%

$$\begin{array}{lllllllll} \mathsf{b}_1 & = & (0,0,1) & \mathsf{b}_2 & = & (0,1,-2) & \mathsf{b}_3 & = & (2,0,2) \\ \mathsf{b}_4 & = & (0,2,-2) & \mathsf{b}_5 & = & (1,0,3) & \mathsf{b}_6 & = & (1,1,3) \\ \mathsf{b}_7 & = & (1,1,1) & \mathsf{b}_8 & = & (2,0,1) & \mathsf{b}_9 & = & (2,0,0) \end{array}$$

$K[x,y,z]$, $\prec$ the deglex ordering induced by $x \prec y \prec z$.

1. $\mathsf{b}_1 = (0,0,1)$,
   $t_1 := 1,\ q_1 := 1,$
   $f_x := x, f_y := y, f_z := z-1,$
   $G_1 := \{f_x, f_y, f_z\}$

2. $\mathsf{b}_2 = (0,1,-2),\ f_x(\mathsf{b}_2) = 0, f_y(\mathsf{b}_2) = 1,\ f_z(\mathsf{b}_2) = -3,$
   $t_2 := y,\ q_2 := y,$
   $f_z := f_z - f_z(\mathsf{b}_2)q_2 = z + 3y - 1;$
   $p = yf_y; f_{y^2} := \left(y - p(\mathsf{b}_2)f_y^{-1}(\mathsf{b}_2)\right) f_y = y^2 - y$
   $G_2 := \{f_x, f_z, f_{y^2}\}$

3. $\mathsf{b}_3 = (2,0,2),\ f_x(\mathsf{b}_3) = 2,\ f_z(\mathsf{b}_3) = 1,\ f_{y^2}(\mathsf{b}_3) = 0;$
   $t_3 := x,\ q_3 := x/2,$
   $f_z := f_z - f_z(\mathsf{b}_3)q_3 = z + 3y - \frac{1}{2}x - 1;$
   $p = xf_x; f_{x^2} := \left(x - p(\mathsf{b}_3)f_x^{-1}(\mathsf{b}_3)\right) f_x = x^2 - 2x$
   $p = yf_x; f_{xy} := \left(y - p(\mathsf{b}_3)f_x^{-1}(\mathsf{b}_3)\right) f_x = xy$
   $G_3 := \{f_z, f_{x^2}, f_{xy}.f_{y^2}\}$

4. $\mathsf{b}_4 = (0,2,-2),\ f_z(\mathsf{b}_4) = 3,\ f_{x^2}(\mathsf{b}_4) = 0, f_{xy}(\mathsf{b}_4) = 0,\ f_{y^2}(\mathsf{b}_4) = 2;$
   $t_4 := z,\ q_4 := f_z/3 = \frac{1}{3}z + y - \frac{1}{6}x - \frac{1}{3}$
   $f_{y^2} := f_{y^2} - f_{y^2}(\mathsf{b}_4)q_4 = y^2 - \frac{2}{3}z - 3y + \frac{1}{3}x + \frac{2}{3};$
   $p = xf_z; f_{xz} := \left(x - p(\mathsf{b}_4)f_z^{-1}(\mathsf{b}_4)\right) f_z = xz + 3xy - \frac{1}{2}x^2 - x$
   $p = yf_z; f_{yz} := \left(y - p(\mathsf{b}_4)f_z^{-1}(\mathsf{b}_4)\right) f_z = yz + 3y^2 - \frac{1}{2}xy - 2z - 7y + x + 2$
   $p = zf_z; f_{z^2} := \left(z - p(\mathsf{b}_4)f_z^{-1}(\mathsf{b}_4)\right) f_z = z^2 + 3yz - \frac{1}{2}xz + z + 6y - x - 2$
   $G_4 := \{f_{x^2}, f_{xy}.f_{y^2}, f_{xz}, f_{yz}, f_{z^2}\}$

5. $\mathsf{b}_5 = (1,0,3),\ f_{x^2}(\mathsf{b}_5) = -1, f_{xy}(\mathsf{b}_5) = 0,\ f_{y^2}(\mathsf{b}_5) = -1,\ f_{xz}(\mathsf{b}_5) = \frac{3}{2},\ f_{yz}(\mathsf{b}_5) = -3,\ f_{z^2}(\mathsf{b}_5) = \frac{15}{2},$
   $t_5 := x^2,\ q_5 := -x^2 + x$
   $f_{y^2} := f_{y^2} - f_{y^2}(\mathsf{b}_5)q_5 = y^2 - x^2 - \frac{2}{3}z - 3y + \frac{7}{3}x + \frac{2}{3};$
   $f_{xz} := f_{xz} - f_{xz}(\mathsf{b}_5)q_5 = xz + 3xy + x^2 - 4x;$
   $f_{yz} := f_{yz} - f_{yz}(\mathsf{b}_5)q_5 = yz + 3y^2 - \frac{1}{2}xy - 3x^2 - 2z - 7y + 7x + 2;$
   $f_{z^2} := f_{z^2} - f_{z^2}(\mathsf{b}_5)q_5 = z^2 + 3yz - \frac{1}{2}xz + \frac{15}{2}x^2 + z + 6y - 16x - 2$
   $p = xf_{x^2}; f_{x^3} := \left(x - p(\mathsf{b}_5)f_{x^2}^{-1}(\mathsf{b}_5)\right) f_{x^2} = x^3 - 3x^2 + 2x$
   $G_5 := \{f_{xy}, f_{y^2}, f_{xz}, f_{yz}, f_{z^2}, f_{x^3}\}$

6. $\mathsf{b}_6 = (1,1,3),\ f_{xy}(\mathsf{b}_6) = 1,\ f_{y^2}(\mathsf{b}_6) = -2,\ f_{xz}(\mathsf{b}_6) = 3,\ f_{yz}(\mathsf{b}_6) = -\frac{3}{2},\ f_{z^2}(\mathsf{b}_6) = 15,\ f_{x^3}(\mathsf{b}_6) = 0,$
   $t_6 := xy,\ q_6 := -x^2 + x$
   $f_{y^2} := f_{y^2} - f_{y^2}(\mathsf{b}_6)q_6 = y^2 + 2xy - x^2 - \frac{2}{3}z - 3y + \frac{7}{3}x + \frac{2}{3};$
   $f_{xz} := f_{xz} - f_{xz}(\mathsf{b}_6)q_6 = xz + x^2 - 4x;$
   $f_{yz} := f_{yz} - f_{yz}(\mathsf{b}_6)q_6 = yz + 3y^2 + xy - 3x^2 - 2z - 7y + 7x + 2;$
   $f_{z^2} := f_{z^2} - f_{z^2}(\mathsf{b}_6)q_6 = z^2 + 3yz - \frac{1}{2}xz - 15xy + \frac{15}{2}x^2 + z + 6y - 16x - 2$
   $p = xf_{xy}; f_{x^2y} := \left(x - p(\mathsf{b}_6)f_{xy}^{-1}(\mathsf{b}_6)\right) f_{xy} = x^2y - xy$
   $G_6 := \{f_{y^2}, f_{xz}, f_{yz}, f_{z^2}, f_{x^3}, f_{x^2y}\}$

7. $\mathsf{b}_7 = (1,1,1),\ f_{y^2}(\mathsf{b}_7) = \frac{2}{3},\ f_{xz}(\mathsf{b}_7) = -2,\ f_{yz}(\mathsf{b}_7) = 2,\ f_{z^2}(\mathsf{b}_7) = -15,\ f_{x^3}(\mathsf{b}_7) = 0,\ f_{x^2y}(\mathsf{b}_8) = 0,$
   $t_7 := y^2,\ q_7 := \frac{3}{4}y^2 + \frac{3}{2}xy - \frac{3}{4}x^2 - \frac{1}{2}z - 3y - \frac{9}{4}y + \frac{7}{4}x + \frac{1}{2};$
   $f_{xz} := f_{xz} - f_{xz}(\mathsf{b}_7)q_7 = xz + \frac{3}{2}y^2 + 3xy - \frac{1}{2}x^2 - z - \frac{9}{2}y - \frac{1}{2}x + 1;$
   $f_{yz} := f_{yz} - f_{yz}(\mathsf{b}_7)q_7 = yz + \frac{3}{2}y^2 - 2xy - \frac{3}{2}x^2 - z - \frac{5}{2}y + \frac{7}{2}x + 1;$

$f_{z^2} := f_{z^2} - f_{z^2}(\mathsf{b}_7)q_7 = z^2 + 3yz - \frac{1}{2}xz + \frac{45}{4}y^2 + \frac{15}{2}xy - \frac{15}{4}x^2 - \frac{13}{2}z - \frac{111}{4}y + \frac{41}{4}x + \frac{11}{2};$

$p = xf_{y^2}; f_{xy^2} := \left(x - p(\mathsf{b}_7)f_{y^2}^{-1}(\mathsf{b}_7)\right) f_{y^2} = xy^2 + 2x^2y - x^3 - \frac{2}{3}xz - y^2 - 5xy + \frac{10}{3}x^2 + \frac{2}{3}z + 3y - \frac{5}{3}x - \frac{2}{3}$

$p = yf_{y^2}; f_{y^3} := \left(y - p(\mathsf{b}_7)f_{y^2}^{-1}(\mathsf{b}_7)\right) f_{y^2} = y^3 + 2xy^2 - x^2y - \frac{2}{3}yz - 4y^2 + \frac{1}{3}xy + x^2 + \frac{2}{3}z + \frac{11}{3}y - \frac{7}{3}x - \frac{2}{3}$

$G_7 := \{f_{xz}, f_{yz}, f_{z^2}, f_{x^3}, f_{x^2y}, f_{xy^2}, f_{y^3}\}$

8. $\mathsf{b}_8 = (2, 0, 1)$, $f_{xz}(\mathsf{b}_8 = -1$, $f_{yz}(\mathsf{b}_8) = 1$, $f_{z^2}(\mathsf{b}_8) = \frac{9}{2}$, $f_{x^3}(\mathsf{b}_8) = 0$, $f_{x^2y}(\mathsf{b}_8) = 0$, $f_{xy^2}(\mathsf{b}_8) = \frac{2}{3}$,$f_{y^3}(\mathsf{b}_8) = -\frac{2}{3}$,

$t_8 := xz$, $q_8 := -xz - \frac{3}{2}y^2 - 3xy + \frac{1}{2}x^2 + z + \frac{9}{2}y + \frac{1}{2}x - 1;$

$f_{yz} := f_{yz} - f_{yz}(\mathsf{b}_8)q_8 = yz + xz + 3y^2 + xy - 2x^2 - 2z - 7y + 3x + 2;$

$f_{z^2} := f_{z^2} - f_{z^2}(\mathsf{b}_8)q_8 = z^2 + 3yz + 4xz + 18y^2 + 21xy - 6x^2 - 11z - 48y + 8x + 10;$

$f_{xy^2} := f_{xy^2} - f_{xy^2}(\mathsf{b}_8)q_8 = xy^2 + 2x^2y - x^3 - 3xy + 3x^2 - 2x;$

$f_{y^3} := f_{y^3} - f_{y^3}(\mathsf{b}_8)q_8 = y^3 + 2xy^2 - x^2y - \frac{2}{3}yz - \frac{2}{3}xz - 5y^2 - \frac{5}{3}xy + \frac{4}{3}x^2 + \frac{4}{3}z + \frac{20}{3}y - 2x - \frac{4}{3}$

$p = xf_{xz}; f_{x^2z} := \left(x - p(\mathsf{b}_8)f_{xz}^{-1}(\mathsf{b}_8)\right) f_{xz} = x^2z + \frac{3}{2}xy^2 + 3x^2y - \frac{1}{2}x^3 - 3xz - 3y^2 - \frac{21}{2}xy + \frac{1}{2}x^2 + 2z + 9y + 2x - 2$

$G_8 := \{f_{yz}, f_{z^2}, f_{x^3}, f_{x^2y}, f_{xy^2}, f_{y^3}, f_{x^2z}, \}$

9. $\mathsf{b}_9 = (2, 0, 0)$, $f_{yz}(\mathsf{b}_9) = 0$, $f_{z^2}(\mathsf{b}_9) = 2$, $f_{x^3}(\mathsf{b}_9) = 0$, $f_{x^2y}(\mathsf{b}_9) = 0$, $f_{xy^2}(\mathsf{b}_9) = 0$,$f_{y^3}(\mathsf{b}_9) = 0$, $f_{x^2z}(\mathsf{b}_9 = 0$,

$t_9 := z^2$, $q_9 := \frac{1}{2}z^2 + \frac{3}{2}yz + 2xz + 98y^2 + \frac{21}{2}xy - 3x^2 - \frac{11}{2}z - 24y + 4x + 5;$

$p = xf_{z^2}; f_{xz^2} := \left(x - p(\mathsf{b}_9)f_{z^2}^{-1}(\mathsf{b}_9)\right) f_{z^2} = xz^2 + 3xyz + 4x^2z + 18xy^2 + 21x^2y - 6x^3 - 2z^2 - 6yz - 19xz - 36y^2 - 90xy + 20x^2 + 22z + 96y - 6x - 20$

$p = zf_{z^2}; f_{z^3} := \left(z - p(\mathsf{b}_9)f_{z^3}^{-1}(\mathsf{b}_9)\right) f_{z^3} = z^3 + 3yz^2 + 4xz^2 + 18y^2z + 21xyz - 6x^2z - 11z^2 - 48yz + 8xz + 10z$

$G_8 := \{f_{yz}, f_{x^3}, f_{x^2y}, f_{xy^2}, f_{y^3}, f_{x^2z}, f_{xz^2}, f_{z^3}\}$

The corresponding Gröbner basis G_9 is not reduced; reduction gives the following reduced Gröbner basis:

$$
\begin{aligned}
g_1 &= yz + xz + 3y^2 + xy - 2x^2 - 2z - 7y + 3x + 2 \\
g_2 &= x^3 - 3x^2 + 2x \\
g_3 &= x^2y - xy \\
g_4 &= xy^2 - xy \\
g_5 &= y^3 - 3y^2 + 2y \\
g_6 &= x^2z - 3xz - 3y^2 - 6xy - x^2 + 2z + 9y + 3x - 2 \\
g_7 &= xz^2 - 2z^2 - 4xz - 15y^2 - 30xy + 8z + 45y + 3x - 6 \\
g_8 &= z^3 - 3z^2 + 3xz - 3y^2 - 6xy - 4z + 9y - 3x + 6
\end{aligned}
$$

Consider the nine functionals, defined

$$\begin{array}{rclrclrcl} \ell_1(f) &=& c(1,f) & \ell_2(f) &=& c(x,f) & \ell_3(f) &=& c(y,f) \\ \ell_4(f) &=& c(z,f) & \ell_5(f) &=& c(x^2,f) & \ell_6(f) &=& c(xy,f) \\ \ell_7(f) &=& c(y^2,f) & \ell_8(f) &=& c(xz,f) & \ell_9(f) &=& c(z^2,f) \end{array}$$

for each polynomial $f \in k[x,y,z]$ where

$$NF(f,\mathsf{I}) := c(z^2,f)z^2 + c(xz,f)xz + c(xy,f)xy + c(x^2,f)x^2 + c(z,f)z + c(y,f)y + c(x,f)x + c(1,f)$$

is the normal form of f w.r.t. the ideal computed above and the deglex ordering $\prec$.

Let us now compute the Gröbner basis of the same ideal w.r.t. the lex ordering $<$ induced by $z < y < x$.

- $t_1 := 1$, $NF(t_1,\mathsf{I}) = 1$, $\lambda_1 = \ell_1$.

  $q_1 := 1$, $\mathrm{vect}(1) = (1,0,0,0,0,0,0,0,0)$

- $t_2 := z$, $NF(t_2,\mathsf{I}) = z$, $\lambda_2 = \ell_4$.

  $q_2 := x$, $\mathrm{vect}(2) = (0,0,0,1,0,0,0,0,0)$

	ℓ_1	ℓ_4	ℓ_2	ℓ_3	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9
vect(1)	1	0	0	0	0	0	0	0	0
vect(2)	0	1	0	0	0	0	0	0	0

- $t_3 := z^2$, $NF(t_3,\mathsf{I}) = z^2$,

  $\lambda_3 = \ell_9$. $q_3 := x^2$,

  $\mathrm{vect}(3) = (0,0,0,0,0,0,0,0,1)$

	ℓ_1	ℓ_4	ℓ_9	ℓ_2	ℓ_3	ℓ_5	ℓ_6	ℓ_7	ℓ_8
vect(1)	1	0	0	0	0	0	0	0	0
vect(2)	0	1	0	0	0	0	0	0	0
vect(3)	0	0	1	0	0	0	0	0	0

- $t_4 := z^3$, $NF(t_4,\mathsf{I}) = 3z^2 - 3zx + 3y^2 + 6yx + 4z - 9y + 3x - 6$, $v = (-6,3,-9,4,0,6,3,-3,3)$

  $v = v + 6\,\mathrm{vect}(1) - 4\,\mathrm{vect}(2) - 3\,\mathrm{vect}(3) = (0,3,-9,0,0,6,3,-3,0)$, $q = z^3 + 6 - 4z - 3z^2$

  $\lambda_4 = \ell_2$, $q_4 := \frac{1}{3}z^3 - z^2 - \frac{4}{3}z + 2$,

  $\mathrm{vect}(4) = (0,1,-3,0,0,2,1,-1,0)$

	ℓ_1	ℓ_4	ℓ_9	ℓ_2	ℓ_3	ℓ_5	ℓ_6	ℓ_7	ℓ_8
vect(1)	1	0	0	0	0	0	0	0	0
vect(2)	0	1	0	0	0	0	0	0	0
vect(3)	0	0	1	0	0	0	0	0	0
vect(4)	0	0	0	1	−3	0	2	1	−1

- $t_5 := z^4$, $NF(t_5,\mathsf{I}) = 7z^2 - 18zx - 42y^2 - 84yx + 30z + 126y + 18x - 36$, $v = (-36,18,126,30,0,-84,-42,-18,7)$

  $v = v + 36\,\mathrm{vect}(1) - 30\,\mathrm{vect}(2) - 7\,\mathrm{vect}(3) - 18\,\mathrm{vect}(4) = (0,0,180,0,0,120,-60,0,0)$

  $q = z^4 + 36 - 30z - 7z^2 - 18q_4 = z^4 - 6z^3 + 11z^2 - 6z$

  $\lambda_5 = \ell_3$, $q_5 := \frac{1}{180}z^4 - \frac{1}{30}z^3 + \frac{11}{180}z^2 - \frac{1}{30}z$,

  $\mathrm{vect}(5) = (0,0,1,0,0,-\frac{2}{3},-\frac{1}{3},0,0)$

	ℓ_1	ℓ_4	ℓ_9	ℓ_2	ℓ_3	ℓ_5	ℓ_6	ℓ_7	ℓ_8
vect(1)	1	0	0	0	0	0	0	0	0
vect(2)	0	1	0	0	0	0	0	0	0
vect(3)	0	0	1	0	0	0	0	0	0
vect(4)	0	0	0	1	−3	0	2	1	−1
vect(5)	0	0	0	0	1	0	$-\frac{2}{3}$	$-\frac{1}{3}$	0

- $t_6 := z^5$, $NF(t_6,\mathsf{I}) = 15z^2 - 75zx - 165y^2 - 330yx + 136z + 495y + 75x - 150$, $v = (-150,75,495,136,0,-330,-165,-75,15)$

  $v = v + 150\,\mathrm{vect}(1) - 136\,\mathrm{vect}(2) - 15\,\mathrm{vect}(3) - 75\,\mathrm{vect}(4) = (0,0,720,0,0,-480,-240,0,0)$

  $v = v - 720\,\mathrm{vect}(5) = 0$

  $q = z^5 + 150 - 136z - 15z^2 - 75q_4 - 720q_5 = z^5 - 4z^4 - z^3 + 16z^2 - 12z$

  $q = z(z-1)(z+2)(z-2)(z-3)$

  $G := \{z^5 - 4z^4 - z^3 + 16z^2 - 12z\}$

- $t_6 := y$, $NF(t_6,\mathsf{I}) = y$ $v = (0,0,1,0,0,0,0,0,0)$

$v = v - \text{vect}(5) = (0,0,0,0,0,\frac{1}{3},\frac{2}{3},0)$

$q = y - q_5 = y - \frac{1}{180}z^4 + \frac{1}{30}z^3 - \frac{11}{180}z^2 + \frac{1}{30}z$

$\lambda_6 = \ell_6,\ q_6 := \frac{3}{2}y - \frac{1}{120}z^4 + \frac{1}{20}z^3 - \frac{11}{120}z^2 + \frac{1}{20}z,$

$\text{vect}(6) = (0,0,0,0,0,1,\frac{1}{2},0,0)$

	ℓ_1	ℓ_4	ℓ_9	ℓ_2	ℓ_3	ℓ_6	ℓ_5	ℓ_7	ℓ_8
vect(1)	1	0	0	0	0	0	0	0	0
vect(2)	0	1	0	0	0	0	0	0	0
vect(3)	0	0	1	0	0	0	0	0	0
vect(4)	0	0	0	1	-3	2	0	1	-1
vect(5)	0	0	0	0	1	$-\frac{2}{3}$	0	$-\frac{1}{3}$	0
vect(6)	0	0	0	0	0	1	0	$\frac{1}{2}$	0

- $t_7 := yz,\ NF(t_7, \mathsf{I}) = -zx - 3y^2 - yx + 2x^2 + 2z + 7y - 3x - 2\ v = (-2,-3,7,2,2,-1,-3,-1,0)$

  $v = v + 2\,\text{vect}(1) - 2\,\text{vect}(2) + 3\,\text{vect}(4) = (0,0,-2,0,2,5,0,-4,0)$

  $v = v + 2\,\text{vect}(5) = (0,0,0,0,2,\frac{11}{3},-\frac{2}{3},-4,0)$

  $v = v - \frac{11}{3}\,\text{vect}(6) = (0,0,0,0,2,0,-\frac{5}{2},-4,0)$

  $\lambda_7 = \ell_5,\ q_7 := \frac{1}{2}(yz + 2 - 2z + 3q_4 + 2q_5 - \frac{11}{3}q_6) = \frac{1}{2}zy + \frac{1}{48}z^4 + \frac{3}{8}z^3 - \frac{61}{48}z^2 - \frac{25}{8}z - \frac{11}{4}y + 4,$

  $\text{vect}(7) = (0,0,0,0,1,0,-\frac{5}{4},-2,0)$

	ℓ_1	ℓ_4	ℓ_9	ℓ_2	ℓ_3	ℓ_6	ℓ_5	ℓ_7	ℓ_8
vect(1)	1	0	0	0	0	0	0	0	0
vect(2)	0	1	0	0	0	0	0	0	0
vect(3)	0	0	1	0	0	0	0	0	0
vect(4)	0	0	0	1	-3	2	0	1	-1
vect(5)	0	0	0	0	1	$-\frac{2}{3}$	0	$-\frac{1}{3}$	0
vect(6)	0	0	0	0	0	1	0	$\frac{1}{2}$	0
vect(7)	0	0	0	0	0	0	1	$-\frac{5}{4}$	-2

- $t_8 := z^2y,\ NF(t_8, \mathsf{I}) = -4zx - 12y^2 - 19yx + 8x^2 + 8z + 40y - 12x - 8,\ v = (-8,-12,40,8,8,-19,-12,-4,0)$

  $v = v + 8\,\text{vect}(1) - 8\,\text{vect}(2) + 12\,\text{vect}(4) = (0,0,4,0,8,5,0,-16,0)$

  $v = v - 4\,\text{vect}(5) = (0,0,0,0,8,\frac{23}{3},\frac{4}{3},-16,0)$

  $v = v - \frac{23}{3}\,\text{vect}(6) - 8\,\text{vect}(7) = (0,0,0,0,0,0,\frac{15}{2},0)$

  $\lambda_8 = \ell_8,\ q_8 := \frac{2}{15}z^2y - \frac{8}{15}zy + \frac{7}{5}y - \frac{1}{60}z^4 + \frac{1}{10}z^3 - \frac{11}{60}z^2 - \frac{1}{10}z,$

  $\text{vect}(8) = (0,0,0,0,0,0,1,0,0)$

	ℓ_1	ℓ_4	ℓ_9	ℓ_2	ℓ_3	ℓ_6	ℓ_5	ℓ_8	ℓ_7
vect(1)	1	0	0	0	0	0	0	0	0
vect(2)	0	1	0	0	0	0	0	0	0
vect(3)	0	0	1	0	0	0	0	0	0
vect(4)	0	0	0	1	-3	2	0	-1	1
vect(5)	0	0	0	0	1	$-\frac{2}{3}$	0	0	$-\frac{1}{3}$
vect(6)	0	0	0	0	0	1	0	0	$\frac{1}{2}$
vect(7)	0	0	0	0	0	0	1	-2	$-\frac{5}{4}$
vect(8)	0	0	0	0	0	0	0	1	0

- $t_9 := z^3y, NF(t_9, \mathsf{I}) = -13zx - 39y^2 - 43yx + 26x^2 + 26z + 109y - 39x - 26,\ v = (-26,-39,109,26,26,-43,-39,-13,0)$

  $v = v + 26\,\text{vect}(1) - 26\,\text{vect}(2) + 39\,\text{vect}(4) = (0,0,-8,0,26,35,-39,-52,0)$

  $v = v + 8\,\text{vect}(5) = (0,0,0,0,26,\frac{89}{3},-\frac{8}{3},-52,0)$

  $v = v - \frac{89}{3}\,\text{vect}(6) - 26\,\text{vect}(7) = (0,0,0,0,0,0,15,0,0)$

  $q = z^3y + 26 - 26z + 39q_4 + 8q_5 - \frac{89}{3}q_6 + 26q_7 - 15q_8 = z^3y - 2z^2y - 5zy + 6y$

  $G = G \cup \{z^3y - 2z^2y - 5zy + 6y\}$

- $t_9 := y^2,\ NF(t_9, \mathsf{I}) = y^2\ v = (0,0,0,0,0,0,1,0,0)$

  $g = y^2 - q_8 = y^2 - \frac{2}{15}z^2y + \frac{8}{15}zy - \frac{7}{5}y + \frac{1}{60}z^4 - \frac{1}{10}z^3 + \frac{11}{60}z^2 - \frac{1}{10}z$

  $G = G \cup \{g\}$

- $t_9 := x,\ NF(t_9, \mathsf{I}) = x,\ v = (0,1,0,0,0,0,0,0,0)$

  $v - \text{vect}(4) = (0,0,3,0,0,-2,-1,1,0)$

$v - 3\,\mathrm{vect}(45) = (0,0,0,0,0,0,0,1,0) =: \mathrm{vect}(9)$

$q_9 = x - q_4 - 3q_5 = x - \frac{1}{60}z^4 - \frac{7}{30}z^3 + \frac{49}{60}z^2 + \frac{43}{30}z - 2$

	ℓ_1	ℓ_4	ℓ_9	ℓ_2	ℓ_3	ℓ_6	ℓ_5	ℓ_8	ℓ_7
vect(1)	1	0	0	0	0	0	0	0	0
vect(2)	0	1	0	0	0	0	0	0	0
vect(3)	0	0	1	0	0	0	0	0	0
vect(4)	0	0	0	1	-3	2	0	-1	1
vect(5)	0	0	0	0	1	$-\frac{2}{3}$	0	0	$-\frac{1}{3}$
vect(6)	0	0	0	0	0	1	0	0	$\frac{1}{2}$
vect(7)	0	0	0	0	0	0	1	-2	$-\frac{5}{4}$
vect(8)	0	0	0	0	0	0	0	1	0
vect(9)	0	0	0	0	0	0	0	0	1

- $t := zx,\ NF(t, \mathsf{I}) = zx\ \ v = (0,0,0,0,0,0,0,1,0)$

  $g = zx - q_9 = zx - x + \frac{1}{60}z^4 + \frac{7}{30}z^3 - \frac{49}{60}z^2 - \frac{43}{30}z + 2$

  $G = G \cup \{g\}$

- $t := yx,\ NF(t, \mathsf{I}) = yx\ \ v = (0,0,0,0,0,1,0,0,0)$

  $v - \mathrm{vect}(6) = (0,0,0,0,0,0,-\frac{1}{2},0,0)$

  $g = yx - q_6 + \frac{1}{2}q_8 = yx + \frac{1}{15}z^2y - \frac{4}{15}zy - \frac{4}{5}y$

  $G = G \cup \{g\}$

- $t := x^2,\ NF(t, \mathsf{I}) = x^2\ \ v = (0,0,0,0,1,0,0,0,0)$

  $v - \mathrm{vect}(5) = (0,0,0,0,0,0,\frac{5}{4},2,0) = \frac{5}{4}\,\mathrm{vect}(8) + 2\,\mathrm{vect}(9)$

  $g = x^2 - q_7 + \frac{5}{4}q_8 - 2q_9 = x^2 - 2x - \frac{1}{6}z^2y + \frac{1}{6}zy + y + \frac{1}{30}z^4 - \frac{1}{30}z^3 - \frac{2}{15}z^2 + \frac{2}{15}z$

  $G = G \cup \{g\}$

- $r = 1, \tau_1 = 1, \mathbf{N}_\prec := \{\tau_1\}$, $\mathsf{B} = \{x, y, z\}$
- $\omega := x = x\tau_1 \notin \mathbf{T}_\prec(\mathsf{I})$,
  $r = 2, \tau_2 = x, \mathbf{N}_\prec := \{\tau_i, 1 \leq 2\}$, $\mathsf{B} = \{y, z, x^2, xy, xz\}$
- $\omega := y = y\tau_1 \notin \mathbf{T}_\prec(\mathsf{I})$,
  $r = 3, \tau_3 = y, \mathbf{N}_\prec := \{\tau_i, 1 \leq 3\}$, $\mathsf{B} = \{z, x^2, xy, xz, y^2, yz\}$
- $\omega := z = z\tau_1 \notin \mathbf{T}_\prec(\mathsf{I})$,
  $r = 4, \tau_4 = z, \mathbf{N}_\prec := \{\tau_i, 1 \leq 4\}$, $\mathsf{B} = \{x^2, xy, xz, y^2, yz, z^2\}$
- $\omega := x^2 = x\tau_2 \notin \mathbf{T}_\prec(\mathsf{I})$,
  $r = 5, \tau_5 = x^2, \mathbf{N}_\prec := \{\tau_i, 1 \leq 5\}$, $\mathsf{B} = \{xy, xz, y^2, yz, z^2, x^3, x^2y, x^2z\}$
- $\omega := xy = x\tau_3 = y\tau_2 \notin \mathbf{T}_\prec(\mathsf{I})$,
  $r = 6, \tau_6 = xy, \mathbf{N}_\prec := \{\tau_i, 1 \leq 6\}$, $\mathsf{B} = \{xz, y^2, yz, z^2, x^3, x^2y, xy^2, x^2z, xyz\}$
- $\omega := xz = x\tau_4 = z\tau_2 \notin \mathbf{T}_\prec(\mathsf{I})$,
  $r = 7, \tau_7 = xz, \mathbf{N}_\prec := \{\tau_i, 1 \leq 7\}$, $\mathsf{B} = \{y^2, yz, z^2, x^3, x^2y, xy^2, x^2z, xyz, xz^2\}$
- $\omega := y^2 = y\tau_3 \notin \mathbf{T}_\prec(\mathsf{I})$,
  $r = 8, \tau_8 = y^2, \mathbf{N}_\prec := \{\tau_i, 1 \leq 8\}$, $\mathsf{B} = \{yz, z^2, x^3, x^2y, xy^2, y^3, x^2z, xyz, y^2z, xz^2\}$
- $M_x =$

	1	x	y	z	x^2	xy	y^2	xz	z^2
1	0	1	0	0	0	0	0	0	0
x	0	0	0	0	1	0	0	0	0
y	0	0	0	0	0	1	0	0	0
z	0	0	0	0	0	0	0	1	0
x^2									
xy									
y^2									
xz									
z^2									

- $M_y =$

	1	x	y	z	x^2	xy	y^2	xz	z^2
1	0	0	1	0	0	0	0	0	0
x	0	0	0	0	0	1	0	0	0
y	0	0	0	0	0	0	1	0	0
z									
x^2									
xy									
y^2									
xz									
z^2									

- $M_z =$

	1	x	y	z	x^2	xy	y^2	xz	z^2
1	0	0	0	1	0	0	0	0	0
x	0	0	0	0	0	0	0	1	0
y									
z									
x^2									
xy									
y^2									
xz									
z^2									

- $\omega := yz = y\tau_4 = z\tau_3 \in \mathbf{G}_\prec(\mathsf{I})$,
  $\omega = -2 - 3x + 7y + 2z + 2x^2 - xy - 3y^2 - xz$
  $\mathsf{B} = \{z^2, x^3, x^2y, xy^2, y^3, x^2z, xyz, y^2z, xz^2, yz^2\}$
- $\omega := z^2 = z\tau_4 \notin \mathbf{T}_\prec(\mathsf{I})$,
  $r = 9, \tau_9 = z^2, \mathbf{N}_\prec := \{\tau_i, 1 \leq 9\}$, $\mathsf{B} = \{x^3, x^2y, xy^2, y^3, x^2z, xyz, y^2z, xz^2, yz^2, z^3\}$
- $\omega := x^3 = x\tau_5 \in \mathbf{G}_\prec(\mathsf{I})$,
  $\omega = -2x + 3x^2$
  $\mathsf{B} = \{x^2y, xy^2, y^3, x^2z, xyz, y^2z, xz^2, yz^2, z^3\}$
- $\omega := x^2y = x\tau_6 = y\tau_5 \in \mathbf{G}_\prec(\mathsf{I})$,
  $\omega = xy$
  $\mathsf{B} = \{xy^2, y^3, x^2z, xyz, y^2z, xz^2, yz^2, z^3\}$
- $\omega := xy^2 = x\tau_7 = y\tau_6 \in \mathbf{G}_\prec(\mathsf{I})$,
  $\omega = xy$
  $\mathsf{B} = \{y^3, x^2z, xyz, y^2z, xz^2, yz^2, z^3\}$
- $\omega := y^3 = y\tau_7 \in \mathbf{G}_\prec(\mathsf{I})$,
  $\omega = -2y + 3y^2$
  $\mathsf{B} = \{x^2z, xyz, y^2z, xz^2, yz^2, z^3\}$
- $\omega := x^2z = x\tau_8 = z\tau_5 \in \mathbf{G}_\prec(\mathsf{I})$,
  $\omega = 2 - 3x - 9y - 2z + x^2 + 6xy + 3y^2 + 3xz$
  $\mathsf{B} = \{xyz, y^2z, xz^2, yz^2, z^3\}$
- $M_x =$

	1	x	y	z	x^2	xy	y^2	xz	z^2
1	0	1	0	0	0	0	0	0	0
x	0	0	0	0	1	0	0	0	0
y	0	0	0	0	0	1	0	0	0
z	0	0	0	0	0	0	0	1	0
x^2	0	−2	0	0	3	0	0	0	0
xy	0	0	0	0	0	1	0	0	0
y^2	0	0	0	0	0	1	0	0	0
xz	2	−3	−9	−9	1	6	3	3	0
z^2									

- $M_y =$

	1	x	y	z	x^2	xy	y^2	xz	z^2
1	0	0	1	0	0	0	0	0	0
x	0	0	0	0	0	1	0	0	0
y	0	0	0	0	0	0	1	0	0
z	−2	−3	7	2	2	−1	−3	−1	0
x^2	0	0	0	0	0	1	0	0	0
xy	0	0	0	0	0	1	0	0	0
y^2	0	0	−2	0	0	0	3	0	0
xz									
z^2									

- $M_z =$

	1	x	y	z	x^2	xy	y^2	xz	z^2
1	0	0	0	1	0	0	0	0	0
x	0	0	0	0	0	0	0	1	0
y	−2	−3	7	2	2	−1	−3	−1	0
z	0	0	0	0	0	0	0	0	1
x^2	2	−3	−9	−2	1	6	3	3	0
xy									
y^2									
xz									
z^2									

- $\omega := xyz = y\tau_8 = z\tau_6 = x \cdot yz \in \mathbf{T}_{\prec}(\mathsf{I}) \setminus \mathbf{G}_{\prec}(\mathsf{I})$,

$$\begin{aligned} x \cdot yz &\equiv x\,(-2\tau_1 - 3\tau_2 + 7\tau_3 + 2\tau_4 + 2\tau_5 - \tau_6 - 3\tau_7 - \tau_8) \\ &= -2\tau_2 - 3\tau_5 + 7\tau_6 + 2\tau_8 + 2(-2\tau_2 + 3\tau_5) - \tau_6 - 3\tau_6 \\ &- (2\tau_1 - 3\tau_2 - 9\tau_3 - 2\tau_4 + \tau_5 + 6\tau_6 + 3\tau_7 + 3\tau_8 \\ &= -2\tau_1 - 3\tau_2 + 9\tau_3 + 2\tau_4 + 2\tau_5 - 3\tau_6 - 3\tau_7 - \tau_8 \end{aligned}$$

$\mathsf{B} = \{y^2z, xz^2, yz^2, z^3\}$

- $\omega := y^2zy^2z = z\tau_7 = y \cdot yz \in \mathbf{T}_{\prec}(\mathsf{I}) \setminus \mathbf{G}_{\prec}(\mathsf{I})$,

$$\begin{aligned} y \cdot yz &\equiv y\,(-2\tau_1 - 3\tau_2 + 7\tau_3 + 2\tau_4 + 2\tau_5 - \tau_6 - 3\tau_7 - \tau_8) \\ &= -2\tau_3 - 3\tau_6 + 7\tau_7 + 2\tau_8 + y(2\tau_4 - \tau_8) + 2\tau_6 - 3(-2\tau_3 + 3\tau_7) \\ &= -2\tau_1 - 3\tau_2 + 9\tau_3 + 2\tau_4 + 2\tau_5 - \tau_6 - 5\tau_7 - \tau_8 \end{aligned}$$

$\mathsf{B} = \{xz^2, yz^2, z^3\}$

- $\omega := xz^2 = x\tau_9 = z\tau_9 \in \mathbf{G}_{\prec}(\mathsf{I})$,
  $\omega = 6 - 3x - 45y - 8z + 30xy + 15y^2 + 4xz + 2z^2$
- $\omega := yz^2 = y\tau_9 = z \cdot yz \in \mathbf{T}_{\prec}(\mathsf{I}) \setminus \mathbf{G}_{\prec}(\mathsf{I})$,
  $\omega = -8 - 12x + 40y + 8z + 8x^2 - 19xy - 12y^2 - 4xz$
- $\omega := z^3 = z\tau_9 \in \mathbf{G}_{\prec}(\mathsf{I})$,
  $\omega = -6 + 3x - 9y + 4z + 6xy + 3y^2 - 3xz + 3z^2$
- $M_x =$

	1	x	y	z	x^2	xy	y^2	xz	z^2
1	0	1	0	0	0	0	0	0	0
x	0	0	0	0	1	0	0	0	0
y	0	0	0	0	0	1	0	0	0
z	0	0	0	0	0	0	0	1	0
x^2	0	-2	0	0	3	0	0	0	0
xy	0	0	0	0	0	1	0	0	0
y^2	0	0	0	0	0	1	0	0	0
x^2	2	-3	-9	-2	1	6	3	3	0
z^2	6	-3	-45	-8	0	30	15	4	2

- $M_y =$

	1	x	y	z	x^2	xy	y^2	xz	z^2
1	0	0	1	0	0	0	0	0	0
x	0	0	0	0	0	1	0	0	0
y	0	0	0	0	0	0	1	0	0
z	-2	-3	7	2	2	-1	-3	-1	0
x^2	0	0	0	0	0	1	0	0	0
xy	0	0	0	0	0	1	0	0	0
y^2	0	0	-2	0	0	0	3	0	0
xz	-2	-3	9	2	2	-3	-3	-1	0
z^2	-8	-12	40	8	8	-19	-12	-4	0

- $M_z =$

	1	x	y	z	x^2	xy	y^2	xz	z^2
1	0	0	0	1	0	0	0	0	0
x	0	0	0	0	0	0	0	1	0
y	-2	-3	7	2	2	-1	-3	-1	0
z	0	0	0	0	0	0	0	0	1
x^2	2	-3	-9	-2	1	6	3	3	0
xy	-2	-3	9	2	2	-3	-3	-1	0
y^2	-2	-3	9	2	2	-1	-5	-1	0
xz	6	-3	-45	-8	0	30	15	4	2
z^2	-6	3	-9	4	0	6	3	-3	3