathsf{a}_1 := (0,0), \Phi(\mathsf{a}_1) := t_1 := 1;$

[3] If $s = 1$ the only possible solution is $\mathbf{N} = \{1\}, \Phi(\mathsf{x}_1) = 1$.

(0,1) $\mathsf{a}_2 := (0,1), m = 1, d = \#\{(0,0)\} = 1, \mathsf{W} = \{(0,1)\}$,
$\omega = 1, \Phi(\mathsf{a}_2) := t_2 := X_2$,
(2,0) $\mathsf{a}_3 := (2,0), m = 0, d = \#\{(0,0)\} = 1, \mathsf{W} = \{(2,0)\}$,
$\omega = 1, \Phi(\mathsf{a}_3) := t_3 := X_1$,
(0,2) $\mathsf{a}_4 := (0,2), m = 1, d = \#\{(0,0),(0,1)\} = 2, \mathsf{W} = \{(0,2)\}$,
$\omega = 1, \Phi(\mathsf{a}_4) := t_4 := X_2^2$,
(1,0) $\mathsf{a}_5 := (1,0), m = 0, d = \#\{(0,0),(2,0)\} = 2, \mathsf{W} = \{(1,0)\}$,
$\omega = 1, \Phi(\mathsf{a}_5) := t_5 := X_1^2$,
(1,1) $\mathsf{a}_6 := (1,1), m = 1, d = \#\{(1,0)\} = 1, \mathsf{W} = \{(0,1),(1,1)\}$,
$\omega = X_1, \Phi(\mathsf{a}_6) := t_6 := X_1 X_2$.

$\boxed{\odot}$

Example 33.2.2. Let us consider the set $\mathsf{X} := \{\mathsf{b}_i, 1 \le i \le 9\}$ where

$$\begin{array}{lllllllll} \mathsf{b}_1 & = & (0,0,1) & \mathsf{b}_2 & = & (0,1,-2) & \mathsf{b}_3 & = & (2,0,2) \\ \mathsf{b}_4 & = & (0,2,-2) & \mathsf{b}_5 & = & (1,0,3) & \mathsf{b}_6 & = & (1,1,3) \\ \mathsf{b}_7 & = & (1,1,1) & \mathsf{b}_8 & = & (2,0,1) & \mathsf{b}_9 & = & (2,0,0) \end{array}$$

and let us set $\mathsf{a}_i := \pi_2(\mathsf{b}_i)$, for each i, so that $\pi_2(\mathsf{X}) = \mathsf{Y}$, where Y is the set of points discussed in Example 33.2.1.

Clearly Cerlienco–Mureddu Correspondence returns $\Phi(\mathsf{b}_i) = \Phi(\mathsf{a}_i)$ for each $i \le 6$ and

$$t_7 \;:=\; X_3, \quad t_8 \;:=\; X_1 X_3, \quad t_9 \;:=\; X_3^2.$$

$\boxed{\odot}$

Let $\mathbb{L} := \{\lambda_1, \dots, \lambda_s\}$ and

$$\mathsf{X} := \{\mathsf{x}_1, \dots, \mathsf{x}_s\} \subset k^n \times \mathcal{T}, \mathsf{x}_i = (\mathsf{a}_i, \upsilon_i), \mathsf{a}_i := (a_{i1}, \dots, a_{in}), \upsilon_i = \prod_{l=1}^{n} X_l^{\alpha_{il}}$$

be the Macaulay representation and the CM-scheme of a (zero-dimensional) CM-ideal $\mathsf{I} \subset \mathcal{P}$ so that, for each i,

$$\lambda_i = M(\lambda) = M(\upsilon_i)\lambda_{\mathsf{a}_i}, \text{ for each } i, 1 \le i \le s,$$

and let $\mathbf{N} := \mathbf{N}(\mathsf{X})$ and $\Phi := \Phi(\mathsf{X})$ the result of Cerlienco–Mureddu Correspondence. Then

Lemma 33.2.2. *If* $\mathsf{Y} = \{\mathsf{x}_1, \dots, \mathsf{x}_r\} \subset \mathsf{X}$ *is an initial segment of* X *then*

- Y *is a CM-scheme,*
- $\mathbf{N}(\mathsf{Y}) \subset \mathbf{N}(\mathsf{X})$,
- *for each* $j \le r < s, \Phi(\mathsf{Y})(\mathsf{x}_j) = \Phi(\mathsf{X})(\mathsf{x}_j)$.

$\boxed{\odot}$

Remark 33.2.1. Let us denote, for each $\nu, 1 \le \nu < n$, and each $\mathsf{y} \in \pi_\nu(\mathsf{X})$,

$$\mu(\mathsf{y}) := \#\{\mathsf{x} \in \mathsf{X} : \mathsf{y} = \pi_\nu(\mathsf{x})\},$$

and for each $\nu, 1 \le \nu < n$, and each $\delta \in \mathbb{N}$,

$$\mathsf{Y}_{\nu\delta} := \{\pi_\nu(\mathsf{x}) : \text{ exists } \omega \in \mathcal{T}[1,\nu] : \Phi(\mathsf{x}) = \omega X_{\nu+1}^{\delta}\}.$$

Then

- $\mathsf{Y}_{\nu\delta} = \{\mathsf{y} \in \pi_\nu(\mathsf{X}) : \delta < \mu(\mathsf{y})\}$,
- $\pi_\nu(\mathsf{X}) = \mathsf{Y}_{\nu 0} \supset \mathsf{Y}_{\nu 1} \supset \cdots \supset \mathsf{Y}_{\nu\delta} \supset \mathsf{Y}_{\nu\delta+1} \supset \cdots$,
- $\mathsf{I}(\pi_\nu(\mathsf{X})) = \mathsf{I}(\mathsf{Y}_{\nu 0}) \subset \mathsf{I}(\mathsf{Y}_{\nu 1}) \subset \cdots \subset \mathsf{I}(\mathsf{Y}_{\nu\delta}) \subset \mathsf{I}(\mathsf{Y}_{\nu\delta+1}) \subset \cdots$.

The result is essentially a specialization of Theorem 26.2.2 $\boxed{\odot}$

Let $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathcal{T} \setminus \mathbf{N}(\mathsf{X})$ be any term such that $\mathbf{N} \cup \{\tau\}$ is an order ideal and let us define, for each $m, 1 \le m \le n$:

$\mathsf{N}_m(\tau) := \mathsf{N}_m(\mathsf{X},\tau) := \{\omega \in \mathcal{T}[1,m] : \tau > \omega X_{m+1}^{d_{m+1}} \cdots X_n^{d_n} \in \mathbf{N}\}$,
$\mathsf{A}_m(\tau) := \mathsf{A}_m(\mathsf{X},\tau) := \{\Phi^{-1}(\omega X_{m+1}^{d_{m+1}} \cdots X_n^{d_n}) : \omega \in \mathsf{N}_m(\tau)\} \subset \mathsf{X} \subset k^n \times \mathcal{T}$,
$\mathsf{B}_m(\tau) := \mathsf{B}_m(\mathsf{X},\tau) := \pi_m(\mathsf{A}_m(\tau)) \subset k^m \times \mathcal{T}[1,m]$,
$\mathsf{C}_m(\tau) := \mathsf{C}_m(\mathsf{X},\tau) := \{\pi_m(\mathsf{x}) \in \mathsf{B}_m(\tau) : \pi_{m-1}(\mathsf{x}) \notin \mathsf{B}_{m-1}(\tau)\} \subset k^m \times \mathcal{T}[1,m]$,
$\mathsf{D}_m(\tau) := \mathsf{D}_m(\mathsf{X},\tau) := \{\mathsf{x} \in \mathsf{X} : \pi_m(\mathsf{x}) \in \mathsf{C}_m(\tau)\} \subset \mathsf{X}$;
$\mathsf{L}_m(\tau) := \mathsf{L}_m(\mathsf{X},\tau) := \{\lambda_i \in \mathbb{L} : \pi_m(\mathsf{x}_i) = (\pi_m(\mathsf{a}_i), \pi_m(\upsilon_i)) \in \mathsf{C}_m(\tau)\} \subset \mathbb{L}$;
$\mathsf{M}_m(\tau) := \mathsf{M}_m(\mathsf{X},\tau) := \{\omega \in \mathcal{T}[1,m] : \omega < X_m^{d_m}, \omega X_{m+1}^{d_{m+1}} \cdots X_n^{d_n} \in \mathbf{N}\}$,

where, with slight abuse of notation, we have

$$\mathsf{N}_n(\tau) := \{\omega \in \mathcal{T} : \omega < \tau\}, \mathsf{A}_n(\tau) := \mathsf{B}_n(\tau) := \{\mathsf{a} : \Phi(\mathsf{a}) < \tau\}, \mathsf{C}_1(\tau) := \mathsf{B}_1(\tau).$$

Example 33.2.3. With respect to Example 33.2.2, if we choose $\tau := X_2X_3$ we have

$$\mathsf{N}_1 = \mathsf{A}_1 = \mathsf{B}_1 = \mathsf{C}_1 = \mathsf{D}_1 = \mathsf{M}_1 = \emptyset,$$

and

$$\begin{array}{lllllll}
\mathsf{N}_2 & = & \{1, X_1\}, & & \mathsf{N}_3 & = & \mathbf{N} \setminus \{X_3^2\}, \\
\mathsf{A}_2 & = & \{\mathsf{b}_7, \mathsf{b}_8\}, & & \mathsf{A}_3 & = & \{\mathsf{b}_i, 1 \le i \le 8\}, \\
\mathsf{B}_2 & = & \{(1,1),(2,0)\}, & & \mathsf{B}_3 & = & \{\mathsf{b}_i, 1 \le i \le 8\}, \\
\mathsf{C}_2 & = & \{(1,1),(2,0)\}, & & \mathsf{C}_3 & = & \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_4, \mathsf{b}_5\}, \\
\mathsf{D}_2 & = & \{\mathsf{b}_3, \mathsf{b}_6, \mathsf{b}_7, \mathsf{b}_8, \mathsf{b}_9\}, & & \mathsf{D}_3 & = & \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_4, \mathsf{b}_5\}, \\
\mathsf{M}_2 & = & \{1, X_1\}, & & \mathsf{M}_3 & = & \{1, X_1, X_1^2, X_2, X_1X_2, X_2^2\}.
\end{array}$$

If we instead choose $\tau := X_1X_3^2$ we have

$$\begin{array}{lllllllll}
\mathsf{N}_1 & = & \{1\}, & \mathsf{N}_2 & = & \{1\}, & \mathsf{N}_3 & = & \mathbf{N}, \\
\mathsf{A}_1 & = & \{\mathsf{b}_9\}, & \mathsf{A}_2 & = & \{\mathsf{b}_9\}, & \mathsf{A}_3 & = & \{\mathsf{b}_i, 1 \le i \le 9\}, \\
\mathsf{B}_1 & = & \{2\}, & \mathsf{B}_2 & = & \{(2,0)\}, & \mathsf{B}_3 & = & \{\mathsf{b}_i, 1 \le i \le 9\}, \\
\mathsf{C}_1 & = & \{2\}, & \mathsf{C}_2 & = & \emptyset, & \mathsf{C}_3 & = & \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_4, \mathsf{b}_5, \mathsf{b}_6, \mathsf{b}_7\}, \\
\mathsf{D}_1 & = & \{\mathsf{b}_3, \mathsf{b}_8, \mathsf{b}_9\}, & \mathsf{D}_2 & = & \emptyset, & \mathsf{D}_3 & = & \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_4, \mathsf{b}_5, \mathsf{b}_6, \mathsf{b}_7\}, \\
\mathsf{M}_1 & = & \{1\}, & \mathsf{M}_2 & = & \emptyset, & \mathsf{M}_3 & = & \mathbf{N} \setminus \{X_3^2\}.
\end{array}$$

Lemma 33.2.3. *With the notation above, it holds*

(1) $\#(\mathsf{B}_m(\tau)) = \#(\mathsf{A}_m(\tau)) = \#(\mathsf{N}_m(\tau))$;
(2) *Cerlienco–Mureddu Correspondence associates to* $\mathsf{B}_m(\tau)$ *the order ideal*

$$\mathbf{N}(\mathsf{B}_m(\tau)) = \mathsf{N}_m(\tau)$$

and the bijection $\Phi(\mathsf{B}_m(\tau))$ *defined by*

$$\Phi(\mathsf{B}_m(\tau))(\pi_m(\mathsf{x}))X_{m+1}^{d_{m+1}} \cdots X_n^{d_n} = \Phi(\mathsf{x}), \text{ for each } \mathsf{x} \in \mathsf{A}_m;$$

(3) $\#(\mathsf{L}_m(\tau)) = \#(\mathsf{C}_m(\tau)) \le \#(\mathsf{M}_m(\tau))$;
(4) *under Cerlienco–Mureddu Correspondence one has*

$$\mathbf{N}(\mathsf{C}_m(\tau)) \subset \{\omega \in \mathcal{T}[1,m] : \omega < X_m^{d_m}\};$$

(5) $\mathsf{X} = \cup_m \mathsf{D}_m(\tau)$.

Proof.

(1) is trivial;
(2) Cerlienco–Mureddu Algorithm when applied to the ordered set X associates each element $\mathsf{x} \in \mathsf{A}_m(\tau)$ to the term

$$\Phi(\mathsf{x}) = \Phi(\pi_m(\mathsf{A}_m(\tau)))(\pi_m(\mathsf{x}))X_{m+1}^{d_{m+1}} \cdots X_n^{d_n};$$

(3) in order to obtain $\mathsf{M}_m(\tau)$ one has to remove form $\mathsf{N}_m(\tau)$ the subset

$$\{\omega X_m^{d_m} \in \mathsf{N}_m(\tau) : \omega \in \mathcal{T}[1, m-1]\} = \{\omega X_m^{d_m} : \omega \in \mathsf{N}_{m-1}(\tau)\}$$

while for each $\omega \in \mathsf{N}_{m-1}(\tau)$ there are $d_m + 1$ elements $\mathsf{y} \in \mathsf{B}_m(\tau)$ such that

$$\Phi(\mathsf{B}_{m-1}(\tau))(\pi_{m-1}(\mathsf{y})) = \omega.$$

(4) In order that there is $\omega \in \mathbf{N}(\mathsf{C}_m(\tau))$ such that $\omega \ge X_m^{d_m}$, Cerlienco–Mureddu Algorithm requires the existence of at least $d_m + 1$ elements $\mathsf{y}_0, \dots, \mathsf{y}_{d_m}$ such that

$$\pi_m(\mathsf{y}_0) = \cdots = \pi_m(\mathsf{y}_i) = \cdots = \pi_m(\mathsf{y}_{d_m}),$$

so that $\pi_{m-1}(\mathsf{y}_0) \in \mathsf{B}_{m-1}(\tau)$.
(5) If $\mathsf{x} \in \mathsf{X}$ is such that $\Phi(\mathsf{x}) < \tau$, then there is a minimal value $m \le n$ for which $\mathsf{x} \in \mathsf{A}_m(\tau)$, $\pi_m(\mathsf{x}) \in \mathsf{B}_m(\tau)$, $\pi_m(\mathsf{x}) \in \mathsf{C}_m(\tau)$, $\mathsf{x} \in \mathsf{D}_m(\tau)$.
If $\mathsf{x} \in \mathsf{X}$ is such that $\Phi(\mathsf{x}) = X_1^{e_1} \cdots X_n^{e_n} > \tau$, there is $m \le n$ such that $e_m > d_m$, while $e_i = d_i$, for each $i > m$; this implies that there is $\mathsf{y} \in \mathsf{A}_m(\tau)$ such that $\pi_m(\mathsf{y}) = \pi_m(\mathsf{x})$ so that $\mathsf{x} \in \mathsf{D}_m(\tau)$.

$\boxed{\odot}$

33.3 Lazard Structural Theorem

Let $\mathsf{I} \subset \mathcal{P}$ be a CM-ideal, and, using the same notation as above, $\mathbb{L} := \{\lambda_1, \dots, \lambda_s\}$ and

$$\mathsf{X} := \{\mathsf{x}_1, \dots, \mathsf{x}_s\} \subset k^n \times \mathcal{T}, \mathsf{x}_i = (\mathsf{a}_i, \upsilon_i), \mathsf{a}_i := (a_{i1}, \dots, a_{in}), \upsilon_i = \prod_{l=1}^{n} X_l^{\alpha_{il}}$$

a Macaulay representation and a CM-scheme of I so that, for each i,

$$\lambda_i = M(\lambda_i) = M(\upsilon_i)\lambda_{\mathsf{a}_i}, \text{ for each } i, 1 \leq i \leq s;$$

let us now denote $\mathbf{N} := \mathbf{N}(\mathsf{X})$ and $\Phi := \Phi(\mathsf{X})$ the result of Cerlienco–Mureddu Correspondence which satisfies

Fact 33.3.1. *It holds*

(A) $\mathbf{N} := \mathbf{N}(\mathsf{I})$.

$\boxed{\odot}$

Since $\mathbf{N}$ is an order ideal, $\mathbf{T} := \mathcal{T} \setminus \mathbf{N}$ is a monomial ideal whose minimal basis $\mathbf{G} := \{\mathsf{t}_1, \dots, \mathsf{t}_r\}$ will be ordered so that $\mathsf{t}_1 < \mathsf{t}_2 < \dots < \mathsf{t}_r$.

Denoting further

$$\mathbf{B} := (\{1\} \cup \{X_i\tau : \tau \in \mathbf{N}\}) \setminus \mathbf{N}$$

we obviously obtain

Corollary 33.3.1. *It holds*

(B) $\mathbf{G}(\mathsf{I}) = \mathbf{G} = \{\mathsf{t}_1, \dots, \mathsf{t}_r\}, \mathsf{t}_1 < \mathsf{t}_2 < \dots < \mathsf{t}_r$;
(C) $\mathbf{B}(\mathsf{I}) = \mathbf{B}$.

$\boxed{\odot}$

Let us extend the ordering of X to $\mathbf{N} = \{\tau_1, \dots, \tau_s\}$ enumerating it so that $\tau_\sigma = \Phi(\mathsf{x}_\sigma)$, for each σ and let us denote the ordering of X and $\mathbf{N}$ by $\prec$ so that

$$\text{for each } \alpha, \beta, \tau_\alpha \prec \tau_\beta, \mathsf{x}_\alpha \prec \mathsf{x}_\beta \iff \alpha < \beta.$$

Denote for each $\tau \in \mathbf{N}$

- $\mathfrak{X}(\tau) := \{\mathsf{x} \in \mathsf{X} : \mathsf{x} \prec \Phi^{-1}(\tau)\} = \{\mathsf{x} \in \mathsf{X} : \Phi(\mathsf{x}) \prec \tau\}$,
- $\mathbb{L}(\tau) := \{\lambda_j : \mathsf{x}_j \in \mathfrak{X}(\tau)\}$,
- $\mathsf{I}(\mathfrak{X}(\tau)) := \mathfrak{P}(\mathrm{Span}_k(\mathbb{L}(\tau)))$,

and, for each $\tau \in \mathbf{N} \cup \mathbf{B}$:

- $\mathfrak{N}(\tau) := \{\omega \in \mathbf{N} : \omega \prec \tau\}$,
- $\mathfrak{M}_m(\tau) := \{\omega \in \mathsf{M}_m : \omega \prec \tau\}$,

so that

Corollary 33.3.2. *It holds*

(D) *For each $\tau \in \mathbf{N}$ there is a unique polynomial*

$$f_\tau := \tau - \sum_{\omega \in \mathfrak{N}(\tau)} c(f_\tau, \omega)\omega$$

such that $\lambda(f_\tau) = 0$, for each $\lambda \in \mathbb{L}(\tau)$.

(E) *For each $\tau \in \mathbf{G}$ there is a unique polynomial*

$$f_\tau := \tau - \sum_{\omega \in \mathbf{N}} c(f_\tau, \omega)\omega$$

such that $\lambda(f_\tau) = 0$, for each $\lambda \in \mathbb{L}$.

Proof. Since $\#\mathbb{L}(\tau) = \#\mathfrak{X}(\tau) = \#\mathfrak{N}(\tau)$ and $\#\mathbb{L} = \#\mathsf{X} = \#\mathbf{N}$, f_τ can be computed by interpolation. $\boxed{\odot}$

In the same mood, but interpolation is not sufficient to prove it, we can state

Fact 33.3.2. *It holds*

(F) *For each $\tau \in \mathbf{B}$ there is a polynomial*

$$f_\tau := \tau - \sum_{\omega \in \mathfrak{N}(\tau)} c(f_\tau, \omega)\omega$$

such that $\lambda(f_\tau) = 0$, for each $\lambda \in \mathbb{L}$.

$\boxed{\odot}$

Corollary 33.3.3. *It holds:*

(G) *The reduced Gröbner basis of I is*

$$\mathcal{G}(\mathsf{I}) := \{f_\tau : \tau \in \mathbf{G}\};$$

moreover, for each $\tau \in \mathbf{N}$, $\mathbf{T}(f_\tau) = \tau$.

(H) *The border basis of I is*

$$\mathcal{B}(\mathsf{I}) := \{f_\tau : \tau \in \mathbf{B}\}.$$

Proof. For each $\tau \in \mathbf{G} \cup \mathbf{B}$, τ is the only term in f_τ which is not a member of $\mathbf{N}$ so that $\mathbf{T}(f_\tau) = \tau$.

For any $\tau \in \mathbf{N}$, $\mathbf{T}(f_\tau) = \tau$ because Cerlienco–Mureddu Correspondence grants $\tau \in \mathbf{G}(\mathsf{I}(\mathfrak{X}(\tau)))$ and $\mathbf{N}(\mathsf{I}(\mathfrak{X}(\tau))) = \mathfrak{N}(\tau)$. $\boxed{\odot}$

Linear interpolation, again, is all one needs to prove

Proposition 33.3.1. *With the same notation as in Lemma 33.2.3, it holds*

(U) *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{G}$, *and each* $m, 1 \leq m \leq n$, *there are polynomials*

$$g_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathsf{M}_m(\tau)} c(g_{m\tau}, \omega)\omega$$

such that $g_{m\tau}(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_m(\tau)$*;*

(T) *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{N}$ *and each* $m, 1 \leq m \leq n$, *there are polynomials*

$$g_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathfrak{M}_m(\tau)} c(g_{m\tau}, \omega)\omega$$

such that $g_{m\tau}(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_m(\tau), \mathsf{a} \prec \Phi^{-1}(\tau)$.

Proof.

(U) Since $\#(\mathsf{C}_m(\tau)) \leq \#(\mathsf{M}_m(\tau))$, we can evaluate each $c(g_{m\tau}, \omega)$ by interpolation, so that $g_{m\tau}(\mathsf{b}) = 0$, for each $\mathsf{b} \in \mathsf{C}_m(\tau)$ and $g_{m\tau}(\mathsf{a}) = g_{m\tau}(\pi_m(\mathsf{a}))$, for each $\mathsf{a} \in \mathsf{D}_m(\tau)$.

(T) One has just to apply **(U)** to the set $\mathfrak{X}(\tau)$.

⊡

For each $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{N}$, let us denote $\nu := \nu(\tau) \leq n$ the value such that $d_\nu \neq 0$ while $d_\mu = 0$ for each $\mu > \nu$ so that $\tau \in \mathcal{T}[1, \nu]$, $g_{m\tau} = 1$ for $m > \nu$, and, denoting

$$\begin{aligned} h_\tau &:= \prod_{m=1}^{n} g_{m\tau} \in k[X_1, \ldots, X_{\nu-1}][X_\nu], \\ l_\tau &:= \prod_{m=1}^{\nu(\tau)-1} g_{m\tau} \in k[X_1, \ldots, X_{\nu-1}], \\ p_\tau &:= g_{\nu\tau} \in k[X_1, \ldots, X_{\nu-1}][X_\nu], \end{aligned}$$

it holds

$$h_\tau = l_\tau p_\tau = l_\tau X_\nu^{d_\nu} + \cdots$$

so that $l_\tau \in k[X_1, \ldots, X_{\nu-1}]$ is the leading polynomial and the content of h_τ while the monic polynomial p_τ is the primitive component of h_τ.

Therefore we have[4]

Corollary 33.3.4. *With the notation above, under the assumption that* I *is radical, it holds*

[4] This justifies why we need to require that I is radical: in this restricted setting, each functional λ_i is evaluation at a point and distributes with product.

(W) *for each* $\tau = X_1^{d_1} \cdots X_\nu^{d_\nu} \in \mathbf{N}$, *there are*

$$l_\tau \in k[X_1, \dots, X_{\nu-1}]$$

and a monic polynomial

$$p_\tau = X_\nu^{d_\nu} + \sum_{\omega \in \mathfrak{M}_\nu(\tau)} c(p_\tau, \omega)\omega \in k[X_1, \dots, X_{\nu-1}][X_\nu]$$

so that $h_\tau := l_\tau p_\tau$ *are such that*

- $\mathbf{T}(h_\tau) = \tau$,
- $l_\tau(\pi_{\nu-1}(\mathsf{a})) = 0$, *for all* $\mathsf{a} \in \mathfrak{X}(\tau)$,
- $p_\tau(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_\nu(\tau)$,
- $h_\tau(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{X}$ *such that* $\mathsf{a} \prec \Phi^{-1}(\tau)$.

(X) *for each* $i, 1 \leq i \leq r$ *there are*

$$l_i \in k[X_1, \dots, X_{\nu-1}]$$

and a monic polynomial

$$p_i = X_\nu^{d_\nu} + \sum_{\omega \in \mathsf{M}_\nu(\mathsf{t}_i)} c(p_i, \omega)\omega \in k[X_1, \dots, X_{\nu-1}][X_\nu]$$

so that $h_i := l_i p_i$ *are such that*

- $\mathbf{T}(h_i) = \mathsf{t}_i = X_1^{d_1} \cdots X_\nu^{d_\nu} \in \mathbf{G} \cap \mathcal{T}[1, \nu]$,
- $l_i(\pi_{\nu-1}(\mathsf{a})) = 0$, *for each* $\mathsf{a} \in \cup_{m=1}^{\nu-1} \mathsf{D}_m(\mathsf{t}_i)$,
- $p_i(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_\nu(\mathsf{t}_i)$,
- $h_i(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{X}$.

$\boxed{\odot}$

While $\#(\mathsf{C}_m(\tau)) \leq \#(\mathsf{M}_m(\tau))$, in general equality does not hold and the polynomials $g_{m\tau}$ are not unique. However, uniqueness can be forced via Cerlienco–Mureddu Corespondence in such a way that the result does not require the assumption that I is radical.

We begin by remark that, however, $\#(\mathsf{C}_1(\tau)) = \#(\mathsf{M}_1(\tau))$ so that $g_{1\tau}$ is unique. We therefore begin our construction by setting $\gamma_{1\tau} := g_{1\tau}$ and, inductively, for $m, 1 < m \leq n$,

- $\zeta_{m\tau} := \prod_{\nu=1}^{m-1} \gamma_{\nu\tau}$,
- $\mathsf{Q}_m(\tau) := \{M(\omega)\lambda_{\mathsf{a}} : \omega \in \mathcal{T}[1, m-1], \mathsf{a} \in \mathsf{Z} := \mathcal{Z}(\mathsf{I}), M(\omega)\lambda_{\mathsf{a}}(\zeta_{m\tau}) \neq 0\}$,
- $\mathsf{P}_m(\tau) := \{M\left(\pi_m\left(\frac{\upsilon_i}{\omega}\right)\right)\lambda_{\mathsf{a}_i} : M(\upsilon_i))\lambda_{\mathsf{a}_i} \in \mathsf{L}_m(\tau), M(\omega)\lambda_{\mathsf{a}_i} \in \mathsf{Q}_m(\tau)\}$,
- $\mathsf{R}_m(\tau) := \{\left(\pi_m(\mathsf{a}_i), \pi_m\left(\frac{\upsilon_i}{\omega}\right)\right) : M\left(\pi_m\left(\frac{\upsilon_i}{\omega}\right)\right)\lambda_{\mathsf{a}_i} \in \mathsf{P}_m(\tau)\}$,
- $\mathsf{E}_m(\tau) := \mathbf{N}(\mathsf{R}_m(\tau))$,
- $\mathsf{S}_m(\tau) := \{\left(\pi_m(\mathsf{a}_i), \pi_m\left(\frac{\upsilon_i}{\omega}\right)\right) \in \mathsf{R}_m(\tau)(\mathsf{a}_i, \upsilon_i) \prec \Phi^{-1}(\tau)\}$,
- $\mathsf{F}_m(\tau) := \mathbf{N}(\mathsf{S}_m(\tau))$.

Then:

Corollary 33.3.5. *With this notation it holds*

(L) *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{G}$, *and each* $m, 1 \le m \le n$ *there are unique polynomials*

$$\gamma_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathsf{E}_m(\tau)} c(\gamma_{m\tau}, \omega)\omega$$

such that $\pi_m(\lambda_i)(\gamma_{m\tau}) = 0$, *for each* $\mathsf{x}_i \in \mathsf{D}_m(\tau)$*;*

(I) *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{N}$, *and each* $m, 1 \le m \le n$ *there are unique polynomials*

$$\gamma_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathsf{F}_m(\tau)} c(\gamma_{m\tau}, \omega)\omega$$

such that $\pi_m(\lambda_i)(\gamma_{m\tau}) = 0$, *for each* $\mathsf{x}_i \in \mathsf{D}_m(\tau), \mathsf{x}_i \prec \Phi^{-1}(\tau)$;

(M) *for each* $\tau = X_1^{d_1} \cdots X_\nu^{d_\nu} \in \mathbf{N}$, *there are*

$$L_\tau \in k[X_1, \ldots, X_{\nu-1}]$$

and a unique monic polynomial

$$P_\tau = X_\nu^{d_\nu} + \sum_{\omega \in \mathsf{F}_\nu(\tau)} c(P_\tau, \omega)\omega \in k[X_1, \ldots, X_{\nu-1}][X_\nu]$$

so that $H_\tau := L_\tau P_\tau$ *are such that*

- $\mathbf{T}(H_\tau) = \tau$, $\mathrm{Lp}(H_\tau) = L_\tau$,
- $\pi_{\nu-1}(\lambda)(L_\tau) = 0$, *for each* $\lambda \in \mathbb{L}(\tau)$,
- $\pi_\nu(\lambda)(P_\tau) = 0$, *for each* $\lambda \in \mathsf{L}_\nu(\tau)$,
- $\pi_\nu(\lambda_i)(H_\tau) = 0$, *for each* $\lambda_i \in \mathbb{L} : \mathsf{x}_i \prec \Phi^{-1}(\tau)$.

(N) *for each* $i, 1 \le i \le r$ *there are*

$$L_i \in k[X_1, \ldots, X_{\nu-1}]$$

and a unique monic polynomial

$$P_i = X_\nu^{d_\nu} + \sum_{\omega \in \mathsf{E}_\nu(\mathsf{t}_i)} c(P_i, \omega)\omega \in k[X_1, \ldots, X_{\nu-1}][X_\nu]$$

so that $H_i := L_i P_i$ *are such that*

- $\mathbf{T}(H_i) = \mathsf{t}_i = X_1^{d_1} \cdots X_\nu^{d_\nu} \in \mathbf{G} \cap \mathcal{T}[1, \nu]$, $\mathrm{Lp}(H_i) = L_i$,
- $\pi_{\nu-1}(\lambda)(L_i) = 0$, *for each* $\lambda \in \cup_{m=1}^{\nu-1} \mathsf{L}_m(\mathsf{t}_i)$,
- $\pi_\nu(\lambda)(P_i) = 0$, *for each* $\lambda \in \mathsf{L}_\nu(\mathsf{t}_i)$,
- $\pi_\nu(\lambda)(H_i) = 0$, *for each* $\lambda_i \in \mathbb{L}$.

Proof. The only non trivial statements, i.e. the vanishing of $\pi_{\nu-1}(\lambda)(L)$ and $\pi_\nu(\lambda)(H)$ are an elementary consequence of Leibniz Formula (Proposition 31.4.1). $\boxed{\odot}$

Corollary 33.3.6. *It holds*

(O) $\{H_1, \ldots, H_r\}$ *is a minimal Gröbner basis of* I*;*

(Q) *For each $\nu, 1 \le \nu < n$, and each $\delta \in \mathbb{N}$ let $j(\nu\delta)$ the value such that $\mathsf{t}_{j(\nu\delta)} < X_{\nu+1}^{\delta} \le \mathsf{t}_{j(\nu\delta)+1}$; then $\{L_1, \dots, L_{j_{\nu\delta})}\}$ is a Gröbner basis of $\mathsf{I}(\mathsf{Y}_{\nu\delta})$;*

(P) *For each $\nu, 1 \le \nu < n$ let j_ν the value such that $\mathsf{t}_{j_\nu} < X_{\nu+1} \le \mathsf{t}_{j_\nu+1}$; then $\{H_1, \dots, H_{j_\nu}\}$ is a minimal Gröbner basis of $\mathsf{I} \cap k[X_1, \dots, X_\nu]$ and of $\mathsf{I}(\pi_\nu(\mathsf{X}))$.*

$\odot$

Proof. **(O)** is obvious;
(Q) is a direct application of **(O)** to the set of points $\mathsf{Y}_{\nu\delta}$ via Remark 33.2.1
(P) is a particular instance of **(Q)**; minimality is trivial.

$\odot$

Clearly, if I is radical similar statements hold for

$$\{h_1, \dots, h_r\}, \{l_1, \dots, l_{j_{\nu\delta})}\} \text{ and } \{h_1, \dots, h_{j_\nu}\}.$$

Remark 33.3.1. The only difference between the three bases

$$\{f_1, \dots, f_r\}, \{h_1, \dots, h_r\} \text{ and } \{H_1, \dots, H_r\}$$

is that the first is reduced unlike the others. On the other side, for each i, we have

$$\mathbf{T}(f_i) = \mathbf{T}(h_i) = \mathbf{T}(H_i) = \mathsf{t}_i.$$

Therefore we have

- $f_1 = h_1 = H_1$ and
- $f_i - h_i \in (h_1, \dots, h_{i-1}), f_i - H_i \in (H_1, \dots, H_{i-1})$ for each $i, 1 < i \le r$,

whence

- $f_i \in (h_1, \dots, h_i), f_i \in (H_1, \dots, H_i)$ for each $i, 1 \le i \le r$.

Fact 33.3.3. *It holds*

(R) *For each $i, 2 \le i \le r$, $P_i \in (H_j, j < i) : L_i$.*

$\odot$

Fact 33.3.4. *It holds*

(S) *for each $j, 1 \le j \le s$, $\lambda_j(f_{\tau_j}) \ne 0$ and $\lambda_j(H_{\tau_j}) \ne 0$ so that $\mathbb{L}(\mathsf{I})$ is triangular to $\{\lambda_j(f_{\tau_j})^{-1} f_{\tau_j}, 1 \le j \le s\}$ and $\{\lambda_j(H_{\tau_j})^{-1} H_{\tau_j}, 1 \le j \le s\}$.*

$\odot$

Corollary 33.3.7. *If I is radical, moreover*

(Z) *l_i, p_i, h_i, $1 \le i \le r$ satisfy*
$\{h_1, \dots, h_r\}$ is a minimal Gröbner basis of I;

for each $\nu, 1 \le \nu < n$, $\{h_1, \dots, h_{j_\nu}\}$ *is a minimal Gröbner basis of* $\mathsf{I} \cap k[X_1, \dots, X_\nu]$ *and of* $\mathsf{I}(\pi_\nu(\mathsf{X}))$;
for each $\nu, 1 \le \nu < n$, $\{l_1, \dots, l_{j_{\nu\delta}}\}$ *is a Gröbner basis of* $\mathsf{I}(\mathsf{Y}_{\nu\delta})$;
for each $i, 2 \le i \le r$, $p_i \in (h_j, j < i) : l_i$;
for each $j, 1 \le j \le s$, $\lambda_j(h_{\tau_j}) \ne 0$;
$\mathbb{L}(\mathsf{I})$ *is triangular to* $\{\lambda_j(h_{\tau_j})^{-1} h_{\tau_j}, 1 \le j \le s\}$.

⊙

The construction which led to Corollary 33.3.5 can be refined as follows: for each $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{G}$, for each $\nu \le n$, iteratively for increasing $\delta \le d_\nu$, with initial value $\mathsf{P}_{\nu d_n+1}(\tau) := \mathsf{P}_{\nu-1} := \mathsf{P}_{\nu-12}$ we compute

$$\begin{aligned}
\mathsf{Y}_{\nu\delta}(\tau) &:= \{\pi_\nu(\mathsf{x}) : \exists \omega \in \mathcal{T}[1,\nu] : \Phi(\mathsf{x}) = \omega X_{\nu+1}^{\delta}, \mathsf{x} \in \mathsf{P}_{\nu\delta+1}(\tau)\} \\
\mathsf{E}_{\nu\delta}(\tau) &:= \mathsf{N}(\mathsf{Y}_{\nu\delta}(\tau)) \\
\mathsf{P}_{\nu\delta}(\tau) &:= \{M\left(\pi_\nu\left(\frac{\upsilon_i}{\omega}\right)\right)\lambda_{\mathsf{a}_i} : M(\upsilon_i))\lambda_{\mathsf{a}_i} \in \mathsf{L}_\nu(\tau), M(\omega)\lambda_{\mathsf{a}_i} \in \mathsf{Y}_{\nu\delta}(\tau)\},
\end{aligned}$$

so that

Corollary 33.3.8. *For each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{G}$, *each* $m, 1 \le m \le n$, *and each* $\delta \le d_m$ *there is a unique polynomial*

$$\gamma_{m\delta\tau} := X_m + \sum_{\omega \in \mathsf{E}_{\nu\delta}(\tau)} c(\gamma_{m\tau}, \omega)\omega$$

such that $\pi_m(\lambda_i)(\gamma_{m\tau}) = 0$, *for each* $\lambda_i \in \mathsf{Y}_{\nu\delta}(\tau)$.

Also $\gamma_{m\tau} = \prod_\delta \gamma_{m\delta\tau}$.

⊙

33.4 Some examples

Example 33.4.1. Let us consider the set Y introduced in Example 33.2.1.

A direct application of the Algorithm of Figure 28.1 returns

(0,0) $t_1 := 1$,
$G_1 := \{X_1, X_2\}$;
(0,1) $t_2 = X_2$,
$G_2 = \{X_1, X_2^2 - X_2\}$;
(2,0) $t_3 := X_1$,
$G_3 = \{X_1^2 - 2X_1, X_1X_2, X_2^2 - X_2\}$;
(0,2) $t_4 = X_2^2$,
$G_4 = \{X_1^2 - 2X_1, X_1X_2, X_2^3 - 3X_2^2 + 2X_2\}$;
(1,0) $t_5 = X_1^2$,
$G_5 = \{X_1^3 - 3X_1^2 + 2X_1, X_1X_2, X_2^3 - 3X_2^2 + 2X_2\}$;
(1,1) $t_6 = X_1X_2$,
$G_6 = \{X_1^3 - 3X_1^2 + 2X_1, X_1^2X_2 - X_1X_2, X_1X_2^2 - X_1X_2, X_2^3 - 3X_2^2 + 2X_2\}$.

Remark that we have

$$\begin{array}{rcl} X_1^3 - 3X_1^2 + 2X_1 &=& (X_1-2)(X_1-1)X_1 \\ X_1^2X_2 - X_1X_2 &=& X_2(X_1-1)X_1, \\ X_1X_2^2 - X_1X_2 &=& X_2(X_2-1)X_1, \\ X_2^3 - 3X_2^2 + 2X_2 &=& X_2(X_2-1)(X_2-2), \end{array}$$

illustrating Lazard Theorem and Corollary 33.3.8. The fact that Möller's Algorithm returns Cerlienco–Mureddu Correspondence is not a coincidence. ⊙

Example 33.4.2. The result of the application of the Algorithm of Figure 28.1 to the set X of Example 33.2.2 returns, again, Cerlienco–Mureddu Correspondence and the Gröbner basis $G_6 \cup \{f_1, f_2, f_3, f_4\}$ where

$$\begin{array}{rcl} f_1 &:=& X_3X_1^2 - 3X_3X_1 + 2X_3 - 3X_2^2 - 6X_2X_1 + 9X_2 - X_1^2 + 3X_1 - 2, \\ f_2 &:=& X_3X_2 + X_3X_1 - 2X_3 + 3X_2^2 + X_2X_1 - 7X_2 - 2X_1^2 + 3X_1 + 2, \\ f_3 &:=& X_3^2X_1 - 2X_3^2 - 4X_3X_1 + 8X_3 - 15X_2^2 - 30X_2X_1 + 45X_2 + 3X_1 - 6, \\ f_4 &:=& X_3^3 - 3X_3^2 + 3X_3X_1 - 4X_3 - 3X_2^2 - 6X_2X_1 + 9X_2 - 3X_1 + 6, \end{array}$$

and (modulo $\mathsf{I}(\mathsf{Y})$)

$$\begin{array}{rcl} f_1 &=& (X_1-2)(X_1-1)(X_3 - \frac{3}{2}X_2^2 + \frac{9}{2}X_2 - 1) \\ f_2 &=& (X_2+X_1-2)(X_3+3X_2-2X_1-1) \\ f_3 &=& (X_1-2)(X_3-1)(X_3-5X_1+2) \\ f_4 &=& (X_3-1)X_3(X_3+3X_1^2-8X_1+2) \end{array}$$

where

- $(X_1^2 - 3X_1 + 2, X_2 + X_1 - 2, X_3 - 1)$ is the Gröbner basis of the ideal whose roots are $\{\pi_2(\mathsf{b}_7), \pi_2(\mathsf{b}_8)\}$,
- $\{\mathsf{b} \in \mathsf{X} : (X_1^2 - 3X_1 + 2)(\mathsf{b}) \neq 0\} = \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_4\}$ to which Cerlienco–Mureddu Correspondence associates $\{1, X_2, X_2^2\}$
- $\{\mathsf{b} \in \mathsf{X} : (X_2 + X_1 - 2)(\mathsf{b}) \neq 0\} = \{\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_5\}$ to which Cerlienco–Mureddu Correspondence associates $\{1, X_1, X_2\}$
- $\{\mathsf{b} \in \mathsf{X} : (X_1 - 2)(X_3 - 1)(\mathsf{b}) \neq 0\} = \{\mathsf{b}_2, \mathsf{b}_4, \mathsf{b}_5, \mathsf{b}_6\}$ to which Cerlienco–Mureddu Correspondence associates $\{1, X_1, X_2, X_1X_2\}$.
- $\{\mathsf{b} \in \mathsf{X} : (X_3^2 - X_3))(\mathsf{b}) \neq 0\} = \{\mathsf{b}_2, \mathsf{b}_3, \mathsf{b}_4, \mathsf{b}_5, \mathsf{b}_6\}$ to which Cerlienco–Mureddu Correspondence associates $\{1, X_1, X_1^2, X_2, X_1X_2\}$.

⊙

Example 33.4.3. Let us set $\mathsf{a} := (0,0,0)$, $\mathsf{b} := (1,0,1)$, $\mathsf{c} := (0,-1,-1)$,

$$\begin{array}{rcl} \lambda_{\mathsf{a}}(\mathsf{q}_{\mathsf{a}}) & := & (X_1^4, X_1X_2^2, X_1^2X_2, X_1X_3, X_2X_3, X_3^2) \\ \lambda_{\mathsf{b}}(\mathsf{q}_{\mathsf{b}}) & := & (X_1, X_2^3, X_1X_3, X_3^2) \\ \lambda_{\mathsf{c}}(\mathsf{q}_{\mathsf{c}}) & := & (X_1, X_2^2, X_3^2), \\ \mathsf{I} & := & \mathsf{q}_{\mathsf{a}} \cup \mathsf{q}_{\mathsf{b}} \cup \mathsf{q}_{\mathsf{c}}. \end{array}$$

so that $s := \deg(\mathsf{I}) = 8 + 4 + 4 = 16$.

In the table below we properly list the sets $\mathsf{X}(\mathsf{I})$, $\mathbb{L}(\mathsf{I})$ and the result $\mathbf{N}(\mathsf{X})$ of Cerlienco–Mureddu Correspondence.

i	1	2	3	4	5	6	7	8
a_i	a	a	a	a	a	a	a	a
υ_i	1	X_1	X_2	X_3	X_1^2	X_1X_2	X_2^2	X_1^3
$\Phi(\lambda_i)$	1	X_1	X_2	X_3	X_1^2	X_1X_2	X_2^2	X_1^3
i	9	10	11	12	13	14	15	16
a_i	b	b	b	b	c	c	c	c
υ_i	1	X_2	X_3	X_2^2	1	X_2	X_3	X_2X_3
$\Phi(\lambda_i)$	X_1^4	$X_1^2X_2$	X_1X_3	$X_1X_2^2$	X_2^3	X_2^4	X_2X_3	$X_2^2X_3$

The lex reduced Gröbner basis of I is $\mathcal{G}(\mathsf{I}) = \{f_i, 1 \le i \le 9\}$ where

$$\begin{array}{rcl} f_1 & := & X_1^5 - X_1^4 \\ f_2 & := & X_1^3X_2 - X_1^2X_2 \\ f_3 & := & X_1^2X_2^2 - X_1X_2^2 \\ f_4 & := & X_1X_2^3 \\ f_5 & := & X_2^5 + 2X_2^4 + X_2^3 \\ f_6 & := & X_1^2X_3 - X_1X_3 \\ f_7 & := & X_1X_2X_3 - X_1^2X_2 \\ f_8 & := & X_2^3X_3 + 2X_2^2X_3 + X_2X_3 - 2X_1X_2^2 - X_1^2X_2 \\ f_9 & := & X_3^2 - 2X_2^2X_3 - 4X_2X_3 - 2X_1X_3 - 3X_2^4 + 2X_1X_2^2 + 4X_1^2X_2 + X_1^4 \end{array}$$

and we have the following factorization of each f_i modulo $(f_1, \ldots, f_{i-1})$:

$$\begin{array}{rcl} f_1 & = & X_1^4(X_1 - 1) \\ f_2 & = & X_1^2(X_1 - 1)X_2 \\ f_3 & = & X_1(X_1 - 1)X_2^2 \\ f_4 & = & X_1X_2^3 \\ f_5 & = & X_2^3(X_2 + 1)^2 \\ f_6 & = & X_1(X_1 - 1)X_3 \\ f_7 & = & X_1X_2(X_3 - X_2) \\ f_8 & \equiv & X_2(X_2 + 1)^2(X_3 - X_1^2) \\ f_9 & \equiv & (X_3 - X_1^2 - 2X_2 - X_2^2)(X_3 + 3X_2^2 + 2X_2^3 - X_1^2). \end{array}$$

Remark that for

$$\begin{array}{ll} f_2 & \mathsf{Q}_2(\mathsf{t}_2) = \{M(X_1^2)\lambda_{\mathsf{a}}, M(X_1)\lambda_{\mathsf{b}}, M(X_1^2)\lambda_{\mathsf{c}}\}, \\ & \mathsf{L}_2(\mathsf{t}_2) = \{\lambda_5, \lambda_8\}, \\ & \mathsf{P}_2(\mathsf{t}_2) = \{\lambda_1, \lambda_2\}, \end{array}$$

$\mathsf{E}_2(\mathsf{t}_2) = \{1, X_1\};$

f_3 $\mathsf{Q}_2(\mathsf{t}_3) = \{M(X_1)\lambda_\mathsf{a}, M(X_1)\lambda_\mathsf{b}, M(X_1)\lambda_\mathsf{c}\},$
$\mathsf{L}_2(\mathsf{t}_3) = \{\lambda_2, \lambda_5, \lambda_8\},$
$\mathsf{P}_2(\mathsf{t}_3) = \{\lambda_1, \lambda_2, \lambda_5, \lambda_3\},$
$\mathsf{E}_2(\mathsf{t}_3) = \{1, X_1, X_1^2, X_2\};$

f_4 $\mathsf{Q}_2(\mathsf{t}_4) = \{M(X_1)\lambda_\mathsf{a}, M(1)\lambda_\mathsf{b}, M(X_1)\lambda_\mathsf{c}\},$
$\mathsf{L}_2(\mathsf{t}_4) = \{\lambda_2, \lambda_5, \lambda_8, \lambda_9\},$
$\mathsf{P}_2(\mathsf{t}_4) = \{\lambda_1, \lambda_2, \lambda_5, \lambda_3, \lambda_9, \lambda_{10}, \lambda_{12}\},$
$\mathsf{E}_2(\mathsf{t}_4) = \{1, X_1, X_1^2, X_1^3, X_2, X_1X_2, X_2^2\};$

f_5 $\mathsf{R}_2(\mathsf{t}_5) = \{\lambda_1, \lambda_3, \lambda_7, \lambda_{13}, \lambda_{15}\},$

f_6 $\mathsf{Q}_3(\mathsf{t}_6) = \{M(X_1)\lambda_\mathsf{a}, M(X_1)\lambda_\mathsf{b}, M(X_1)\lambda_\mathsf{c}\},$
$\mathsf{L}_3(\mathsf{t}_6) = \{\lambda_2, \lambda_5, \lambda_6, \lambda_8\},$
$\mathsf{P}_3(\mathsf{t}_6) = \{\lambda_1, \lambda_2, \lambda_5, \lambda_3\},$
$\mathsf{E}_3(\mathsf{t}_6) = \{1, X_1, X_1^2, X_2\};$

f_7 $\mathsf{Q}_2(\mathsf{t}_7) = \{M(X_1)\lambda_\mathsf{a}, M(1)\lambda_\mathsf{b}, M(X_1)\lambda_\mathsf{c}\},$
$\mathsf{L}_2(\mathsf{t}_7) = \{\lambda_2, \lambda_5, \lambda_8, \lambda_9\},$
$\mathsf{P}_2(\mathsf{t}_7) = \{\lambda_1, \lambda_2, \lambda_5\},$
$\mathsf{E}_2(\mathsf{t}_7) = \{1, X_1, X_1^2\};$
$\mathsf{Q}_3(\mathsf{t}_7) = \{M(X_1X_2)\lambda_\mathsf{a}, M(X_2)\lambda_\mathsf{b}, M(X_1)\lambda_\mathsf{c}\},$
$\mathsf{L}_3(\mathsf{t}_7) = \{\lambda_6, \lambda_{10}, \lambda_{12}\},$
$\mathsf{P}_3(\mathsf{t}_7) = \{\lambda_1, \lambda_9, \lambda_{10}\},$
$\mathsf{E}_3(\mathsf{t}_7) = \{1, X_1, X_2\};$

f_8 $\mathsf{Q}_2(\mathsf{t}_8) = \{M(1)\lambda_\mathsf{a}, M(1)\lambda_\mathsf{b}, M(1)\lambda_\mathsf{c}\},$
$\mathsf{L}_2(\mathsf{t}_8) = \{\lambda_1, \lambda_{13}, \lambda_{14}\},$
$\mathsf{P}_2(\mathsf{t}_8) = \{\lambda_1, \lambda_{13}, \lambda_{14}, \},$
$\mathsf{E}_2(\mathsf{t}_8) = \{1, X_2, X_2^2\};$
$\mathsf{Q}_3(\mathsf{t}_8) = \{M(X_2)\lambda_\mathsf{a}, M(X_2)\lambda_\mathsf{b}, M(X_2^2)\lambda_\mathsf{c}\},$
$\mathsf{L}_3(\mathsf{t}_8) = \{\lambda_2, \lambda_3, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{12}\},$
$\mathsf{P}_3(\mathsf{t}_8) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_9, \lambda_{10}\},$
$\mathsf{E}_3(\mathsf{t}_8) = \{1, X_1, X_2, X_1^2, X_1X_2\}$

f_9 $\mathsf{P}_3(\mathsf{t}_9) = \{\lambda_i, i \leq 16\}\},$
$\mathsf{Y}_{32}(\mathsf{t}_9) = \{\lambda_1, \lambda_{19}, \lambda_{13}, \lambda_{14}\},$
$\mathsf{E}_{32}(\mathsf{t}_9) = \{1, X_1, X_2, X_2^2\},$
$\gamma_{32\mathsf{t}_9} = X_3 - X_1 - 2X_2 - X_2^2,$
$\mathsf{P}_{32}(\mathsf{t}_9) = \{\lambda_i, i \in \{1, 2, 3, 5, 9, 10, 13, 14\}\},$
$\mathsf{Y}_{31}(\mathsf{t}_9) = \{\lambda_i, i \in \{1, 2, 3, 5, 9, 10, 13, 14\}\},$
$\mathsf{E}_{31}(\mathsf{t}_9) = \{1, X_1, X_2, X_1^2, X_1X_2, X_2^2, X_1^3, X_2^3\},$
$\gamma_{31\mathsf{t}_9} = X_3 - X_1^3 + 3X_2^2 + 2X_2^3,$

and that each factor is obtained by interpolation as stated in Corollary 33.3.5.

Example 33.4.4. If, in the example above, we now add, where $\mathsf{d} = (1, 0, 0)$,

$$\begin{array}{lllll} \lambda_{17} & := & M(X_3^2)\lambda_\mathsf{a} & \Phi(\lambda_{17}) & = & X_3 \\ \lambda_{18} & := & M(1)\lambda_\mathsf{d} & \Phi(\lambda_{18}) & = & X_1X_3 \end{array}$$

the corresponding lex reduced Gröbner basis is

$$\{f_i, 1 \le i \le 8\} \cup \{f_{10}, f_{11}\}$$

where

$$\begin{aligned} f_{10} &:= X_2X_3^2 + 2X_2X_3 + 2X_2^4 + 3X_2^3 - 3X_1^2X_2 \\ &\equiv X_2(X_3 - 1 - 4X_2 - 2X_2^2)(X_3 - X_1^2 + 3X_2^2 + 2X_2^3); \\ f_{11} &:= X_3^3 - 2X_1X_3^2 + 3X_2^2X_3 + 6X_2X_3 + X_1X_3 \\ &\equiv X_3(X_3 - X_1 - 2X_2 - X_2^2)(X_3 - X_1^2 + 3X_2^2 + 2X_2^3). \end{aligned}$$

The factorization is justified by

f_{10}
$\mathsf{Q}_2(\mathsf{t}_{10}) = \{M(1)\lambda_{\mathsf{a}}, M(1)\lambda_{\mathsf{b}}, M(1)\lambda_{\mathsf{c}}, M(1)\lambda_{\mathsf{d}}\},$
$\mathsf{L}_2(\mathsf{t}_{10}) = \{\lambda_1\},$
$\mathsf{P}_2(\mathsf{t}_{10}) = \{\lambda_1\},$
$\mathsf{E}_2(\mathsf{t}_{10}) = \{1\},$
$\gamma_{2\mathsf{t}_{10}} = X_2;$
$\mathsf{Q}_3(\mathsf{t}_{10}) = \{M(X_2)\lambda_{\mathsf{a}}, M(1)\lambda_{\mathsf{b}}, M(1)\lambda_{\mathsf{c}}, M(1)\lambda_{\mathsf{d}}\},$
$\mathsf{L}_3(\mathsf{t}_{10}) = \{\lambda_i, i \le 18, 1 \ne i \ne 4\},$
$\mathsf{P}_3(\mathsf{t}_{10}) = \{\lambda_i, i \notin \{4,5,6,7,8,18\}\},$
$\mathsf{Y}_{32}(\mathsf{t}_{10}) = \{\lambda_9, \lambda_{13}, \lambda_{14}, \lambda_{18}\},$
$\mathsf{E}_{32}(\mathsf{t}_{10}) = \{1, X_1, X_2, X_2^2\},$
$\gamma_{32\mathsf{t}_{10}} = X_3 - 1 - 4X_2 - 2X_2^2,$
$\mathsf{P}_{32}(\mathsf{t}_{10}) = \{\lambda_i, i \in \{1,2,3,9,10,13,14\}\},$
$\mathsf{Y}_{31}(\mathsf{t}_{10}) = \{\lambda_i, i \in \{1,2,3,9,10,13,14\}\},$
$\mathsf{E}_{31}(\mathsf{t}_{10}) = \{1, X_1, X_2, X_1^2, X_1X_2, X_2^2, X_2^3\},$
$\gamma_{31\mathsf{t}_{10}} = X_3 - X_1^2 + 3X_2^2 + 2X_2^3,$

f_{11}
$\mathsf{P}_3(\mathsf{t}_{11}) = \{\lambda_i, i \le 18\}\},$
$\mathsf{Y}_{33}(\mathsf{t}_{11}) = \{\lambda_1, \lambda_{18}\},$
$\mathsf{E}_{33}(\mathsf{t}_{11}) = \{1, X_1, \},$
$\gamma_{33\mathsf{t}_{11}} = X_3,$
$\mathsf{P}_{33}(\mathsf{t}_{11}) = \{\lambda_i, i \notin \{6,7,8\}\},$
$\mathsf{Y}_{32}(\mathsf{t}_{11}) = \{\lambda_1, \lambda_9, \lambda_{13}, \lambda_{14}\},$
$\mathsf{E}_{32}(\mathsf{t}_{11}) = \{1, X_1, X_2, X_2^2\},$
$\gamma_{32\mathsf{t}_{11}} = X_3 - X_1 - 2X_2 - X_2^2,$
$\mathsf{P}_{32}(\mathsf{t}_{11}) = \{\lambda_i, i \in \{1,2,3,9,10,13,14\}\},$
$\mathsf{Y}_{31}(\mathsf{t}_{11}) = \{\lambda_i, i \in \{1,2,3,9,10,13,14\}\},$
$\mathsf{E}_{31}(\mathsf{t}_{11}) = \{1, X_1, X_2, X_1^2, X_1X_2, X_2^2, X_2^3\},$
$\gamma_{31\mathsf{t}_{11}} = X_3 - X_1^2 + 3X_2^2 + 2X_2^3.$

33.5 An algorithmic proof

The fact that Möller's Algorithm returns Cerlienco–Mureddu Correspondence suggests that a proof can be obtained by a directly application of it[5].

[5] of which a simplified version in this setting is presented in Figure 33.1.

Fig. 33.1. Möller Algorithm for Macaulay representation

$r := 1, \mathsf{B} := \emptyset$
$t_1 := 1, \mathbf{N} := \{t_1\}, q_1 := t_1,\ \mathbf{q} := \{q_1\},$
For $h = 1..n$ **do**
    $t := X_h, b_t := X_h - a_{h1}, \mathbf{B} := \mathbf{B} \cup \{t\}$
While $r \leq s$ **do**
    Let $t := \min_<\{t \in \mathbf{B} : \lambda_{r+1}(b_t) \neq 0\}$
    $r := r+1,\ \mathbf{B} := \mathbf{B} \setminus \{t\},$
    $t_r := t, \mathbf{N} := \mathbf{N} \cup \{t_r\},\ q_r := \lambda_r(b_t)^{-1} b_t, \mathbf{q} := \mathbf{q} \cup \{q_r\},$
    For each $\tau \in \mathbf{B}$ **do** $b_\tau := b_\tau - \lambda_r(b_\tau) q_r,$
    For $h = 1..n$ **do**
        If $X_h t_r \notin \mathbf{B}$ **then**
            $t := X_h t_r,$
            $f := X_h b_{t_r} - \sum_{\substack{\tau \in \mathbf{N} \\ X_h \tau \in \mathbf{B}}} c(b_{t_r}, \tau) b_{X_h \tau}$
            $b_t := f - \lambda_r(f) q_r$
        $\mathbf{B} := \mathbf{B} \cup \{X_h t_r, h = 1..n\}$
$\mathbf{q}, \{b_\tau : \tau \in \mathbf{B}\}$

The proof being by induction, we begin with

Lemma 33.5.1. *If* $\#\mathsf{X} = 1$ *conditions* **(A)**, **(F)**, **(W)**, **(X)** *hold.*

Proof. When we have a single point $(a_1, \ldots, a_n) \in k^n$, we have

- $\mathbf{N} = \{1\}$,
- $\mathbf{B} = \mathbf{G} = \{X_1, \ldots, X_n\}$,
- $f_1 = 1$,
- $f_{X_i} = X_i - a_i$, for each i,

and the properties are obviously satisfied. ⊙

This giving a starting point for induction, let us assume we have a Macaulay representation $\mathbb{L} := \{\lambda_1, \ldots, \lambda_s\}$ and a CM-scheme

$$\mathsf{X} := \{\mathsf{x}_1, \ldots, \mathsf{x}_s\} \subset k^n \times \mathcal{T}, \mathsf{x}_i = (\mathsf{a}_i, \upsilon_i), \mathsf{a}_i := (a_{i1}, \ldots, a_{in}), \upsilon_i = \prod_{l=1}^{n} X_l^{\alpha_{il}}$$

of a CM-ideal I, so that, for each i,

$$\lambda_i = M(\lambda) = M(\upsilon_i)\lambda_{\mathsf{a}_i},\ \text{for each } i, 1 \leq i \leq s,$$

and let us denote

$$\mathsf{X}' := \{\mathsf{x}_1, \ldots, \mathsf{x}_{s-1}\}, \mathbb{L}' := \{\lambda_1, \ldots, \lambda_{s-1}\} \text{ and } \mathsf{I}' := \mathfrak{P}(\mathrm{Span}_k(\mathbb{L}'),$$

for which we assume conditions **(A-Z)** hold.

In particular:

$\Phi' := \mathbf{N}' \mapsto \mathsf{X}'$ is Cerlienco–Mureddu Correspondence,

$\mathbf{G}' := \mathbf{G}(\mathsf{I}(\mathsf{X}')) = \{\omega_1, \ldots, \omega_r\}, \omega_1 < \omega_2 < \ldots < \omega_r$,
$\mathbf{B}' := \mathbf{B}(\mathsf{I}(\mathsf{X}'))$,
$f'_\omega, \omega \in \mathbf{B}'$, are the polynomials whose existence is implied by **(F)**,
$F_i := f'_{\omega_i}$ are the polynomials whose existence is implied by **(E)**, so that $\{F_i : 1 \leq i \leq r\}$ is the reduced Gröbner basis of $\mathsf{I}(\mathsf{X}')$;
L'_i, P'_i, H'_i are the polynomials whose existence is implied by **(N)**,
l'_i, p'_i, h'_i are the polynomials whose existence is implied by **(X)**.

Setting

$$I := \min_{<}\{j, 1 \leq j \leq r : \lambda_s(F_j) \neq 0\}$$

then it holds

Lemma 33.5.2. *If* X' *satisfies conditions* **(A-Z)** *then*

$$\Psi(\mathsf{X})(\mathsf{x}_s) = \omega_I.$$

Proof. Let $\omega_I = X_1^{d_1} \ldots X_n^{d_n}$ and let $m+1 := \max(i : d_i \neq 0)$, so that

$$F_I \in k[X_1, \ldots, X_{m+1}].$$

Since, by **(P)**, for each ν,

$$\mathsf{I}(\mathsf{X}') \cap k[X_1, \ldots, X_\nu] = \mathsf{I}(\pi_\nu(\mathsf{X}')),$$

and

$$F_j \in k[X_1, \ldots, X_\nu], \nu \leq m \implies j < I$$

we deduce that

$\pi_\nu(\lambda_s)(F_j) = \lambda_s(F_j) = 0$, for each $F_j \in k[X_1, \ldots, X_\nu], \nu \leq m$, while
$\pi_{m+1}(\lambda_s)(F_I) = \lambda_s(F_I) \neq 0$.

This allows to deduce that

$$m = \max\left(j : \text{ exists } i < s : \pi_j(\mathsf{x}_i) = \pi_j(\mathsf{x}_s)\right).$$

Therefore $\pi_{m+1}(\mathsf{x}_s) \notin \{\pi_{m+1}(\mathsf{x}), \mathsf{x} \in \mathsf{X}'\}$; also

$$d_m = \#\{\mathsf{x}_i, i < s : \pi_m(\mathsf{x}_i) = \pi_m(\mathsf{x}_s)\};$$

in fact, for each $\delta < d_m$, since

$$\mathbf{T}(F_j) = \omega_j < X_m^\delta < X_m^{d_m} \implies j < I,$$

and $\pi_m(\lambda_s)(F_j) = 0$, **(Q)** allows to deduce that

$$\pi_m(\mathsf{x}_s) \in \mathsf{Y}_{m\delta} := \left\{\mathsf{y} \in \pi_m(\mathsf{X}') : \delta < \#\left\{\mathsf{x} \in \mathsf{X}' : \mathsf{y} = \pi_m(\mathsf{x})\right\}\right\}$$

and $\pi_m(\mathsf{x}_s) \notin \mathsf{Y}_{m d_m}$.

As a consequence we consider the sets of points

$$\mathsf{W} := \{\mathsf{x}_i : \Phi'(\mathsf{x}_i) = \tau_i X_{m+1}^{d_m}, \tau_i \in \mathcal{T}[1,m]\} \cup \{\mathsf{x}_s\} \text{ and } \mathsf{V} := \pi_m(\mathsf{W});$$

in this setting Cerlienco–Mureddu Correspondence gives a relation between each point $\pi_m(\mathsf{x}_i)$ and the corresponding term τ_i; also, by **(Q)**, the ideal $\mathsf{I}(\pi_{m+1}(\mathsf{W}))$ has the Gröbner basis $\{l'_1, \dots, l'_{j_{m\,d_m}}\}$ where

$$\pi_m(\lambda_s)(l'_j)) = 0, \forall j < I \text{ while } \pi_m(\lambda_s)(l'_I) \neq 0.$$

So the same argument grants that Cerlienco–Mureddu Correspondence returns $\Phi(\pi_m((\mathsf{x}_s)) = X_1^{d_1} \dots X_{m-1}^{d_{m-1}}$. ⊙

As a consequence, the application of Möller Algorithm to $\mathsf{X} = \mathsf{X}' \cup \{\mathsf{x}_s\}$ produces

$q_s := c^{-1} F_I$, with $c = \lambda_s(F_I)$;
$\mathbf{N} := \mathbf{N}' \cup \{\omega_I\}$;
$\mathbf{B} := \mathbf{B}' \setminus \{\omega_I\} \cup \{X_i\omega_I, 1 \le i \le n\}$;
$f_\tau := f'_\tau - \lambda_s(f'_\tau) q_s$ for each $\tau \in \mathbf{B}' \setminus \{\omega_I\}, \tau > \omega_I$ and
$f_\tau := f'_\tau$, for each $\tau \in \mathbf{B}' \setminus \{\omega_I\}, \tau < \omega_I$ since $\lambda_s(f'_\tau) = 0$;
for each $\tau := X_i\omega_I \notin \mathbf{B}'$

$$f_\tau := (X_i - a_{is})F_I - \sum_{X_i\omega \in \mathbf{B}'} c(F_I, \omega) f_{X_i\omega}$$

where

$$F_I = \omega_I + \sum_{\omega \in \mathbf{N}'} c(F_I, \omega)\omega.$$

Proposition 33.5.1. *If* X' *satisfies conditions* **(A-Z)** *then* X *satisfies conditions* **(A), (F), (R), (S)**.

Proof. **(A)** is obvious;
(F) is obvious.
(R) On the basis of Remark 33.3.1 we know that $F_I \in (h'_1, \dots, h'_I)$; also all we need to prove is that, for each i,

$$H_i \in (H_1, \dots, H_{i-1}) = \{H_j, \mathbf{T}(H_j) < \mathbf{T}(H_i)\}.$$

Therefore

- if $\mathbf{T}(H_i) = \mathsf{t}_i \in \mathbf{G}', i < I$, we have

$$H_i = H'_i \in (H'_1, \dots, H'_{i-1}) = (H_1, \dots, H_{i-1});$$

- if $\mathbf{T}(H_i) = \mathsf{t}_i \in \mathbf{G}', i > I$, we have

$$H_i = H'_i - aF_I \in (H'_1, \dots, H'_{i-1}) = (H_1, \dots, H_{i-1})$$

  so that, also $(H'_1, \dots, H'_i) = (H_1, \dots, H_i)$.

- Finally, for $\tau = X_i \mathbf{t}_I$ we have $L_\tau = L'_I$, and

$$L_\tau P_\tau = H_\tau \equiv f_\tau \equiv (X_i - a_{is})F_I \equiv (X_i - a_{is})L'_I P'_I \equiv 0$$

modulo $(H'_1, \ldots, H'_I) = (H_1, \ldots, H_I)$

The same argument proofs the claim for $\{h_1, \ldots, h_r\}$.

(S) $\lambda_s(f_{\omega_I}) \neq 0$ for construction; $\lambda_s(H_{\omega_I}) \neq 0$ and $\lambda_s(h_{\omega_I}) \neq 0$ because both $H_{\omega_I} - f_{\omega_I}$ and $h_{\omega_I} - f_{\omega_I}$ have a representation in terms of $\{F_i, i < I\}$ and $\lambda_s(F_i) = 0$, for each $i < I$. $\boxed{\odot}$

In conclusion we have:

Theorem 33.5.1. *For a CM-ideal* I*, given by a CM-scheme* X *of CM-conditions, using the same notation as above, it holds*

(A) $\mathbf{N} := \mathbf{N}(\mathsf{I})$.
(B) $\mathbf{G}(\mathsf{I}) = \mathbf{G} = \{\mathsf{t}_1, \ldots, \mathsf{t}_r\}, \mathsf{t}_1 < \mathsf{t}_2 < \ldots < \mathsf{t}_r$;
(C) $\mathbf{B}(\mathsf{I}) = \mathbf{B}$.
(D) *For each* $\tau \in \mathbf{N}$ *there is a unique polynomial*

$$f_\tau := \tau - \sum_{\omega \in \mathfrak{N}(\tau)} c(f_\tau, \omega)\omega$$

such that $\lambda(f_\tau) = 0$*, for each* $\lambda \in \mathbb{L}(\tau)$.

(E) *For each* $\tau \in \mathbf{G}$ *there is a unique polynomial*

$$f_\tau := \tau - \sum_{\omega \in \mathbf{N}} c(f_\tau, \omega)\omega$$

such that $\lambda(f_\tau) = 0$*, for each* $\lambda \in \mathbb{L}$.

(F) *For each* $\tau \in \mathbf{B}$ *there is a polynomial*

$$f_\tau := \tau - \sum_{\omega \in \mathfrak{N}(\tau)} c(f_\tau, \omega)\omega$$

such that $\lambda(f_\tau) = 0$*, for each* $\lambda \in \mathbb{L}$.

(G) *The reduced Gröbner basis of* I *is*

$$\mathcal{G}(\mathsf{I}) := \{f_\tau : \tau \in \mathbf{G}\};$$

moreover, for each $\tau \in \mathbf{N}$, $\mathbf{T}(f_\tau) = \tau$.

(H) *The border basis of* I *is*

$$\mathcal{B}(\mathsf{I}) := \{f_\tau : \tau \in \mathbf{B}\};$$

(I) *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{N}$*, and each* $m, 1 \leq m \leq n$ *there are unique polynomials*

$$\gamma_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathsf{F}_m(\tau)} c(\gamma_{m\tau}, \omega)\omega$$

such that $\pi_m(\lambda_i)(\gamma_{m\tau}) = 0$*, for each* $\mathsf{x}_i \in \mathsf{D}_m(\tau), \mathsf{x}_i \prec \Phi^{-1}(\tau)$;

(L) *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{G}$, *and each* $m, 1 \le m \le n$ *there are unique polynomials*

$$\gamma_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathsf{E}_m(\tau)} c(\gamma_{m\tau}, \omega)\omega$$

such that $\pi_m(\lambda_i)(\gamma_{m\tau}) = 0$, *for each* $\mathsf{x}_i \in \mathsf{D}_m(\tau)$;

(M) *for each* $\tau = X_1^{d_1} \cdots X_\nu^{d_\nu} \in \mathbf{N}$, *there are*

$$L_\tau \in k[X_1, \ldots, X_{\nu-1}]$$

and a unique monic polynomial

$$P_\tau = X_\nu^{d_\nu} + \sum_{\omega \in \mathsf{F}_\nu(\tau)} c(P_\tau, \omega)\omega \in k[X_1, \ldots, X_{\nu-1}][X_\nu]$$

so that $H_\tau := L_\tau P_\tau$ *are such that*

- $\mathbf{T}(H_\tau) = \tau$, $\mathrm{Lp}(H_\tau) = L_\tau$,
- $\pi_{\nu-1}(\lambda)(L_\tau) = 0$, *for each* $\lambda \in \mathbb{L}(\tau)$,
- $\pi_\nu(\lambda)(P_\tau) = 0$, *for each* $\lambda \in \mathsf{L}_\nu(\tau)$,
- $\pi_\nu(\lambda_i)(H_\tau) = 0$, *for each* $\lambda_i \in \mathbb{L} : \mathsf{x}_i \prec \Phi^{-1}(\tau)$.

(N) *for each* $i, 1 \le i \le r$ *there are*

$$L_i \in k[X_1, \ldots, X_{\nu-1}]$$

and a unique monic polynomial

$$P_i = X_\nu^{d_\nu} + \sum_{\omega \in \mathsf{E}_\nu(\mathsf{t}_i)} c(P_i, \omega)\omega \in k[X_1, \ldots, X_{\nu-1}][X_\nu]$$

so that $H_i := L_i P_i$ *are such that*

- $\mathbf{T}(H_i) = \mathsf{t}_i = X_1^{d_1} \cdots X_\nu^{d_\nu} \in \mathbf{G} \cap \mathcal{T}[1, \nu]$, $\mathrm{Lp}(H_i) = L_i$,
- $\pi_{\nu-1}(\lambda)(L_i) = 0$, *for each* $\lambda \in \cup_{m=1}^{\nu-1} \mathsf{L}_m(\mathsf{t}_i)$,
- $\pi_\nu(\lambda)(P_i) = 0$, *for each* $\lambda \in \mathsf{L}_\nu(\mathsf{t}_i)$,
- $\pi_\nu(\lambda)(H_i) = 0$, *for each* $\lambda_i \in \mathbb{L}$.

(O) $\{H_1, \ldots, H_r\}$ *is a minimal Gröbner basis of* I;

(P) *For each* $\nu, 1 \le \nu < n$ *let* j_ν *the value such that* $\mathsf{t}_{j_\nu} < X_{\nu+1} \le \mathsf{t}_{j_\nu+1}$; *then* $\{H_1, \ldots, H_{j_\nu}\}$ *is a minimal Gröbner basis of* $\mathsf{I} \cap k[X_1, \ldots, X_\nu]$ *and of* $\mathsf{I}(\pi_\nu(\mathsf{X}))$;

(Q) *For each* $\nu, 1 \le \nu < n$, *and each* $\delta \in \mathbb{N}$ *let* $j(\nu\delta)$ *the value such that* $\mathsf{t}_{j(\nu\delta)} < X_{\nu+1}^\delta \le \mathsf{t}_{j(\nu\delta)+1}$; *then* $\{L_1, \ldots, L_{j_{\nu\delta}}\}$ *is a Gröbner basis of* $\mathsf{I}(\mathsf{Y}_{\nu\delta})$;

(R) *For each* $i, 2 \le i \le r$, $P_i \in (H_j, j < i) : L_i$.

(S) *for each* $j, 1 \le j \le s$, $\lambda_j(f_{\tau_j}) \ne 0$ *and* $\lambda_j(H_{\tau_j}) \ne 0$ *so that* $\mathbb{L}(\mathsf{I})$ *is triangular to* $\{\lambda_j(f_{\tau_j})^{-1} f_{\tau_j}, 1 \le j \le s\}$ *and* $\{\lambda_j(H_{\tau_j})^{-1} H_{\tau_j}, 1 \le j \le s\}$.

(T) *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{N}$ *and each* $m, 1 \le m \le n$, *there are polynomials*

$$g_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathfrak{M}_m(\tau)} c(g_{m\tau}, \omega)\omega$$

such that $g_{m\tau}(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_m(\tau), \mathsf{a} \prec \Phi^{-1}(\tau)$.

(U) *for each* $\tau := X_1^{d_1} \cdots X_n^{d_n} \in \mathbf{G}$, *and each* $m, 1 \le m \le n$, *there are polynomials*

$$g_{m\tau} := X_m^{d_m} + \sum_{\omega \in \mathsf{M}_m(\tau)} c(g_{m\tau}, \omega)\omega$$

such that $g_{m\tau}(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_m(\tau)$*;*

If moreover I *is radical:*

(W) *for each* $\tau = X_1^{d_1} \cdots X_\nu^{d_\nu} \in \mathbf{N}$, *there are*

$$l_\tau \in k[X_1, \ldots, X_{\nu-1}]$$

and a monic polynomial

$$p_\tau = X_\nu^{d_\nu} + \sum_{\omega \in \mathfrak{M}_\nu(\tau)} c(p_\tau, \omega)\omega \in k[X_1, \ldots, X_{\nu-1}][X_\nu]$$

so that $h_\tau := l_\tau p_\tau$ *are such that*

- $\mathbf{T}(h_\tau) = \tau$,
- $l_\tau(\pi_{\nu-1}(\mathsf{a})) = 0$, *for all* $\mathsf{a} \in \mathfrak{X}(\tau)$,
- $p_\tau(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_\nu(\tau)$,
- $h_\tau(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{X}$ *such that* $\mathsf{a} \prec \Phi^{-1}(\tau)$.

(X) *for each* $i, 1 \le i \le r$ *there are*

$$l_i \in k[X_1, \ldots, X_{\nu-1}]$$

and a monic polynomial

$$p_i = X_\nu^{d_\nu} + \sum_{\omega \in \mathsf{M}_\nu(\mathsf{t}_i)} c(p_i, \omega)\omega \in k[X_1, \ldots, X_{\nu-1}][X_\nu]$$

so that $h_i := l_i p_i$ *are such that*

- $\mathbf{T}(h_i) = \mathsf{t}_i = X_1^{d_1} \cdots X_\nu^{d_\nu} \in \mathbf{G} \cap \mathcal{T}[1, \nu]$,
- $l_i(\pi_{\nu-1}(\mathsf{a})) = 0$, *for each* $\mathsf{a} \in \cup_{m=1}^{\nu-1} \mathsf{D}_m(\mathsf{t}_i)$,
- $p_i(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{D}_\nu(\mathsf{t}_i)$,
- $h_i(\mathsf{a}) = 0$, *for each* $\mathsf{a} \in \mathsf{X}$.

(Z) l_i, p_i, h_i, $1 \le i \le r$ *satisfy*

$\{h_1, \ldots, h_r\}$ *is a minimal Gröbner basis of* I*;*

for each $\nu, 1 \le \nu < n$, $\{h_1, \ldots, h_{j_\nu}\}$ *is a minimal Gröbner basis of* $\mathsf{I} \cap k[X_1, \ldots, X_\nu]$ *and of* $\mathsf{I}(\pi_\nu(\mathsf{X}))$*;*

for each $\nu, 1 \le \nu < n$, $\{l_1, \ldots, l_{j_{\nu\delta})}\}$ *is a Gröbner basis of* $\mathsf{I}(\mathsf{Y}_{\nu\delta})$*;*

for each $i, 2 \le i \le r$, $p_i \in (h_j, j < i) : l_i$*;*

for each $j, 1 \le j \le s$, $\lambda_j(h_{\tau_j}) \ne 0$*;*

$\mathbb{L}(\mathsf{I})$ *is triangular to* $\{\lambda_j(h_{\tau_j})^{-1} h_{\tau_j}, 1 \le j \le s\}$. ⊙