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Non-strict don't care algebras and specifications†

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Non-strict don't care functions, whose foremost representative is the ubiquitous *if_then_else*, play an essential role in computer science. As far as the semantics is concerned, they can be modelled by their totalizations with the appropriate use of elements representing undefinedness, as D. Scott has shown in his denotational approach. The situation is not so straightforward when we consider non-strict functions in the context of an algebraic framework; this point is discussed in the last section, where we explore the relationship between non-strict don't care and total algebras. The central part of this paper, after presenting the basic properties of the category of non-strict algebras, is an investigation of conditional algebraic specifications. It is shown that non-strict conditional specifications are equivalent to disjunctive specifications, and necessary and sufficient conditions for the existence of initial models are given. Since non-strict don't care specifications generalize both the total and the partial case, it is shown how the results for initiality can be obtained as specializations.

1. Introduction

Functions like the well known *if_then_else* are called non-strict since they can be defined even over tuples where some argument is undefined. For example *if true then a else* $b = a$ is defined no matter what b is. Note the difference with partial functions, which satisfy the strictness condition $D(f(x_1,\ldots,x_n)) \supset D(x_i)$ for $i = 1,\ldots,n$. What is important to stress in the case of the *if_then_else* operation is the don't care feature; if the first argument is *true*, the third argument can be whatever and even missing (corresponding for example to a non-terminating computation). Non-strict don't care functions are a common feature of programming languages; indeed a user-defined function built over non-strict predefined functions like *if_then_else* may be non-strict, and, moreover, non-strict built-in functions abound in many languages, notably Ada.

We can model this situation defining partial tuples and an ordering over them. For example the tuple $(true, a, ?)$, where the symbol ? denotes a missing argument, is less

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than or equal to $(true, a, b)$ for any b. Thus a don't care function f can be characterized by a monotonicity condition (denoting tuples by an underline)

$$\underline{d_1} \leq \underline{d_2} \text{ and } f(\underline{d_1}) \text{ defined } \supset f(\underline{d_1}) = f(\underline{d_2}).$$

All this is well known in denotational semantics where undefinedness is denoted by $\perp$, domains are cpo's with least element $\perp$ and functions are monotonic.

Since it is a widespread misbelief that the above totalization by $\perp$ settles the question of the relationship between the non-strict and the total frame, it may be fruitful to tackle first this point, which is one of the motivations of our work: this issue is discussed in the final section of this paper, after presenting the theory of non-strict algebras and specifications, which is an essential tool for this discussion.

If we consider algebras with non-strict (don't care) functions, we cannot just identify them with a subclass of total algebras, if we want algebraic properties to be preserved. Indeed we can think of associating with a non-strict algebra A its totalization $A_{\perp}$; however, this association does not define a functor from non-strict into total algebras. It is shown that the correct relationship is provided by a free construction, similar to that indicated by Poigné (Poigné 1987) in the case of a partial to total translation.

A deeper characterization of that correspondence is then given in terms of the recently introduced concepts of the simulation of an institution (Astesiano and Cerioli 1990; Cerioli 1993) and of the map of institutions (Meseguer 1989). The existence of a simulation from one institution into another means that signatures and sentences are translated from the first into the second institution in such a way that each model in the first institution is represented by a (generally more than one) model in the second institution that satisfies exactly (the translation of) the same sentences. Thus each model in the first institution may be seen as an abstraction of its implementations in the second institution. This generalizes to classes of models, but in general not to types (*i.e.*, theories), that is, in general, for a given theory (specification) in the first institution there does not exist a theory in the second institution whose models are exactly all the implementations of its models. This becomes true whenever the class of all models of the second institution representing some model of the first one are the model class of some theory, and, in particular, whenever the simulation is also a map of institutions; in this case the logical aspect is preserved also.

On the basis of the above concepts, the correspondence between non-strict and total algebras is completely characterized on the semantic level by giving two simulations:

— non-strict algebras (without sentences) by total algebras, via trivial totalization;
— non-strict algebras (without sentences) by total first-order structures, using typing predicates.

Then the logical side is addressed and it is shown that the trivial totalization cannot deal with the logics, in the sense that equalities in the non-strict frames become sets of atoms and negated atoms, while the second simulation is extended to deal with both conditional and disjunctive sentences. Thus two more simulations are defined:

— non-strict algebras with conditional formulas by first-order structures with conditional formulas;

— non-strict algebras with disjuntive formulas by first-order structures with disjunctive formulas.

Only the third of these simulations is a correspondence between logical theories, and indeed it is also a map of institutions.

We can now go back to the central part of the paper, which is an investigation of non-strict algebras and of algebraic specifications with non-strict functions (henceforth non-strict specifications).

In Section 2 the category of non-strict algebras and its basic properties are presented, with particular attention paid to the concepts of subalgebra, product and term-generated algebras, which are basic for the investigation of initiality.

Having settled the model theoretical aspects, in Section 3 we turn our attention to non-strict specifications. Many problems arise: first, even simple equational specifications are not always consistent (*i.e.*, admit models), the usual conditional specifications do not always admit initial models and, moreover, no simple equational deduction system, generalizing those for the partial and the total case, seems to hand.

We try to clarify the matter as follows. It is shown that the monotonicity condition necessarily introduces hidden disjunctions. Hence we pass on to consider disjunctive specifications, giving necessary and sufficient conditions for the existence of initial models, from which, by specialization, necessary and sufficient conditions for the non-strict conditional case are obtained. Since the non-strict paradigm encompasses both the partial and the total one, we can show that the known results about initiality for those cases, including some recent results on non-positive partial conditional specifications, are obtained by rather simple specializations.

Note that in our framework we are not dealing with error handling. It seems to us that error handling is in a sense an orthogonal problem. We think that it should be interesting and possible to include in an overall non-strict framework the treatment of error handling presented by Poigné (Poigné 1987).

Problems connected with non-strictness have been addressed by Broy and Wirsing (Broy and Wirsing 1984) in the context of generalized algebras, where total algebras are enriched by definedness predicates and special morphisms. However, since the aim of that paper is much broader, as its title indicates, the treatment there of non-strictness is rather indirect, and the issues of initiality and of the relationship with total algebras are not addressed either. Some stimulating considerations about non-strictness, but in a context with different aims, can also be found in Broy (1986).

2. Non-strict algebras

2.1. *The category of non-strict algebras*

In the following we will deal with *partial objects*, that is, with meta-terms not necessarily denoting concrete elements. Thus, as is usual in partial frames, two kinds of equalities may be defined between them: the *existential* equality, which holds iff both sides are defined and equal, and the *strong* equality, which holds iff either both sides are undefined or both sides are defined and equal. We will denote the *existential* equality by $=_e$ and the *strong* equality by $=$.

The basic idea we start from is the one of *partial product*. Usually, the product $A_1 \times \ldots \times A_n$ is the set of all (total) functions g from $\{1 \ldots n\}$ into $A_1 \cup \ldots \cup A_n$ such that $g(i) \in A_i$. We generalize this concept by allowing *partial functions*.

In order to keep the notation as similar as possible to the usual one, we use the symbol ? to denote the 'undefined' elements.

Definition 2.1. Let $A_1, \ldots, A_n$ be sets. The *partial product* of $A_1, \ldots, A_n$, denoted by $\times_p\{A_1, \ldots, A_n\}$, consists of all partial functions from $\{1, \ldots, n\}$ into $A_1 \cup \ldots \cup A_n$ such that if $g(i)$ is defined, then $g(i) \in A_i$. If $n \geq 2$, instead of $\times_p\{A_1, \ldots, A_n\}$, we also use the infix notation $A_1 \times_p \ldots \times_p A_n$.
A partial order $\leq$ over $\times_p\{A_1, \ldots, A_n\}$ is naturally defined by $a \leq b$ iff $a(i) \in A_i$ implies $b(i) =_e a(i)$ for all $i = 1 \ldots n$.
A partial function g from $\times_p\{A_1, \ldots, A_n\}$ into a set A is called *strict* iff $g(a) \in A$ implies $a(i) \in A_i$ for all $i = 1, \ldots, n$, and is called *monotonic* if $a \leq b$ and $g(a) \in A$ implies $g(a) =_e g(b)$.
Later in this paper, we often denote an element $a \in \times_p\{A_1, \ldots, A_n\}$ by $a_1, \ldots, a_n$, where $a_i = a(i)$ if $a(i) \in A_i$ and $a_i =?$ otherwise.

Two remarks are in order here. First note that A and $\times_p\{A\}$ are not in general isomorphic; for example, if A has finite cardinality k, then in $\times_p\{A\}$ there is one more element, the totally undefined tuple, so that $\times_p\{A\}$ has cardinality $k+1$.

Moreover, while the usual product coincides with the categorical product in the category of sets with total functions as arrows, the *partial* product is not the categorical product in the category of sets with *partial* functions as arrows, because the uniqueness of the factorization through the partial product fails. Indeed let us consider a singleton set X and its binary *partial* product $Y = X \times_p X$, with projections $\pi_i(x) = x(i)$; since the projections are partial functions, the factorization throughout Y of the couple of functions $h =< \perp, \perp >$, where $\perp$ is the totally undefined function on X, is not unique, because $\pi_i \cdot f = \perp = \pi_i \cdot g$ for both $f, g \colon X \to Y$, respectively defined by $f(\cdot)$ is undefined and $g(\cdot)$ is the partial function x, where both $x(1)$ and $x(2)$ are undefined.

Let us recall the definition of signature, just in order to fix the notation, and then define the non-strict algebras, where function symbols of functionality $(s_1 \ldots s_n, s)$ are interpreted by partial monotonic functions from $s_1^A \times_p \ldots \times_p s_n^A$ into s^A.

Definition 2.2. A *signature* consists of a set S of *sorts* and a family $F = \{F_{w,s}\}_{w \in S^*, s \in S}$ of sets of *function symbols*. We will denote a generic signature by Σ, and use $f \in F$ or $f \colon w \to s$ instead of $f \in F_{w,s}$ if no ambiguity arises.
Let $\Sigma = (S, F)$ be a signature; a *non-strict* Σ*-algebra* consists of a family $\{s^A\}_{s \in S}$ of sets (the *carriers*), and of a family $\{f^A\}_{f \in F_{w,s}, w \in S^*, s \in S}$ of partial functions (the *interpretations of operation symbols*), such that if $f \in F_{\Lambda,s}$, then either f^A is undefined or $f^A \in s^A$, otherwise $f \in F_{s_1 \ldots s_n, s}$ with $n \geq 1$ and $f^A \colon s_1^A \times_p \ldots \times_p s_n^A \to s^A$ is a monotonic function.
We will often denote the non-strict algebra A by the couple $(\{s^A\}, \{f^A\})$, omitting the quantifications for s and f that are associated with the signature.

An algebra A is called *strict* if f^A is strict for each $f \in F$. Moreover, a strict algebra is called *total* if $\underline{a}(i) \in s_i^A$ for all $i = 1 \ldots n$ implies $f^A(\underline{a}) \in s^A$ for all $f \in F_{s_1 \ldots s_n, s}$. The class of all non-strict Σ-algebras will be denoted by $\mathrm{NSAlg}(\Sigma)$.

Then, by definition, *strict algebras* are exactly the partial algebras, and *total algebras* are the usual ones.

Note that in non-strict algebras there are no extra-elements in the carriers to define non-strict functions. For example, we can define the boolean algebras with non-strict functions $\wedge$ and $\vee$ as follows.

Example 2.3.

sig $\Sigma_{bool} =$
    sorts *bool*
    opns
        $T, F\colon \rightarrow bool$
        $\neg\colon bool \rightarrow bool$
        $\wedge, \vee\colon bool \times bool \rightarrow bool$

Algebra $B =$
        $bool^B = \{t, f\}$
        $T^B = t$
        $F^B = f$
        $\neg^B$ is the strict function defined by $\neg^B(t) = f, \neg^B(f) = t$
        $\wedge^B$ is defined by
        if $\underline{b}(1) = t$ then $\wedge^B(\underline{b}) = \underline{b}(2)$; if $\underline{b}(2) = t$ then $\wedge^B(\underline{b}) = \underline{b}(1)$
        if $\underline{b}(1) = f$ or $\underline{b}(2) = f$ then $\wedge^B(\underline{b}) = f$
        otherwise $\wedge^B(\underline{b})$ is undefined
        $\vee^B$ is defined by
        if $\underline{b}(1) = f$ then $\vee^B(\underline{b}) = \underline{b}(2)$; if $\underline{b}(2) = f$ then $\vee^B(\underline{b}) = \underline{b}(1)$
        if $\underline{b}(1) = t$ or $\underline{b}(2) = t$ then $\vee^B(\underline{b}) = t$
        otherwise $\vee^B(\underline{b})$ is undefined

Depending on the partiality of the functions, there are several possibilities to define homomorphisms, each one being useful for a different purpose (see, for example, Broy and Wirsing (1982), Burmeister (1986) and Reichel (1986)). Our choice follows the tradition of partial algebras (see, for example, Astesiano and Cerioli (1989), Burmeister (1986), Broy and Wirsing (1982) and Tarlecki (1986)), where they are used in order to get a no-junk&no-confusion initial object (Meseguer and Goguen 1985).

Definition 2.4. Let $\Sigma = (S, F)$ be a signature, and A and B be non-strict algebras over Σ. Then a *homomorphism* $h\colon A \rightarrow B$ is a family $\{h_s\colon s^A \rightarrow s^B\}_{s \in S}$ of total functions such that $f^A(\underline{a}) \in s^A$ implies $h_s(f^A(\underline{a})) =_e f^B(h \cdot \underline{a})$, where $h \cdot \underline{a}$ is defined by $h \cdot \underline{a}(i) = h_{s_i}(\underline{a}(i))$ for $i = 1 \ldots n$, for all $f \in F_{s_1 \ldots s_n, s}$ and all $\underline{a} \in s_1^A \times_p \ldots \times_p s_n^A$.
The category $\mathbf{NSAlg}(\Sigma)$ is defined by

— the objects of $\mathbf{NSAlg}(\Sigma)$ are $\mathrm{NSAlg}(\Sigma)$,
— the arrows in $\mathbf{NSAlg}(\Sigma)$ are all the homomorphisms,
— composition is done componentwise,
— the identity on A is $\{Id_{s^A}\}_{s \in S}$.

Note that each homomorphism between strict algebras is a total homomorphism of partial algebras and each homomorphism between total algebras is the usual total homomorphism; thus the categories of both total algebras and partial algebras with total homomorphisms are full sub-categories of $\mathbf{NSAlg}(\Sigma)$. Therefore, each result in the non-strict frame applies to the usual ones too.

Remark. Any non-strict algebra may be represented by its trivial totalization, where a special symbol $\perp$ is added to each carrier to denote the *undefined* elements, because each *partial* function $f: s_1^A \times_p \ldots \times_p s_n^A \to s^A$ may be thought of as the usual *total* function $f_\perp: s_1^A \cup \{\perp_1\} \times \ldots \times s_n^A \cup \{\perp_n\} \to s^A \cup \{\perp\}$. But this totalization causes many existential equalities to hold that do not hold in the original algebra. Thus no non-strict homomorphism $h: A \to B$ may be translated into the (usual total) homomorphism between the trivial totalization of A and B. Consider the following example.

sig $\Sigma =$
  sorts s
  opns
    $a, b: \to s$
Algebra $A =$
    $s^A = \{1\}$ a^A, b^A undefined
Algebra $B =$
    $s^B = s^A$ $a^B = 1$; b^B undefined
Algebra $A_\perp =$
    $s^{A_\perp} = \{1, \perp\}$ $a^{A_\perp} = \perp$; $b^{A_\perp} = \perp$
Algebra $B_\perp =$
    $s^{B_\perp} = s^{A_\perp}$ $a^{B_\perp} = 1$; $b^{B_\perp} = \perp$.

Now, $h: A \to B$, defined by $h(1) = 1$, is obviously a homomorphism; but there is no (total) homomorphism from $A_\perp$ into $B_\perp$, because $a^{A_\perp} = b^{A_\perp}$ while $a^{B_\perp} = 1 \neq \perp = b^{B_\perp}$.

Thus, although the *class* of non-strict algebras is in some sense equivalent to a subclass of total algebras, the *category* $\mathbf{NSAlg}(\Sigma)$ cannot be reduced (at least in a trivial way) to a subcategory of total algebras. For a complete discussion of this issue see Section 4.

2.2. *Term algebras*

Let us introduce the term algebras and state some basic results that will be used in the following sections.

Term algebras are defined as in the total frame, so that the concept of substitution as total homomorphism on $T_\Sigma(X)$ is well defined.

Definition 2.5. Let $\Sigma = (S, F)$ be a signature and $X = \{X_s\}_{s \in S}$ be a family of S-sorted variables. The *term sets* $T_\Sigma(X)_s$ on Σ and X are inductively defined by

— $X_s \subseteq T_\Sigma(X)_s$ for all $s \in S$,
— $F_{\Lambda,s} \subseteq T_\Sigma(X)_s$ for all $s \in S$,
— $f \in F_{s_1 \ldots s_n, s}$ and $t_i \in T_\Sigma(X)_{s_i}$ for $i = 1 \ldots n$ imply $f(t_1, \ldots, t_n) \in T_\Sigma(X)_s$.

If X is the empty set, $T_\Sigma(X)$ is denoted by T_Σ and its elements are called *closed* or *ground* terms.

For all $f \in F_{s_1\ldots s_n,s}$, the function $f^{T_\Sigma(X)}: T_\Sigma(X)_{s_1} \times \ldots \times T_\Sigma(X)_{s_n} \to T_\Sigma(X)_s$ is the strict total function defined by $f^{T_\Sigma(X)}(t_1,\ldots,t_n) = f(t_1,\ldots,t_n)$ for all $t_i \in T_\Sigma(X)_{s_i}$.
The *term algebra* $T_\Sigma(X)$ consists of $(\{T_\Sigma(X)_s\}_{s\in S}, \{f^{T_\Sigma(X)} | f \in F_{w,s}\}_{(w,s)\in S^*\times S})$.
Let Y be an S-sorted family of variables. Then a homomorphism $U: T_\Sigma(X) \to T_\Sigma(Y)$ from $T_\Sigma(X)$ into $T_\Sigma(Y)$ is called a *substitution.*

It is well known that when overloading is allowed some conditions on the signatures are needed in order to have the unique decomposition of terms in the functional notation defined above. We simply assume that in case of ambiguity another notation for terms has been chosen that makes the decomposition unique.

The evaluation is defined as in the partial strict frame, but also partial valuations for variables have to be allowed. Valuations being partial functions, it is possible to define in a canonical way an order on them, which has a minimal element: the empty valuation.

Definition 2.6. Let $\Sigma = (S, F)$ be a signature and $X = \{X_s\}_{s\in S}$ be a family of S-sorted variables. For all algebras $A \in \mathrm{NSAlg}(\Sigma)$ and all *valuations* $V = \{V: X_s \to s^A\}_{s\in S}$ for X in A, where V_s are partial functions, the *evaluation* $eval^{A,V}: T_\Sigma(X) \to A$ is inductively defined by

— $eval^{A,V}(x) = V(x)$ for all $x \in X$,
— $eval^{A,V}(f) = f^A$ for all $f \in F_{\Lambda,s}$,
— $eval^{A,V}(f(t_1,\ldots,t_n)) = f^A(\underline{a})$, where $\underline{a}(i) = eval^{A,V}(t_i)$ with $i = 1\ldots n$ for all $f \in F_{s_1\ldots s_n,s}$ and $t_i \in T_\Sigma(X)_{s_i}$ for $i = 1\ldots n$.

For all valuations $V, V': X \to A$, we say that $V \leq V'$ iff $V(x) \in s^A$ implies $V'(x) =_e V(x)$ for all $x \in X$. The valuation $V_?$ for X in A is the empty map, that is, $V_?(x)$ is undefined for all $x \in X_s$ and all $s \in S$.
In the following, we will denote $eval^{A,V}(t)$ by $t^{A,V}$. Moreover, if X is the empty set (so that there exists a unique valuation $V_?$ for X in A), we will denote $eval^{A,V}$ simply by $eval^A$ and $eval^{A,V}(t)$ by t^A. Finally, we denote $\underline{a}$, defined by $\underline{a}(i) = t_i^{A,V}$ for $i = 1,\ldots n$, by $(t_1^{A,V},\ldots,t_n^{A,V})$.

It is worth noting that the order of the valuations is preserved by the evaluation, that is, $V \leq V'$ implies $eval^{A,V} \leq eval^{A,V'}$.

Proposition 2.7. Let $\Sigma = (S, F)$ be a signature, A a non-strict algebra over Σ, X an S-sorted family of variables, and V and V' valuations for X in A such that $V \leq V'$. For all terms $t \in T_\Sigma(X)_s$ if $t^{A,V} \in s^A$, then $t^{A,V'} =_e t^{A,V}$.

Proof. By induction over the definition of terms. □

In the total frame, the term-algebras are the free objects in the class of all total algebras, because of the uniqueness of the evaluation with respect to a valuation. Here, as in the partial case, term-algebras are not free, because the evaluations, being partial functions, are not homomorphisms. However, a derived property holds also in this case, although it is relaxed a bit. Indeed, in the total frame the freedom of the term algebra implies that if the upper triangle of Diagram 1 commutes, the triangle below commutes also. Analogously, in our frame if $h \cdot V \leq V'$, then $h \cdot eval^{A,V} \leq eval^{A',V'}$, so that we have Diagram 2.

$$
\begin{tikzcd}
& X \arrow[dl, "V"'] \arrow[dr, "V'"] \arrow[dd, phantom, "=", pos=0.25] \arrow[dd, phantom, "=", pos=0.75] & \\
A \arrow[rr, "h"] & & A' \\
& T_\Sigma(X) \arrow[ul, "eval^{A,V}"] \arrow[ur, "eval^{A,V'}"'] &
\end{tikzcd}
$$

Diagram 1

$$
\begin{tikzcd}
& X \arrow[dl, "V"'] \arrow[dr, "V'"] \arrow[dd, phantom, "\leq", pos=0.25] \arrow[dd, phantom, "\leq", pos=0.75] & \\
A \arrow[rr, "h"] & & \leq A' \\
& T_\Sigma(X) \arrow[ul, "eval^{A,V}"] \arrow[ur, "eval^{A,V'}"'] &
\end{tikzcd}
$$

Diagram 2

This result also holds in the partial frame and is crucial in order to get that the initial object in a class, if any, satisfies the *no-junk* and *no-confusion* properties (Meseguer and Goguen 1985). Indeed, if we consider Diagram 2 where A is the initial object and h the unique homomorphism from the initial object into A', we have that each term defined in A has to be defined in A' (no-junk), and that two terms existentially equal in A have to be existentially equal in A' (no-confusion).

Proposition 2.8. Let $\Sigma = (S, F)$ be a signature and $X = \{X_s\}_{s\in S}$ be a family of S-sorted variables. For all non-strict algebras $A, A' \in \mathrm{NSAlg}(\Sigma)$, all valuations V for X in A and V' for X in A', and all homomorphisms $h\colon A \to A'$ such that $h \cdot V \leq V'$, we have that

1 $t^{A,V} \in s^A$ implies $h(t^{A,V}) =_e t^{A',V'}$ for all $t \in T_\Sigma(X)$,
2 $t^{A,V} =_e t'^{A,V'}$ implies $t^{A',V'} =_e t'^{A',V'}$ for all $t, t' \in T_\Sigma(X)$.

Proof. By induction over the definition of terms. □

Let us state an *ad hoc* definition of congruence and of quotient on term algebras, whose use is limited to the study of the existence of the initial model. From now on let X denote a family $X = \{X_s\}_{s\in S}$ of variables such that $|X_s| \geq 1$.

Definition 2.9. Let $\Sigma = (S, F)$ be a signature. A *congruence* $\equiv$ is a family $\equiv = \{\equiv_s\}_{s\in S}$ such that:

1 $\equiv_s \subseteq T_\Sigma(X)_s \times T_\Sigma(X)_s$ for all $s \in S$; if $(a, b) \in \equiv_s$, we write $a \equiv_s b$;
2 $\equiv_s$ is symmetric and transitive, that is, $t \equiv_s t'$ implies $t' \equiv_s t$ and $t \equiv_s t'$, $t' \equiv_s t''$ imply $t \equiv_s t''$ for all $t, t', t'' \in T_\Sigma(X)$. Let us denote by $\mathrm{Dom}(\equiv_s)$ the set $\{t \mid t \equiv_s t\}$ and define $t \equiv^D_s t'$ iff either $t \equiv_s t'$ or $t, t' \notin \mathrm{Dom}(\equiv_s)$;
3 $t_i \equiv^D_{s_i} t'_i$ for $i = 1 \ldots n$ and $f \in F_{s_1 \ldots s_n, s}$ imply $f(t_1, \ldots, t_n) \equiv^D_s f(t'_1, \ldots, t'_n)$;
4 $f(t_1, \ldots, t_n) \in \mathrm{Dom}(\equiv)$, $t_i \notin \mathrm{Dom}(\equiv_{s_i})$ imply

$$f(t_1, \ldots, t_n) \equiv_s f(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n)$$

for all $t \in T_\Sigma(X)_s$;
5 $x \notin \mathrm{Dom}(\equiv_s)$ for all $x \in X_s$.

The *quotient* $T_\Sigma/\equiv$ is the non-strict algebra defined by

— $s^{T_\Sigma/\equiv}$ is $T_\Sigma/\equiv_s$ for all $s \in S$ (in the following we denote the class of t in $s^{T_\Sigma/\equiv}$ by $[t]$),

— $f^{T_\Sigma/\equiv}(\underline{t}) = [f(t_1, \ldots, t_n)]$, where if $\underline{t}(i) \in s^{T_\Sigma/\equiv}$, then $t_i \in \underline{t}(i)$, otherwise $t_i \in X_{s_i}$ for all $f \in F_{s_1 \ldots s_n, s}$.

For each non-strict algebra A let $\equiv^A$ be the congruence defined by $t \equiv^A t'$ iff $t^{A,V_?} =_e t'^{A,V_?}$ and $i^A \colon T_\Sigma/\equiv^A \to <A>$ be the isomorphism defined by $i^A([t]) = t^{A,V_?}$, where $<A>$ is the inductive subalgebra of A (see Definition 2.16).

Note that $f^{T_\Sigma/\equiv}$ is well defined. Indeed, let t_i, t'_i belong to $\underline{t}(i)$ for all i such that $\underline{t}(i) \in s^{T_\Sigma/\equiv}$, otherwise t_i, t'_i belong to X_{s_i} and hence $t_i, t'_i \notin \mathrm{Dom}(\equiv_{s_i})$ because of (5); then $t_i \equiv^D t'_i$ for $i = 1 \ldots n$ and hence, because of (3), $f(t_1, \ldots, t_n) \equiv^D f(t'_1, \ldots, t'_n)$, so $[f(t_1, \ldots, t_n)] = [f(t'_1, \ldots, t'_n)]$.

As in more familiar frames, the evaluation of a term in a quotient algebra in this case is the equivalence class of the term where variables have been replaced by (a representative of) their valuation.

Proposition 2.10. Let $\Sigma = (S, F)$ be a signature, $\equiv$ be a congruence, Y be an S-sorted family of variables, V be a valuation for Y in $T_\Sigma/\equiv$, and U be a substitution for $T_\Sigma(Y)$ in $T_\Sigma(X)$ such that $V(y) = [U(y)]$ for all $y \in Y$. Then for each term t we have $t^{T_\Sigma/\equiv, V} = [U(t)]$.

Proof. By induction over the structure of t. □

2.3. *Subalgebras and products*

We investigate the categorical structure of non-strict algebras, with a particular interest into two classical algebraic issues, the notions of *subalgebra* and *product*, which will be used later.

Definition 2.11. Let A be the non-strict algebra $(\{s^A\}_{s \in S}, \{f^A\}_{f \in F})$. The algebra B is a *weak subalgebra* of A iff

— $s^B \subseteq s^A$ for all $s \in S$,

— $f^B(\underline{b}) \le f^A(\underline{b})$ for all $f \in F_{s_1 \ldots s_n, s}$ and all $\underline{b} \in s_1^B \times_p \ldots \times_p s_n^B$.

A weak subalgebra B of A is a *subalgebra* iff $f^B(\underline{b}) = f^A(\underline{b})$ for all $f \in F_{s_1 \ldots s_n, s}$ and all $\underline{b} \in s_1^B \times_p \ldots \times_p s_n^B$.

In other words, a subalgebra is a weak subalgebra whose carriers are closed with respect to the operations of the algebra.

It is easy to check, by induction on the structure of terms, that if B is a weak subalgebra of A and $V \colon X \to B$ is a valuation, then $t^{B,V} \le t^{A, e \cdot V}$ for each $t \in T_\Sigma(X)$, where $e \colon B \to A$ is the embedding, and, analogously, that if B is a subalgebra of A, then $t^{B,V_?} = t^{A, e \cdot V}$.

Weak subalgebras are categorical subobjects, *i.e.*, are (up to isomorphism) the domains of monomorphisms, and subalgebras are regular subobjects, *i.e.*, are (up to isomorphism) the domains of equalizers, as the following propositions state.

Proposition 2.12. Let $h \colon A_1 \to A_2$ be a homomorphism. Then h is a monomorphism iff h_s is injective for all $s \in S$ iff A_1 is isomorphic to a weak subalgebra of A_2.

Proof. Let us assume that m is a monomorphism and show that m_s is injective for all $s \in S$. Assume by contradiction that there exists $\bar{s} \in S$ such that $m_{\bar{s}}$ is not injective,

that is, that there exist $a, b \in \bar{s}^{A_1}$ such that $m_{\bar{s}}(a) = m_{\bar{s}}(b)$ but $a \neq b$, and show that there exist two homomorphisms $g, h: C \to A_1$ such that $m \cdot g = m \cdot h$, but $g \neq h$.
Let C be the algebra defined by $s^C = \emptyset$ for all $s \neq \bar{s}$, $\bar{s}^C = \{x\}$ and f^C totally undefined for all $f \in F$.
Since f^C is totally undefined for all $f \in F$, every function from C into A_1 is a homomorphism. Let g be defined by $g(x) = a$ and h by $h(x) = b$. Then $m \cdot g = m \cdot h$, but $g \neq h$, in contradiction with the assumption that m is a monomorphism.
It is immediate to check that if $m: A_1 \to A_2$ is injective, then m is a monomorphism and A_1 is isomorphic to the weak subalgebra B of A_2, defined by

Algebra $B =$
$s^B = \{m(a) \mid a \in s^{A_1}\}$ for all $s \in S$
$f^B(m \cdot \underline{a}) = m(f^{A_1}(\underline{a}))$ for all $f \in F$

where f^B is well defined, because m is a total injective function, and $f^B \leq f^{A_2}$, because m is a homomorphism.
Finally, if A_1 is (isomorphic to) a weak subalgebra of A_2, then it is the domain of the embedding, which is a monomorphism, because it is injective. □

Proposition 2.13. Any two parallel homomorphisms $g, h: A_1 \to A_2$ have an equalizer $e: E(g, h) \to A_1$, the embedding of $E = E(g, h)$ into A_1, where E is defined by

Algebra $E =$
$s^E = \{a \mid a \in s^{A_1}, g(a) = h(a)\}$ for all $s \in S$
$f^E(\underline{a}) = f^{A_1}(\underline{a})$ for all $f \in F$ and all $\underline{a} \in s_1^E \times_p \ldots \times_p s_n^E$.

Moreover, a homomorphism $e: E \to A$ is an equalizer iff E is (isomorphic to) a subalgebra of A.

Proof. Let us show that such an $E(g, h)$ is the equalizer of g and h. First note that, by definition of homomorphism, $g(\underline{a}) = h(\underline{a})$ and $f^{A_1}(\underline{a}) \in s^{A_1}$ imply $g(f^{A_1}(\underline{a})) = h(f^{A_1}(\underline{a}))$, so f^E is well defined. Moreover, the embedding e of E into A_1 obviously equalizes g and h. Thus we only have to show that each $m: C \to A_1$ such that $g \cdot m = h \cdot m$ factorizes in a unique way throughout e . Since $g \cdot m = h \cdot m$, $m(C) \subseteq E(f, g)$, and hence $m: C \to E(g, h)$ is the unique factorization of m throughout e.
Therefore, if $e: E \to A_1$ is the equalizer of g and h, then, equalizers being unique up to isomorphism, E is isomorphic to the subalgebra $E(g, h)$ of A_1.
Let us show, conversely, that each subalgebra is the domain of an equalizer. Let B be a subalgebra of A and define C as follows:

— $s^C = \{(0, a), (1, a) \mid a \in s^A\}/\equiv$, where $(i, a) \equiv (i', a')$ iff $a = a'$ and ($i = i'$ or $a \in s^B$);
— if for each k either $\underline{c}(k)$ is undefined or there exists $(i, a_k) \in \underline{c}(k)$, then $f^C(\underline{c}) = [(i, f^A(\underline{a}))]$, where $\underline{a}(k) = a_k$ for each k such that $\underline{c}(k) = [(i, a_k)]$ is defined, otherwise $\underline{a}(k)$ is undefined, and otherwise $f^C(\underline{c})$ is undefined, for all $f \in F$.

It is immediate to check that $\equiv$ is an equivalence relation. Thus in order to have that C is an algebra, we only have to show that f^C is well defined.
Let $\underline{c}$ be such that for each k either $\underline{c}(k)$ is undefined or there exist $(i, a_k), (i', a'_k) \in \underline{c}(k)$. Since $(i, a_k) \equiv (i', a'_k)$, $a_k = a'_k$ and $i = i'$, so $(i, f^A(\underline{a})) = (i', f^A(\underline{a}))$, or $a_k \in s^B$ for each k such that $\underline{a}(k)$ is defined, so, B being a subalgebra, $f^A(\underline{a})$ either is undefined

or belongs to s^B and hence in both cases $(i, f^A(\underline{a})) \equiv (i', f^A(\underline{a}))$. Therefore f^C is well defined. Moreover, it is immediate to check that it is monotonic and hence C is an algebra. Let us define $g, h\colon A \to C$ as follows and show that $B = E(g, h)$.

$$g(a) = [(0, a)] \text{ and } h(a) = [(1, a)] \text{ for all } a \in s^A.$$

Let us show that g is a homomorphism. Assume that $f^A(\underline{a})$ is defined. Then $g(f^A(\underline{a})) = [(0, f^A(\underline{a}))] = f^C(\underline{c})$ for $\underline{c}$ is defined by $\underline{c}(k) = [(0, \underline{a}(k))]$, that is, for $\underline{c} = g \cdot \underline{a}$. Thus $g(f^A(\underline{a})) = f^C(g \cdot \underline{a})$. It is analogously easy to check that h is a homomorphism also. Finally, $g(a) = h(a)$ iff $[(0, a)] = [(1, a)]$, that is, iff $a \in s^B$, and, therefore, $B = E(g, h)$. □

The product of non-strict algebras is defined in the usual way.

Definition 2.14. Let $\Sigma = (S, F)$ be a signature and D be a non-empty set of non-strict algebras over Σ. The *product* $\prod^{A \in D} A$ is the non-strict algebra P over Σ defined by:

— for all $s \in S$, let s^P be $\prod^{A \in D} s^A = \{g\colon D \to \cup_{A \in D} s^A \mid g(A) \in s^A \text{ for all } A \in D\}$;
— for all $f \in F_{s_1 \ldots s_n, s}$, let f^P be the function defined by: for each $\underline{p} \in s_1^P \times_p \ldots \times_p s_n^P$ $f^P(\underline{p})$ is defined iff $f^A(\underline{a})$ is defined for all $A \in D$, where $\underline{a}$ is defined by $\underline{a}(i) = \underline{p}(i)(A)$ for $i = 1 \ldots n$, and in this case $f^P(\underline{p})$ is defined by $f^P(\underline{p})(A) = f^A(\underline{a})$ for all $A \in D$.

The *projection* of $\prod^{A \in D} A$ into A, denoted by π^A, is the homomorphism defined by

$$\pi^A(g) = g(A) \text{ for all } g \in s^{\prod^{A \in D} A} \text{ and all } s \in S.$$

In the following, if D is the finite set $\{A_1, \ldots, A_n\}$, we denote $\prod^{A \in D} A$ by $A_1 \times \ldots \times A_n$ also.

Proposition 2.15. Non-strict weak subalgebras coincide with categorical subobjects (*i.e.*, the domain of monomorphisms), non-strict subalgebras coincide with categorical regular objects (*i.e.*, the domain of equalizers), and the product defined in Definition 2.14 coincides with the categorical product.

Proof. Because of Propositions 2.12 and 2.13, B is a weak subalgebra iff it is the domain of a monomorphism, and it is a subalgebra iff it is the domain of an equalizer. Moreover, it is trivial to check that the product defined in Definition 2.14 satisfies the universal property of the categorical product. □

2.4. *Inductive and initial algebras*

We first introduce the concept of an *inductive* algebra (that is, an algebra satisfying the no-junk condition), and relate it to the idea of *term-generated.* Then we show that in every class of algebras closed with respect to inductive subalgebras, the initial object, if any, is characterized by the no-junk and no-confusion properties.

Definition 2.16. Let A be a non-strict algebra. Its *inductive part* $<A>$ is a family $\{s^{<A>}\}_{s\in S}$ of its carrier sub-sets inductively defined by

$$\text{for all } f \in F_{\Lambda,s} \qquad \frac{}{f^A \in s^{<A>}} \qquad f^A \in s^A,$$

$$\text{for all } f \in F_{s_1\dots s_n,s} \qquad \frac{\underline{a} \in s_1^{<A>} \times_p \dots \times_p s_n^{<A>}}{f^A(\underline{a}) \in s^{<A>}} \qquad f^A(\underline{a}) \in s^A.$$

The *inductive subalgebra* B of A consists of

— $s^B = s^{<A>}$ for all $s \in S$,
— $f^B(\underline{b}) = f^A(\underline{b})$ for all $f \in F_{s_1\dots s_n,s}$ and all $\underline{b} \in \times_p\{s_1^{<A>},\dots,s_n^{<A>}\}$.

In the following we will denote by $<A>$ the inductive subalgebra of an algebra A. The *embedding* of $<A>$ into A is the homomorphism $e = \{e_s\}_{s\in S}$ defined by $e_s(a) =_e a$ for all $a \in s^{<A>}$.
A non-strict algebra A is *inductive* iff $A = <A>$. Let C be a class of non-strict algebras on a signature Σ; the subclass $\mathsf{Ind}(C)$ of C consists of $\{A \mid A \in C, A \text{ is inductive}\}$.

Note that the definition of $s^{<A>}$ guarantees both the well definedness of $f^{<A>}$ and that the embedding is a homomorphism.

The usual equivalence between inductive and term-generated algebras has to be a little relaxed, because functions over terms are total, while in inductive algebras they may also be non-strict. Thus some syntactic elements are needed to play the role of the 'undefined' elements that cooperate to build the carriers; we use a family of variables X with the totally undefined valuation $V_?$ over them.

Proposition 2.17. Let $\Sigma = (S, F)$ be a signature, X be any family $\{X_s\}_{s\in S}$ of variables such that $X_s \neq \emptyset$ for all $s \in S$ and A be a non-strict algebra. The following conditions are equivalent:

1 A is inductive;
2 $eval^{A,V_?}: T_\Sigma(X) \to A$ is surjective;
3 for each algebra B there exists at most one homomorphism $k: A \to B$;
4 A has no proper subalgebras.

Proof.

$1 \Rightarrow 2$ Since A is inductive, $s^A = s^{<A>}$, and hence we show by induction that for all $a \in s^{<A>}$ there exists $t \in T_\Sigma(X)$ such that $t^{A,V_?} = a$.
If $a = f^A$ for some $f \in F_{\Lambda,s}$, then $f \in T_\Sigma(X)_s$ by definition of term algebra and $a = f^{A,V_?}$. Otherwise, $a = f^A(\underline{a})$ for some $f \in F_{s_1\dots s_n,s}$ and $\underline{a} \in s_1^{<A>} \times_p \dots \times_p s_n^{<A>}$; because of the induction hypothesis, for each i such that $\underline{a}(i) \in s_i^{<A>}$, there exists $t_i \in T_\Sigma(X)_{s_i}$ such that $\underline{a}(i) = t_i^{A,V_?}$. For all i such that $\underline{a}(i) \notin s_i^{<A>}$, let t_i be any element of X_{s_i}, which exists because X_s is non-empty for all $s \in S$. Then $\underline{a}(i) = t_i^{A,V_?}$ for all $i = 1\dots n$, by definition of t_i and of $V_?$, and hence $f(t_1,\dots,t_n)^{A,V_?} = f^A(\underline{a})$, that is, $f(t_1,\dots,t_n)^{A,V_?} = a$.

$2 \Rightarrow 3$ Let $h, k\colon A \to B$ be homomorphisms. By hypothesis, for each $a \in s^A$ there exists $t \in T_\Sigma(X)$ such that $t^{A,V_?}{=_e}a$. Also, $t^{A,V_?} \in s^A$ implies $h(t^{A,V_?}){=_e}t^{B,V_?}{=_e}k(t^{A,V_?})$, because of Proposition 2.8. Therefore $h(a){=_e}k(a)$ for all $a \in s^A$, and hence $h = k$.

$3 \Rightarrow 4$ Let us assume that E is a subalgebra of A. Then there exist $g, h\colon A \to B$ such that E is their equalizer, because of Proposition 2.13. By hypothesis, $g = h$ and hence, by construction of the equalizer, $E = A$. Thus A has no proper subalgebras.

$4 \Rightarrow 1$ Since A has no proper subalgebras and $<A>$ is a subalgebra of A, $A = <A>$.

□

Definition 2.18. Let Σ be a signature and C be a class of non-strict algebras over Σ. An algebra $I \in C$ is *initial* in C iff for each $A \in C$ there exists exactly one homomorphism from I into A.

Proposition 2.19. Let $\Sigma = (S, F)$ be a signature, X any family $\{X_s\}_{s \in S}$ of variables such that X_s is non-empty for all $s \in S$ and C be a class of non-strict algebras over Σ closed with respect to inductive subalgebras, that is, such that $A \in C$ implies $<A> \in C$. A non-strict algebra $I \in C$ is initial in C iff it satisfies the following two conditions:

1 I is inductive (*no-junk*);

2 $t^{I,V_?}{=_e}t'^{I,V_?}$ implies $t^{A,V_?}{=_e}t'^{A,V_?}$ for all $A \in C$ and all $t, t' \in T_\Sigma(X)$ (*no-confusion*).

Moreover, I is initial in C iff it is initial in $\mathsf{Ind}(C)$.

Proof. Let us show that I is initial in C iff it satisfies Conditions (1) and (2).

$\Rightarrow$ Let I be initial in C. Then $I \in C$ and hence, C being closed with respect to inductive subalgebras, $<I> \in C$, too. Thus, I being initial, there exists one morphism $h\colon I \to <I>$. Let e denote the embedding of $<I>$ into I. Because e is a homomorphism, $e \cdot h$ is a homomorphism also, and hence is the identity, because there is exactly one homomorphism from I into itself, I being initial. Therefore, by definition of e, h is the identity also, and hence $I = <I>$, that is, I satisfies (1).
Let us assume that $t^{I,V_?}{=_e}t'^{I,V_?}$ for certain $t, t' \in T_\Sigma(X)$. Then for each $A \in C$, because of Proposition 2.8 for h the unique homomorphism from I into A and $V = V_? = V'$, $t^{A,V_?}{=_e}t'^{A,V_?}$.

$\Leftarrow$ Let I satisfy Conditions (1) and (2) and $h^A\colon I \to A$ be defined by $h^A(t^{I,V_?}) = t^{A,V_?}$ for all $t \in T_\Sigma(X)$ and all $A \in C$. Then h^A is a well-defined total function from $eval^{I,V_?}(T_\Sigma(X))$ into A, because of Condition (2). Thus, $eval^{I,V_?}$ being surjective because of Proposition 2.17 and Condition (1), h^A is a well-defined total function from I into A. Finally, h^A is a homomorphism by definition and it is unique, because of Proposition 2.17 and Condition (1).

Let us show that I is initial in C iff it is initial in $\mathsf{Ind}(C)$.

$\Rightarrow$ Let us assume that I is initial in C. Then I satisfies Condition (1), and hence $I \in \mathsf{Ind}(C)$. Since I is initial in C, for all $A \in \mathsf{Ind}(C) \subseteq C$ there exists exactly one homomorphism from I into A. Thus I is initial in $\mathsf{Ind}(C)$.

$\Leftarrow$ Let us assume that I is initial in $\mathsf{Ind}(C)$ and A belong to C. Then $<A> \in C$, and hence $<A> \in \mathsf{Ind}(C)$. Therefore, I being initial in $\mathsf{Ind}(C)$, there exists one morphism $h\colon I \to <A>$. Thus the composition $e \cdot h$ of h with the embedding e of $<A>$ into A is a homomorphism, and it is unique, because of Proposition 2.17 and I being inductive.

Therefore for all $A \in C$ there exists exactly one homomorphism from I into A. Thus I is initial in C. □

To check the existence of an initial model in a class closed with respect to inductive subalgebras, because of the above Proposition 2.19, it is sufficient to work on the subclass of inductive models. This is a real simplification, because the (isomorphism classes of) algebras form a proper class, while the subclass of the (isomorphism classes of) inductive algebras is a set. Thus the syntactical characterization of the initial model suggested by Proposition 2.19, as a quotient of a term algebra with respect to the intersection of the kernels of the natural evaluation of terms in *all* models, does not introduce foundational problems, because it is possible to work on the *set* of (canonical representatives for the isomorphism classes of) inductive models.

Definition 2.20. Let C be a non-empty class of non-strict algebras over Σ, and D be the set of non-strict algebras defined by $D = \{T_\Sigma/\equiv^A \mid A \in C\}$. Then $I(C)$ denotes the inductive sub-algebra of the product $\prod^{B \in D} B$.

Theorem 2.21. Let $\Sigma = (S, F)$ be a signature and C be a class of non-strict algebras over Σ closed under isomorphisms and inductive subalgebras. The following conditions are equivalent:

1 there exists an initial object in C;
2 there exists an initial object in $\mathsf{Ind}(C)$;
3 $I(C)$ belongs to C;
4 $I(C)$ is initial in C.

Proof.

1 $\Leftrightarrow$ 2 This follows because of Proposition 2.19.

2 $\Rightarrow$ 3 Let us assume that I is initial in $\mathsf{Ind}(C)$ and show that it is isomorphic to $I(C)$, and thus, C being closed under isomorphism, we will have the thesis.
Since I is initial in $\mathsf{Ind}(C)$ and C is closed under inductive subobjects, for each $A \in C$ there exists (a unique) $h^A: I \to <A>$. Thus, using the notation of Definition 2.9, the composition ${i^A}^{-1} \cdot h^A: I \to T_\Sigma/\equiv^A$ is a homomorphism for each $A \in C$. Therefore, by definition of product in a categorical setting, there exists a morphism $h: I \to \prod^{B \in D} B$, where D is the set $\{T_\Sigma/\equiv^A \mid A \in C\}$.
By Proposition 2.17, I being inductive, such h is unique and $h: I \to < \prod^{B \in D} B >$, that is, $h: I \to I(C)$. Since both I and $I(C)$ are inductive, to show that h is an isomorphism, it is sufficient to show that there exists a homomorphism $k: I(C) \to I$. Indeed, $h \cdot k$ should be the unique homomorphism from $I(C)$ into itself, that is, the identity, and analogously for $k \cdot h$.
Let us consider the composition of the following homomorphisms:

— the embedding $e: I(C) \to \prod^{B \in D} B$;
— the projection $\pi^I: \prod^{B \in D} B \to T_\Sigma/\equiv^I$;
— the isomorphism $i^I: T_\Sigma/\equiv^I \to <I>$;
— the embedding $e_I: <I> \to I$;

and get the thesis for $k = e_I \cdot i^I \cdot \pi^I \cdot e$.

$3 \Rightarrow 4$ Let us assume that $I(C) \in C$ and show that for each $A \in C$ there exists exactly one homomorphism from $I(C)$ into A.

Let A be any element of C. Since $I(C)$ is inductive, there exists at the most one homomorphism from $I(C)$ into A, because of Proposition 2.17, so we just have to prove that there exists a homomorphism h^A from $I(C)$ into A.

To define such a homomorphism, we consider the composition of the following homomorphisms:

- the embedding $e: I(C) \to \prod^{B \in D} B$;
- the projection $\pi^A: \prod^{B \in D} B \to T_\Sigma/\equiv^A$;
- the isomorphism $i^A: T_\Sigma/\equiv^A \to <A>$;
- the embedding $e^A: <A> \to A$.

We then get the thesis for $h^A = e_A \cdot i^A \cdot \pi^A \cdot e$.

$4 \Rightarrow 1$ This is trivial. □

3. Non-strict specifications

It is usual in both the total and the partial frame, to consider logical formulas (equations and positive Horn clauses) such that their model classes are closed with respect to non-empty products and sub-objects, so that the model classes satisfy *a fortiori* the closure with respect to $I(C)$, which is necessary and sufficient for the existence of an initial model, by Theorem 2.21, for classes closed under subobjects and isomorphisms. The same approach cannot be followed in the non-strict frame, because there are finite sets of equations whose model classes are closed neither with respect to $I(C)$, nor with respect to non-empty products. Let us show this claim informally by a simple example.

spec Sp_2 =
  sorts s
  opns
    $k, k': \to s$
    $f: s \to s$
  axioms
    $f(k) =_e k'$

The following two algebras are obviously models of Sp_2:

Algebra A =
  $s^A = \{\cdot\}$
  $k^A = k'^A = \cdot$
  f^A is the total strict function defined by $f^A(\cdot) = \cdot$

Algebra B =
  $s^B = \{\cdot\}$
  $k'^B = \cdot$; k^B is undefined
  $f^B(\underline{b}) = \cdot$ for all $\underline{b} \in \times_p s^B$

Let C be the model class of Sp_2. By definition of product, both $k^{A \times B}$ and $k^{I(C)}$ are undefined, because k^B is undefined, and, analogously, both $f^{A \times B}(g)$ and $f^{I(C)}(g)$, where g is the totally undefined function, are undefined too, because f^A is strict. Therefore both

$f(k)^{A\times B}$ and $f(k)^{I(C)}$ are undefined, and hence neither $A \times B$ nor $I(C)$ are models of Sp_2 .

In the example above the problem arises because of the monotonicity of the interpretation of the function symbols; indeed from $f(a){=_e}b$ we have that $a{=_e}a$ or $f(x){=_e}b$ holds in each model A. Thus equations implicitly introduce disjunctions. Moreover, it is possible to code each disjunction using conditional axioms, and hence in the non-strict frame we can equivalently deal with equations or with disjunctions. Let us support this claim by an informal proof.

Let $\epsilon_1 \vee \ldots \vee \epsilon_n \vee \neg\eta_1 \vee \ldots \vee \neg\eta_m$, where $\epsilon_1, \ldots, \epsilon_n, \eta_1, \ldots, \eta_m$ are all existential equalities. Then $\epsilon_1 \vee \ldots \vee \epsilon_n \vee \neg\eta_1 \vee \ldots \vee \neg\eta_m$ may be coded by the set

$$\alpha_i \qquad D(f_i(x)) \wedge \eta_1 \wedge \ldots \wedge \eta_m \supset \epsilon_i$$

for $i = 1 \ldots n$ and

$$\alpha \qquad D(f_n(f_{n-1}(\ldots(x)\ldots),$$

where $f_1, \ldots, f_n$ are auxiliary unary functions and $D(t)$ stands for $t{=_e}t$. Indeed, each algebra A satisfying α also satisfies at least one $D(f_i(x))$, and hence if A also satisfies α_i, there exists an η_j such that A does not satisfy η_j or A satisfies ϵ_i, so that A satisfies $\epsilon_1 \vee \ldots \vee \epsilon_n \vee \neg\eta_1 \vee \ldots \vee \neg\eta_m$. Conversely, if A satisfies $\epsilon_1 \vee \ldots \vee \epsilon_n \vee \neg\eta_1 \vee \ldots \vee \neg\eta_m$, then A may generalize to a model of $\alpha_1, \ldots, \alpha_n, \alpha$, suitably defining the interpretation of $f_1, \ldots, f_n$.

Therefore, in the following we will focus our attention on *disjunctive* specifications.

3.1. *Formulas, validity and specifications*

In the following we use formulas within an *infinitary* logic (for reference see, for example, Keisler (1971)), with infinitary conjunctions and disjunctions and families of denumerable sets of variables.

Definition 3.1. Let $\Sigma = (S, F)$ be a signature and X be a family of S-sorted variables.

— The set $Eq(\Sigma, X)$ of *equalities* on Σ and X consists of $t{=_e}t'$ for all $t, t' \in T_\Sigma(X)_s$, $s \in S$, and the set $At(\Sigma, X)$ of *atomic* formulas on Σ and X is

$$Eq(\Sigma, X) \cup \{\neg\epsilon] \mid \epsilon \in Eq(\Sigma, X)\}.$$

— The set $Form(\Sigma, X)$ of all *well-formed* formulas is inductively defined by

- $Eq(\Sigma, X) \subseteq Form(\Sigma, X)$,
- $\Phi \cup \{\theta, \theta'\} \subseteq Form(\Sigma, X)$ implies $\wedge\Phi, \vee\Phi, \neg\theta, \theta \supset \theta' \in Form(\Sigma, X)$.

For each well-formed formula ϕ we denote by $Var(\phi)$ the set of variables that appear in ϕ.

— The set $Cond(\Sigma, X)$ of *conditional formulas* on Σ and X is the set

$$\{\wedge\Delta \supset \epsilon \mid \Delta \cup \{\epsilon\} \subseteq Eq(\Sigma, X)\}.$$

If Δ is the empty set, $\wedge\Delta \supset \epsilon$ is an equivalent notation for ϵ, and hence $Eq(\Sigma, X) \subseteq Cond(\Sigma, X)$. For each conditional formula $\phi = (\wedge\Delta \supset \epsilon)$ we denote Δ by $prem(\phi)$ and ϵ by $cons(\phi)$.

— The set $DForm(\Sigma, X)$ of *disjunctive formulas* on Σ and X is the set

$$\{\vee\Delta \mid \Delta \subseteq At(\Sigma, X)\}.$$

If Δ consists of one atomic formula ϵ, then $\vee\Delta$ is an equivalent notation for ϵ, and hence $At(\Sigma, X) \subseteq DForm(\Sigma, X)$.

In the following, a generic equality will be denoted by ϵ or η, a generic atomic formula by γ or δ, and a generic formula by ϕ, θ or ψ. Moreover, we will denote the empty conjunction $\wedge\emptyset$ by *True* and the empty disjunction $\vee\emptyset$ by *False*.

Definition 3.2. Let $\Sigma = (S, F)$ be a signature, X be a family of S-sorted variables, and A be a non-strict Σ-algebra.
If ϕ is a formula and V is a valuation for $Var(\phi)$ in A, we say that ϕ *holds for* V *in* A (equivalently, *is satisfied for* V *by* A), and write $A \models_V \phi$ according to the following definitions:

— $A \models_V t =_e t'$ iff $t^{A,V}, t'^{A,V} \in s^A$ and $t^{A,V} = t'^{A,V}$;
— $A \models_V \wedge \Phi$ iff $A \models_V \phi$ for all $\phi \in \Phi$;
— $A \models_V \vee \Phi$ iff there exists $\phi \in \Phi$ such that $A \models_V \phi$;
— $A \models_V \neg\theta$ iff $A \not\models_V \theta$;
— $A \models_V \theta \supset \theta'$ iff $A \models_V \theta'$ or $A \not\models_V \theta$.

We write $A \models \phi$ for a formula ϕ and say that ϕ *holds in* (equivalently, *is satisfied by* or *is valid in*) A iff $A \models_V \phi$ for all valuations V for $Var(\phi)$ in A. Let us, for short, denote by $D(t)$ the equality $t =_e t$, where both sides are the same term, because $t =_e t$ simply states the definedness of t.

The definition of validity justifies the notation introduced for the empty conjunction and disjunction: indeed $A \models_V \wedge\emptyset$ for all non-strict algebras A and all valuations V because it is obvious that $A \models_V \phi$ for all $\phi \in \emptyset$, so $\wedge\emptyset$ plays the role of the constant *True*; and $A \not\models_V \vee\emptyset$ for all non-strict algebras A and all valuations V because it is obvious that there does not exist $\phi \in \emptyset$ such that $A \models_V \phi$, so $\vee\emptyset$ plays the role of the constant *False*.

Remark. Since valuations are *total* functions in both the total and the partial frame, the relation $\equiv$ on $T_\Sigma(X)$, defined by $t \equiv t'$ iff $A \models t =_e t'$, is not necessarily an equivalence relation if empty carriers are allowed, since it is not necessarily transitive. Indeed, consider the following example, which is a simplified version of a well-known example presented in Meseguer and Goguen (1985).

sig Σ =
  sorts s_1, s_2
  opns
    $a, b \colon \to s_2$
    $f \colon s_1 \to s_2$

In this case T_Σ, as a total algebra, satisfies both $a = f(x)$ and $f(x) = b$, because $T_{\Sigma_{s_2}} = \emptyset$, and hence there does not exist a (total) valuation for $\{x\}$ in T_Σ, but T_Σ does not satisfy $a = b$, so $\equiv$ is not transitive.

This fact has consequences in the case of inference systems, which have to deal very carefully with the elimination of the variables.

However, these problems do not arise in the non-strict frame, because valuations are *partial* functions, so that there exists at least the totally undefined valuation for all families of variables and all non-strict algebras. For instance, in the above example $T_\Sigma \not\models_{V_?} a = f(x)$, and hence $\equiv$ is transitive. Moreover, the following Proposition 3.3 shows that $\equiv$ coincides with the relation $\equiv_?$, defined by $t \equiv_? t'$ iff $A \models_{V_?} t =_e t'$, and hence is an equivalence relation for each non-strict Σ-algebra A.

Manca and Salibra (Manca and Salibra 1990) introduced the partial valuations to solve the empty-carriers problem and keep the original Birkhoff equational calculus by changing the concept of validity. However, note that in our frame the introduction of partial valuations has a completely different motivation: it is not arbitrary and a mere technical device, as in Manca and Salibra (1990), but arises naturally from the setting, since functions are non-strict and variables have to represent *all* the possible arguments.

Proposition 3.3. Let $\Sigma = (S, F)$ be a signature, X be a family of S-sorted variables, and Δ be a set of equalities over Σ and X.
Then $A \models t =_e t'$ iff $A \models_{V_?} t =_e t'$ for all terms t and t' for each non-strict Σ-algebra A, and, moreover, the following conditions are equivalent:

1 $A \models \vee \Delta$;
2 $A \models_{V_?} \vee \Delta$;
3 there exists $\delta \in \Delta$ such that $A \models \delta$.

Proof. Let us show first that $A \models t =_e t'$ iff $A \models_{V_?} t =_e t'$.

$\Rightarrow$ If $A \models t =_e t'$, then, by definition of validity, $A \models_V t =_e t'$ for all valuations V, so, in particular, $A \models_{V_?} t =_e t'$.

$\Leftarrow$ Since $A \models_{V_?} t =_e t'$, $t^{A,V_?}, t'^{A,V_?} \in s^A$ and $V_? \leq V$ for all valuations V by definition of $V_?$. Thus, by Proposition 2.7, $t^{A,V} =_e t^{A,V_?}$ and $t'^{A,V} =_e t'^{A,V_?}$. Therefore, from $t^{A,V_?} =_e t'^{A,V_?}$, we conclude that $t^{A,V} =_e t'^{A,V}$, and hence $A \models t =_e t'$.

Let us now show that the above conditions are equivalent.

$1 \Rightarrow 2$ If $A \models \vee \Delta$, then, by definition of validity, $A \models_V \vee \Delta$ for all valuations V, so, in particular, $A \models_{V_?} \vee \Delta$.

$2 \Rightarrow 3$ Let us assume that $A \models_{V_?} \vee \Delta$. Then, by definition of validity, there exists $t =_e t' \in \Delta$ such that $A \models_{V_?} t =_e t'$. Thus, since we have already shown that $A \models t =_e t'$ iff $A \models_{V_?} t =_e t'$, we get $A \models t =_e t'$.

$3 \Rightarrow 1$ This is trivial. $\square$

Note that if Δ is a set of atoms (that is, a set of equalities and negated equalities), then $A \models_{V_?} \vee \Delta$ does not imply $A \models \vee \Delta$. For example, if Δ is $\{\neg D(x)\}$, it is satisfied for the totally undefined valuation, while each non-empty algebra does not satisfy $\vee \Delta$.

Definition 3.4. A *specification* Sp consists of a signature Σ and a set of well-formed formulas over Σ, called *axioms* of Sp.
A specification is called *disjunctive, conditional* or *equational*, if all the axioms are disjunctions, conditional formulas or equalities, respectively.
Let $Sp = (\Sigma, Ax)$ be a specification. The class $\mathrm{Mod}(Sp)$ of *models* of Sp is the set

$$\{A \mid A \in \mathrm{NSAlg}(\Sigma), A \models \alpha \text{ for all } \alpha \in Ax\}.$$

A model of Sp is *initial* for Sp iff it is initial in $\mathrm{Mod}(Sp)$.

Remark. Note that disjunctive specifications are sufficient to define any class of models definable using well-formed formulas. Indeed each well-formed formula over the usual logical connectives may be expressed in conjunctive normal form, as it is possible to prove directly in the non-strict frame following the same pattern of the proof as in first-order logic. Since a conjunction of formulas is logically equivalent to the set of the formulas in the conjunction, each well-formed formula (in conjunctive normal form) is logically equivalent to a set of disjunctive formulas.

The expressive power of equalities in a non-strict frame is quite different from the usual one – for instance, $\mathrm{Mod}(Sp)$ may be empty even for the special case of equational specifications. For example, if $D(x)$ is an axiom of the specification, no algebra can satisfy this axiom with respect to the valuation completely undefined, so the specification has no models.

Definition 3.5. A specification Sp is *consistent* iff $\mathrm{Mod}(Sp)$ is not empty.

3.2. *Disjunctive specifications and initiality*

Although the class of models of a disjunctive specification is not a variety, because it is not closed under products nor quotients, it is at least closed under (inductive) subalgebras and isomorphisms, and these closures are sufficient to instantiate Proposition 2.19 and Theorem 2.21.

Proposition 3.6. The class of models of a disjunctive specification is closed with respect to subalgebras and isomorphisms.

Proof. Let $Sp = (\Sigma, Ax)$ be a disjunctive specifications and let C denote $\mathrm{Mod}(Sp)$. By definition of validity, if A and B are isomorphic non-strict algebras, $A \models \phi$ iff $B \models \phi$ for all disjunctive formulas ϕ, and thus C is closed with respect to isomorphisms.
Let us assume that B is a subalgebra of A for some $A \in C$, and show that $B \in C$. Let α belong to Ax and V be a valuation for the variables of α in B. Then it is also a valuation for the variables of α in A, and hence $A \models_V \alpha$, because $A \in \mathrm{Mod}(Sp)$. By definition of validity, $A \models_V \epsilon$ iff $B \models_V \epsilon$ for all equalities ϵ. Thus, α being a disjunctive formula, $A \models_V \phi$ implies $B \models_V \phi$. $\square$

Note that model classes of disjunctive specifications may be non-closed with respect to weak subalgebras. Indeed, consider, for example, the axiom $D(a)$, which simply states the definedness of a constant a. In any model A of $D(a)$ the constant a denotes an element a^A of the carrier of A, but a weak subalgebra B of A may exist such that a^B is undefined, so B is not a model of $D(a)$.

Theorem 3.7. Let $\Sigma = (S, F)$ be a signature and $Sp = (\Sigma, Ax)$ be a disjunctive specification. The following conditions are equivalent:

1 there exists an initial model in $\mathrm{Mod}(Sp)$;
2 there exists an initial model in $\mathsf{Ind}(\mathrm{Mod}(Sp))$;
3 $I(\mathrm{Mod}(Sp)) \in \mathrm{Mod}(Sp)$;
4 $I(\mathrm{Mod}(Sp))$ is initial in $\mathrm{Mod}(Sp)$.

Moreover, a non-strict algebra $I \in \mathrm{Mod}(Sp)$ is initial in $\mathrm{Mod}(Sp)$ iff it is initial in $\mathsf{Ind}(\mathrm{Mod}(Sp))$ iff it is isomorphic to $I(\mathrm{Mod}(Sp))$ iff it satisfies the following two conditions

(a) I is inductive;
(b) $I \models t =_e t'$ implies $A \models t =_e t'$ for all $A \in \mathrm{Mod}(Sp)$ and all $t, t' \in T_\Sigma(X)$, where X is an S-sorted family of variables such that X_s is non-empty for all $s \in S$.

Proof. Because of Proposition 3.6, $\mathrm{Mod}(Sp)$ is closed with respect to inductive subalgebras and isomorphisms, so Theorem 2.21 and Proposition 2.19 apply. □

In general, the initial model may not even exist in the special case of consistent equational specifications. Consider the following example.

spec Sp_4 =.
  sorts s
  opns
    $f, g: s \to s$
  axioms
      $D(g(f(x)))$

Sp_4 is a consistent equational specification, because the non-strict algebra A, defined as follows, is a model of Sp_4.

Algebra A =
  $s^A = \{\cdot\}$
  $f^A(\underline{a}) = g^A(\underline{a}) = \cdot$ for all $\underline{a} \in \times_p\{s^A\}$

Now we show that there are two models A and B of Sp_4 such that, respectively, $f^A(?) \notin s^A$ and $g^B(?) \notin s^B$, and hence for any algebra I satisfying Condition (b) of Theorem 3.7 $g^I(f^I(?)) \notin s^I$, so it is not a model of Sp_4, and thus Sp_4 has no initial model, because of Theorem 3.7.

Algebra =
  $s^A = \{\cdot\}$
  $f^A(\cdot)$ and $f^A(?)$ undefined
  $g^A(?) = g^A(\cdot) = \cdot$

Algebra B =
  $s^B = \{\cdot\}$
  $f^B(?) = f^B(\cdot) = \cdot$
  $g^B(\cdot) = \cdot$ and $g^B(?)$ undefined

To give necessary and sufficient conditions for the existence of the initial model, we need some preliminary results.

As we have already noted, variables play the role of the 'undefined' objects and hence, because of monotonicity, they may be replaced by any other term in any formula without affecting its validity. Moreover, this replacement may also be 'asymmetric', changing different occurrences of the same variable in a formula by different terms. For example, from $A \models f(x) = f'(x)$, for the valuation $V_?$, we have $f^A(?) = f'^A(?)$, and hence, by monotonicity, $f^A(a) = f'^A(b)$ for all $a, b \in A$. Thus $A \models f(x) = f'(y)$. We formalize this idea in the following lemma.

Lemma 3.8. Let A be an algebra over a signature $\Sigma = (S, F)$; then $A \models D(f(t_1, \dots, t_n))$ implies, for each $i = 1 \dots n$, that

$$A \models f(t_1, \dots, t_n) =_e f(t_1, \dots, t_{i-1}, x, t_{i+1}, \dots, t_n) \vee D(t_i).$$

Proof. Assume that $A \models D(f(t_1, \dots, t_n))$. Then, in particular, $A \models_{V_?} D(f(t_1, \dots, t_n))$ so that either $A \models_{V_?} D(t_i)$ or $t_i^{A,V_?} = x^{A,V_?}$, and in this case

$$A \models_{V_?} f(t_1, \dots, t_n) =_e f(t_1, \dots, t_{i-1}, x, t_{i+1}, \dots, t_n).$$

Therefore $A \models_{V_?} D(t_i) \vee f(t_1, \dots, t_n) =_e f(t_1, \dots, t_{i-1}, x, t_{i+1}, \dots, t_n)$, and hence, because of Proposition 3.3, $A \models D(t_i) \vee f(t_1, \dots, t_n) =_e f(t_1, \dots, t_{i-1}, x, t_{i+1}, \dots, t_n)$. □

The initial model of a disjunctive specification exists iff each disjunction of *non-negated* atoms that is valid in all models has a privileged element that holds in all models. Thus disjunctions, which potentially cause troubles, may be solved and replaced by atoms. Moreover, it is sufficient to check the property for just two kinds of disjunctions:

1 the disjunctions implicitly introduced because of monotonicity (see Lemma 3.8);
2 the disjunctions coming from instantiations of proper axioms.

Theorem 3.9. Let $Sp = (\Sigma, Ax)$ be a consistent disjunctive specification. The following conditions are equivalent:

1 There exists I initial in $\mathrm{Mod}(Sp)$.
2 There exists I initial in $\mathsf{Ind}(\mathrm{Mod}(Sp))$.
3 $I(\mathrm{Mod}(Sp)) \in \mathrm{Mod}(Sp)$.
4 $I(\mathrm{Mod}(Sp))$ is initial in $\mathrm{Mod}(Sp)$.
5 For all sets Δ of equalities if $A \models \vee \Delta$ for all $A \in \mathrm{Mod}(Sp)$, then there exists $\delta \in \Delta$ such that $A \models \delta$ for all $A \in \mathrm{Mod}(Sp)$.
6 (a) For all $f \in F_{s_1 \dots s_n, s}$ if $A \models D(f(t_1, \dots, t_n))$ for all $A \in \mathrm{Mod}(Sp)$, then one of the following conditions holds:
  — $A \models D(t_i)$ for all $A \in \mathrm{Mod}(Sp)$ or
  — $A \models f(t_1, \dots, t_n) =_e f(t_1, \dots, t_{i-1}, x, t_{i+1}, \dots, t_n)$ for all $A \in \mathrm{Mod}(Sp)$.
  (b) For all $\vee\Delta \in Ax$ and all substitutions $U: T_\Sigma(Var(\vee\Delta)) \to T_\Sigma(X)$ if $A \models t =_e t'$ for all $A \in \mathrm{Mod}(Sp)$ and all $\neg t =_e t' \in U(\Delta)$, then there exists $\delta \in U(\Delta) \cap Eq(\Sigma, X)$ such that $A \models \delta$ for all $A \in \mathrm{Mod}(Sp)$.
7 The relation $\equiv$ over $T_\Sigma(X)$, defined by $t \equiv t'$ iff $A \models t =_e t'$ for all $A \in \mathrm{Mod}(Sp)$, is a congruence and $T_\Sigma / \equiv$ is the initial model.

Proof.

$1 \Leftrightarrow 2$ This follows from Theorem 3.7.
$2 \Leftrightarrow 3$ This follows from Theorem 3.7.
$3 \Leftrightarrow 4$ This follows from Theorem 3.7.
$4 \Rightarrow 5$ Let $I = I(\mathrm{Mod}(Sp))$ be the initial model and Δ be a set of equalities. Thus $A \models \vee\Delta$ for all $A \in \mathrm{Mod}(Sp)$ implies, in particular, $I \models \vee\Delta$. Because of Proposition 3.3, $I \models \vee \Delta$ implies that there exists $t =_e t' \in \Delta$ such that $I \models t =_e t'$, and then, because of Theorem 3.7, $A \models t =_e t'$ for all $A \in \mathrm{Mod}(Sp)$.

$5 \Rightarrow 6$ Assume that $A \models D(f(t_1,\ldots,t_n))$ for all $A \in \mathrm{Mod}(Sp)$.
Then $A \models D(t_i) \vee f(t_1,\ldots,t_n) =_e f(t_1,\ldots,t_{i-1},x,t_{i+1},\ldots,t_n)$ for all $A \in \mathrm{Mod}(Sp)$, by Lemma 3.8, and hence $A \models f(t_1,\ldots,t_n) =_e f(t_1,\ldots,t_{i-1},x,t_{i+1},\ldots,t_n)$ for all $A \in \mathrm{Mod}(Sp)$ or $A \models D(t_i)$ for all $A \in \mathrm{Mod}(Sp)$, because of Condition (5), and thus (6a) holds.
Let us assume that there exists $\vee\Delta \in Ax$ and $U : T_\Sigma(Var(\vee\Delta)) \to T_\Sigma(X)$ such that $A \models t =_e t'$ for all $A \in \mathrm{Mod}(Sp)$ and all $\neg t =_e t' \in U(\Delta)$. Then, since both $A \models t =_e t'$ for all $\neg t =_e t' \in U(\Delta)$ and $A \models \vee U(\Delta)$, A being a model of Sp, $A \models \vee U(\Delta) \cap Eq(\Sigma, X)$ so, because of (5), there exists $\delta \in U(\Delta) \cap Eq(\Sigma, X)$ such that $A \models \delta$ for all $A \in \mathrm{Mod}(Sp)$, and hence (6) holds.

$6 \Rightarrow 7$ It is easy to show that under hypothesis (6a) $\equiv$ is a congruence; we just show that $T_\Sigma/\equiv$ is the initial model.
Let I denote $T_\Sigma/\equiv$, $\alpha = \vee\Phi$ be an axiom of Sp, and V be a valuation for $Var(\alpha)$ in I; if there exists $\neg t =_e t' \in \Phi$ such that $I \not\models_V t =_e t'$, then $I \models_V \neg t =_e t'$ and hence $I \models_V \vee \Phi$.
Thus, let us assume that $I \models_V t =_e t'$ for all $\neg(t =_e t') \in \Phi$, and let U be a substitution for $T_\Sigma(Var(\alpha))$ in $T_\Sigma(X)$ such that $V(y) = [U(y)]$ for all $y \in Var(\alpha)$ and let $U(\phi)$ denote $\phi[U(y)/y \mid y \in Var(\alpha)]$ for all formulas ϕ.
Because of Proposition 2.10, $I \models_V t =_e t'$ implies $U(t) \equiv U(t')$, and hence, by definition of $\equiv$, $A \models U(t =_e t')$ for all $A \in \mathrm{Mod}(Sp)$ and all $\neg t =_e t' \in \Phi$.
Therefore, from (6b), there exists $t =_e t' \in \Phi \cap Eq(\Sigma, Var(\alpha))$ such that $A \models U(t =_e t')$ for all $A \in \mathrm{Mod}(Sp)$, and hence $U(t) \equiv U(t')$, so $I \models_V t =_e t'$.
Therefore $I \models_V \alpha$, and hence I is a model. Moreover, I satisfies Conditions (1) and (2) of Theorem 3.7 by definition and hence it is initial in $\mathrm{Mod}(Sp)$.

$7 \Rightarrow 1$ This is obvious.

$\square$

The results of the above theorem apply to a wide range of specifications, because of the great generality of non-strict disjunctive specifications.

In particular, in the following sections we will show that if all the proper axioms are conditional, Condition (6b) is always satisfied, while if axioms are imposed so that all models are strict, Condition (6a) is satisfied. Thus for total and partial conditional specifications both (6a) and (6b) hold, so that we get the known results about the existence of an initial model in those cases as a specialization of Theorem 3.9.

3.3. *Non-strict conditional specifications*

The conditional specifications are a particular case of disjunctive specifications. Indeed, the conditional formula $\wedge\Delta \supset \epsilon$ is logically equivalent to $\vee(\{\neg\delta \mid \delta \in \Delta\} \cup \{\epsilon\})$, or, in other words, conditional formulas are disjunctions in which exactly one non-negated equality appears, that is, they are positive Horn clauses. In this case the necessary and sufficient conditions for the existence of an initial object are partially simplified.

Theorem 3.10. Let $Sp = (\Sigma, Ax)$ be a consistent conditional specification. The following conditions are equivalent:

1 There exists I initial in $\mathrm{Mod}(Sp)$.

2 There exists I initial in $\mathsf{Ind}(\mathrm{Mod}(Sp))$.
3 $I(\mathrm{Mod}(Sp)) \in \mathrm{Mod}(Sp)$.
4 $I(\mathrm{Mod}(Sp))$ is initial in $\mathrm{Mod}(Sp)$.
5 For all sets Δ of equalities, if $A \models \vee \Delta$ for all $A \in \mathrm{Mod}(Sp)$, then there exists $\delta \in \Delta$ such that $A \models \delta$ for all $A \in \mathrm{Mod}(Sp)$.
6 For all $f \in F_{s_1 \ldots s_n, s}$, if $A \models D(f(t_1, \ldots, t_n))$ for all $A \in \mathrm{Mod}(Sp)$, then one of the following conditions holds:
— $A \models D(t_i)$ for all $A \in \mathrm{Mod}(Sp)$ or
— $A \models f(t_1, \ldots, t_n) =_e f(t_1, \ldots, t_{i-1}, x, t_{i+1}, \ldots, t_n)$ for all $A \in \mathrm{Mod}(Sp)$.
7 The relation $\equiv$ over $T_\Sigma(X)$, defined by $t \equiv t'$ iff $A \models t =_e t'$ for all $A \in \mathrm{Mod}(Sp)$, is a congruence and $T_\Sigma/\equiv$ is the initial model.

Proof. Let us first show that each conditional specification is equivalent to a disjunctive one for which Condition (6b) of Theorem 3.9 is always satisfied. Let Sp be the consistent conditional specification (Σ, Ax) and define $Ax' = \{disj(\alpha) \mid \alpha \in Ax\}$, where $disj(\wedge\Delta \supset \epsilon) = \vee\{\neg\delta \mid \delta \in \Delta\} \cup \{\epsilon\}$, and $Sp' = (\Sigma, Ax')$. Since, by definition of validity, each algebra A satisfies ϕ iff satisfies $disj(\phi)$ for all conditional formulas ϕ, $\mathrm{Mod}(Sp) = \mathrm{Mod}(Sp')$.
Thus, because of Theorem (3.9), we only have to show that for such an Sp', Condition (6) of Theorem 3.10 is equivalent to Conditions (6a) and (6b) of Theorem 3.9, that is, that Condition (6b) of Theorem 3.9 is satisfied also. Let $\wedge\Delta \supset \epsilon$ be an axiom of Sp and $U\colon Var(\wedge\Delta \supset \epsilon) \to X$ be a substitution and assume that $A \models t =_e t'$ for all $\neg t =_e t' \in U(disj(\wedge\Delta \supset \epsilon))$, that is, for all $t =_e t' \in U(\Delta)$, for all $A \in \mathrm{Mod}(Sp)$. Then, for all $A \in \mathrm{Mod}(Sp)$, since $A \models U(\wedge\Delta \supset \epsilon)$ and $A \models t =_e t'$ for all $t =_e t' \in U(\Delta)$, $A \models U(\epsilon)$, so Condition (6b) of Theorem 3.9 is satisfied. □

3.4. *Total and partial specifications*

Since the usual partial (total) algebras coincide with strict (total) algebras in our formalism, Theorem 3.9 applies to those cases too and, since Condition (6a) is always satisfied, we can propose simplified necessary and sufficient conditions for the existence of partial or total initial models.

Moreover, by reducing total conditional and partial positive conditional to non-strict conditional specifications, the well-known results about the existence of an initial object in those cases may be obtained as corollaries of Theorem 3.10. Finally, we show that partial non-positive conditional specifications, *i.e.*, partial specifications whose axioms are in conditional form but involve both *strong* and existential equalities, reduce to non-strict disjunctive specifications, and that the necessary and sufficient conditions for the existence of an initial model, given in Astesiano and Cerioli (1989; 1995) can be deduced from Theorem 3.9.

Lemma 3.11. Let $\Sigma = (S, F)$ be a signature and $Sp = (\Sigma, Ax)$ be a disjunctive specification, and let us denote by Ax_{Str} the set

$$\{D(f(y_1, \ldots, y_n)) \supset D(y_i) \mid f \in F_{s_1 \ldots s_n, s}, i = 1 \ldots n\}.$$

If (a set of axioms first-order equivalent to) $Ax_{Str} \subseteq Ax$, then Condition (6a) of Theorem 3.9 is satisfied.

Proof. Let us assume that $A \models D(f(t_1, \ldots, t_n))$ for all $A \in \mathrm{Mod}(Sp)$.
Since $D(f(y_1, \ldots, y_n)) \supset D(y_i) \in Ax$, we have $A \models D(f(y_1, \ldots, y_n)) \supset D(y_i)$, and hence both $A \models D(f(t_1, \ldots, t_n))$ and $A \models D(f(t_1, \ldots, t_n)) \supset D(t_i)$, so $A \models D(t_i)$ for all $A \in \mathrm{Mod}(Sp)$. □

Let us recall the definition of validity in both total and partial algebras, just in order to fix the notation.

Definition 3.12. Let $\Sigma = (S, F)$ be a signature, A be a strict algebra over Σ and ϕ be a well-formed formula over Σ and an S-sorted family X of variables. $A \overset{t}{\models} \phi$ iff $A \models_V \phi$ for all total valuations V for the variables of ϕ in A.
Let $Sp = (\Sigma, Ax)$ be a specification. $PMod(Sp)$ is the class of partial models of Sp, that is,

$$PMod(Sp) = \{A \mid A \in \mathrm{NSAlg}(\Sigma), A \text{ strict}, A \overset{t}{\models} \alpha \text{ for all } \alpha \in Ax\}$$

and $\mathrm{TMod}(Sp)$ is the class of total models of Sp, that is,

$$\mathrm{TMod}(Sp) = \{A \mid A \in \mathrm{NSAlg}(\Sigma), A \text{ total}, A \overset{t}{\models} \alpha \text{ for all } \alpha \in Ax\}.$$

Since partial algebras coincide with strict algebras, $PMod(Sp)$ is the class of partial models of Sp in the usual sense and, analogously, $\mathrm{TMod}(Sp)$ is the class of total models of Sp in the usual sense also.

Let us show that the validity of a formula with respect to total valuations of its variables is equivalent to the validity with respect to possibly partial valuations of the formula where the definedness of variables has been explicitly required, by adding $D(x)$ in the premises. Thus partial or total validity reduces to validity in non-strict framework under a translation of sentences.

Lemma 3.13. Let $\Sigma = (S, F)$ be a signature, A be a non-strict algebra over Σ and $\phi = (\wedge\Delta \supset \epsilon)$ be a well-formed formula over Σ and X.
Denoting by $tot(\phi)$ the formula $\wedge(\Delta \cup \{D(y) \mid y \in Var(\wedge\Delta \supset \epsilon)\}) \supset \epsilon$,

$$A \overset{t}{\models} \phi \quad \Longleftrightarrow \quad A \models tot(\phi).$$

Proof.

$\Rightarrow$ Let us assume that $A \overset{t}{\models} \phi$ for some non-strict algebra A and show that $A \models tot(\phi)$.
Let V be a partial valuation for $Var(tot(\phi))$ in A such that $A \models_V \delta$ for all δ in the premises of $tot(\phi)$. Then, in particular, $A \models_V D(y)$ for all $y \in Var(\phi)$, that is, V is total, and hence $A \models_V \phi$, because of the assumption $A \overset{t}{\models} \phi$. Thus $A \models_V \phi$ and, since $A \models_V \delta$ for all δ in the premises of $tot(\phi)$, $A \models_V \delta$ for all $\delta \in \Delta$. Therefore $A \models_V \epsilon$, so $A \models_V tot(\phi)$.

$\Leftarrow$ Let us assume that $A \models tot(\phi)$ for some non-strict algebra A and show that $A \overset{t}{\models} \phi$.
Let V be a total valuation for $Var(\phi)$ in A such that $A \models_V \delta$ for all $\delta \in \Delta$. Then V is also a valuation for the variables of $tot(\phi)$ in A, and hence $A \models_V tot(\phi)$. Moreover, $A \models_V D(y)$ for all $y \in Var(\phi)$, V being a total valuation, and $A \models_V \delta$ for all $\delta \in \Delta$, that is, $A \models_V tot(\phi)$ and $A \models_V \delta$ for all $\delta \in prem(tot(\phi))$. Therefore $A \models_V \epsilon$, so $A \models_V \phi$.

□

Using the above results, any total (partial) specification can be translated into a non-strict specification having the same models.

Proposition 3.14. Let $\Sigma = (S, F)$ be a signature and $Sp = (\Sigma, Ax)$ be a conditional specification, and use the notation of Lemma 3.13.

1 Let $Par(Sp)$ be the conditional specification $(\Sigma, Ax_{Str} \cup tot(Ax))$, where

$$tot(Ax) = \{tot(\alpha) \mid \alpha \in Ax\}.$$

The partial model class of Sp coincides with the class of all non-strict models of $Par(Sp)$, that is, $PMod(Sp) = \mathrm{Mod}(Par(Sp))$.

2 Let $Tot(Sp)$ be the conditional specification $(\Sigma, Ax_{Tot} \cup tot(Ax))$, where Ax_{Tot} consists of all the axioms of Ax_{Str} and of $\wedge\{D(y_i) \mid i = 1 \ldots n\} \supset D(f(y_1, \ldots, y_n))$ for all $f \in F_{s_1 \ldots s_n, s}$.

The total model class of Sp coincides with the class of all non-strict models of $Tot(Sp)$, that is, $\mathrm{TMod}(Sp) = \mathrm{Mod}(Tot(Sp))$.

Proof. It is immediate to check that A is a partial/total algebra iff A is a non-strict algebra satisfying Ax_{Str} / Ax_{Tot}. Moreover, because of Lemma 3.13, $A \models^{t} \phi$ iff $A \models tot(\phi)$ for each strict algebra A and each conditional formula ϕ. Therefore a partial/total algebra satisfies Ax iff it is a non-strict algebra satisfying $Ax_{Str} \cup tot(Ax)$ / $Ax_{Tot} \cup tot(Ax)$. □

Now we can get the well-known results of existence of an initial model for partial positive conditional (see, for example, Broy and Wirsing (1982) and Burmeister (1986)) and for total conditional (see, for example, Goguen and Meseguer (1985)) specifications just as a corollary of Theorem 3.10.

Theorem 3.15. Let $\Sigma = (S, F)$ be a signature and $Sp = (\Sigma, Ax)$ be a conditional specification. Using the notation of Proposition 3.14, both $\mathrm{TMod}(Sp)$ and $PMod(Sp)$ have an initial model.

Proof. Because of Proposition 3.14, $PMod(Sp) = \mathrm{Mod}(Par(Sp))$. Since $Par(Sp)$ is a conditional specification, the theorem follows for $Par(Sp)$ by Theorem 3.10 ($5 \Rightarrow 1$) and Lemma 3.11. We can then argue analogously for $Tot(Sp)$. □

Let us finally consider the partial conditional case, *i.e.*, partial models of axioms of the form $\wedge\Delta \supset \epsilon$, where $\Delta \cup \{\epsilon\}$ is a set of possibly *strong* equalities. Let us recall that $t = t'$ holds iff $(\neg D(t) \wedge \neg D(t')) \vee t {=_e} t'$ holds, so strong equality is only a short notation for a particular kind of disjunction.

Definition 3.16. Let $\Sigma = (S, F)$ be a signature and X be an S-sorted family of variables. The set of *non-positive conditional* formulas over Σ and X consists of

$$\{\wedge\Delta \supset \epsilon \mid \Delta \cup \{\epsilon\} \subseteq Eq(\Sigma, X) \cup SEq(\Sigma, X)\},$$

where $SEq(\Sigma, X) = \{(t = t') \mid t, t' \in T_\Sigma(X)\}$.

Let A be a non-strict algebra over Σ. If ϕ is a non-positive conditional formula and V is a valuation for $Var(\phi)$ in A, we say that *ϕ holds for V in A* (equivalently, *is satisfied for V by A*) and we write $A \models_V \phi$ according to the following definition:

$A \models_V t = t'$ iff $A \models_V (\neg D(t) \wedge \neg D(t')) \vee t =_e t'$ and $A \models_V \wedge \Delta \supset \epsilon$ iff $A \models_V \epsilon$ or there exists $\delta \in \Delta$ such that $A \not\models_V \delta$.
We write $A \models \phi$ for a non-positive conditional formula ϕ and say that ϕ ***holds in*** (equivalently, *is satisfied by* or *is valid in*) A iff $A \models_V \phi$ for all partial valuations V for $Var(\phi)$ in A. Moreover, we write $A \overset{t}{\models} \phi$ iff $A \models_V \phi$ for all total valuations V for $Var(\phi)$ in A.

Remark. Let us consider a non-positive conditional formula $\wedge\Delta \supset \epsilon$. If ϵ is the strong equality $t = t'$, then $\wedge\Delta \supset \epsilon$ is logically equivalent to the pair of axioms $\wedge\Delta \cup \{D(t)\} \supset t =_e t'$ and $\wedge\Delta \cup \{D(t')\} \supset t =_e t'$. Thus from now on we assume that the consequences of the non-positive conditional formulas are always existential equalities.

A non-positive conditional axiom is logically equivalent to an implication involving only existential equalities, but with negated existential equalities in its premises, so that it is not a conditional axiom.

Let us consider the simplest example: a conditional axiom having just one strong equality in its premises. Let α be $t = t' \supset \epsilon$. Then α is logically equivalent to $(\neg D(t) \wedge \neg D(t')) \vee t =_e t' \supset \epsilon$ and hence to the set consisting of $\neg D(t) \wedge \neg D(t') \supset \epsilon$ and of $t =_e t' \supset \epsilon$. If more than one strong equality appears in the premises, a little more machinery is needed. Indeed, let us consider a set Γ_S of strong equality and fix a valuation for its variables in an algebra. Since each strong equality is satisfied iff either the existential equality is satisfied or both sides are undefined, the conjunction $\wedge\Gamma_S$ is satisfied iff it is possible to partition Γ_S into two subsets Δ_1 and Δ_2 such that each equality in Δ_1 is satisfied in the existential form and for each equality in Δ_2 both sides are undefined.

Thus $\wedge\Gamma_S$ is for any valuation equivalent to the disjunction of all formulas of the form $\wedge(\{t =_e t' \mid t = t' \in \Delta_1\} \cup \{\neg D(t), \neg D(t') \mid t = t' \in \Delta_2\})$ for some $\Delta_1 \cup \Delta_2 = \Gamma_S$ and $\Delta_1 \cap \Delta_2 = \emptyset$.

For example, let us consider the formula $\wedge\Gamma_S$, for $\Gamma_S = \{t_1 = t'_1, t_2 = t'_2\}$. Then $\wedge\Gamma_S$ is equivalent to the disjunction of the following four formulas:

$$\begin{array}{l}
(\neg D(t_1) \wedge \neg D(t'_1)) \wedge (\neg D(t_2) \wedge \neg D(t'_2)) \\
t_1 =_e t'_1 \wedge (\neg D(t_2) \wedge \neg D(t'_2)) \\
(\neg D(t_1) \wedge \neg D(t'_1)) \wedge t_2 =_e t'_2 \\
t_1 =_e t'_1 \wedge t_2 =_e t'_2.
\end{array}$$

Note that if $t = t' \in \Delta_1 \cap \Delta_2$,

$$\wedge(\{t =_e t' \mid t = t' \in \Delta_1\} \cup \{\neg D(t), \neg D(t') \mid t = t' \in \Delta_2\})$$

is always false, because $t =_e t' \wedge \neg D(t) \wedge \neg D(t')$ cannot be satisfied. Thus the condition $\Delta_1 \cap \Delta_2 = \emptyset$ can be dropped without affecting the validity of the disjunction.

Let us apply the above discussion to non-positive conditional formulas.

Proposition 3.17. Let Σ be a signature, $\Gamma \cup \{\epsilon\}$ a set of (strong and existential) open equalities over Σ, A a non-strict algebra over Σ, and V a valuation for the variables of $\Gamma \cup \{\epsilon\}$ in A.
Then $A \models_V \wedge \Gamma \supset \epsilon$ iff for all Δ_1, Δ_2 such that $\Delta_1 \cup \Delta_2 = \Gamma - Eq(\Sigma, X)$

$$A \models_V \wedge [(\Gamma \cap Eq(\Sigma, X)) \cup (\{t =_e t' \mid t = t' \in \Delta_1\} \cup \{\neg D(t), \neg D(t') \mid t = t' \in \Delta_2\})] \supset \epsilon.$$

Proof. Let us denote by $\Gamma_E = \Gamma \cap Eq(\Sigma, X)$ the set of existential equalities in Γ, and by $\Gamma_S = \Gamma - Eq(\Sigma, X)$ the set of strong equalities in Γ.

$\Leftarrow$ Assume that for all Δ_1, Δ_2 such that $\Delta_1 \cup \Delta_2 = \Gamma_S$ we have

$$(*) \qquad A \models_V \wedge (\Gamma_E \cup \{t =_e t' \mid t = t' \in \Delta_1\} \cup \{\neg D(t), \neg D(t') \mid t = t' \in \Delta_2\}) \supset \epsilon.$$

If there exists $\gamma \in \Gamma$ such that $A \not\models_V \gamma$, then $A \models_V \wedge \Gamma \supset \epsilon$, by definition of validity. Let us assume, then, that $A \models_V \gamma$ for all $\gamma \in \Gamma$. Then, in particular, $A \models_V t = t'$ for each $t = t' \in \Gamma$, that is, either $A \models_V t =_e t'$, or both $A \not\models_V D(t)$ and $A \not\models_V D(t')$, by definition of strong equality.

Then $\bar{\Delta}_1 \cup \bar{\Delta}_2 = \Gamma_S$, for

$$\bar{\Delta}_1 = \{t = t' \mid t = t' \in \Gamma, A \models_V t =_e t'\},$$
$$\bar{\Delta}_2 = \{t = t' \mid t = t' \in \Gamma, A \not\models_V D(t), A \not\models_V D(t')\}.$$

By definition of $\bar{\Delta}_1$ and $\bar{\Delta}_2$,

$$A \models_V \wedge (\{t =_e t' \mid t = t' \in \bar{\Delta}_1\} \cup \{\neg D(t), \neg D(t') \mid t = t' \in \bar{\Delta}_2\})$$

and $A \models_V \gamma$ for all $\gamma \in \Gamma_E$, because $\Gamma_E \subseteq \Gamma$ and $A \models_V \gamma$ for all $\gamma \in \Gamma$.

Thus $A \models_V \wedge (\Gamma_E \cup \{t =_e t' \mid t = t' \in \bar{\Delta}_1\} \cup \{\neg D(t), \neg D(t') \mid t = t' \in \bar{\Delta}_2\})$, and hence $A \models_V \epsilon$, by (*). Thus $A \models_V \wedge \Gamma \supset \epsilon$.

$\Rightarrow$ Assume that $A \models_V \wedge \Gamma \supset \epsilon$ and let Δ_1, Δ_2 be such that $\Delta_1 \cup \Delta_2 = \Gamma_S$.

Assuming that $A \models_V \wedge (\Gamma_E \cup \{t =_e t' \mid t = t' \in \Delta_1\} \cup \{\neg D(t), \neg D(t') \mid t = t' \in \Delta_2\})$, we first show that $A \models_V t = t'$ for each $t = t' \in \Gamma_S$. Indeed, for each $t = t' \in \Gamma_S = \Delta_1 \cup \Delta_2$, if $t = t' \in \Delta_1$, then $A \models_V t =_e t'$, because $A \models_V \wedge \{t =_e t' \mid t = t' \in \Delta_1\}$, otherwise $t = t' \in \Delta_2$, and hence, because $A \models_V \wedge \{\neg D(t), \neg D(t') \mid t = t' \in \Delta_2\}$, both $A \models_V \neg D(t)$ and $A \models_V \neg D(t')$. Thus $A \models_V t = t'$ for each $t = t' \in \Gamma_S$, by definition of strong equality. Since we have assumed that $A \models_V \wedge \Gamma_E$, and shown that $A \models_V \wedge \Gamma_S$, we have that $A \models_V \wedge \Gamma$, so $A \models_V \epsilon$, because of the assumption $A \models_V \wedge \Gamma \supset \epsilon$, and hence

$$A \models_V \wedge [(\Gamma \cap Eq(\Sigma, X)) \cup (\{t =_e t' \mid t = t' \in \Delta_1\} \cup \{\neg D(t), \neg D(t') \mid t = t' \in \Delta_2\})] \supset \epsilon.$$

□

Then, using the notation of Proposition 3.17, each non-positive conditional axiom $\wedge \Gamma \supset \epsilon$ is logically equivalent to the set of all formulas of the form

$$\wedge [(\Gamma \cap Eq(\Sigma, X)) \cup (\{t =_e t' \mid t = t' \in \Delta_1\} \cup \{\neg D(t), \neg D(t') \mid t = t' \in \Delta_2\})] \supset \epsilon$$

for $\Delta_1 \cup \Delta_2 = \Gamma - Eq(\Sigma, X)$ and hence, putting the above formulas in disjunctive form, to the set of all formulas of the form

$$\vee (\{\epsilon\} \cup \{D(t), D(t') \mid t = t' \in \Delta_2\} \cup \{\neg t =_e t' \mid t = t' \in \Delta_1\} \cup \{\neg \gamma \mid \gamma \in \Gamma \cap Eq(\Sigma, X)\})$$

for $\Delta_1 \cup \Delta_2 = \Gamma - Eq(\Sigma, X)$.

Proposition 3.18. Let $\Sigma = (S, F)$ be a signature, $\phi = \wedge \Gamma \supset \epsilon$ be an open non-positive conditional formula and Y denote the set of the variables of ϕ.

Let us denote by $tot(\phi)$ the set of all disjunctive formulas of the form

$$\begin{aligned} \vee \quad & (\{\epsilon\} \cup \{D(t), D(t') \mid t = t' \in \Delta_2\} \cup \{\neg t =_e t' \mid t = t' \in \Delta_1\} \\ & \cup \{\neg \gamma \mid \gamma \in \Gamma \cap Eq(\Sigma, Y)\} \cup \{\neg D(y) \mid y \in Y\}) \end{aligned}$$

for all Δ_1 and Δ_2 such that $\Delta_1 \cup \Delta_2 = \Gamma - Eq(\Sigma, Y)$.
Then for each algebra A we have that $A \models^{t} \phi$ iff $A \models \theta$ for all $\theta \in tot(\phi)$.

Proof. Because of Proposition 3.17, and putting all $\theta \in tot(\phi)$ in conditional form, we get $[A \models \theta$ for all $\theta \in tot(\phi)]$ iff $A \models \wedge \Gamma \cup \{D(y) \mid y \in Y\} \supset \epsilon$. Because of Lemma 3.13, $A \models \wedge \Gamma \cup \{D(y) \mid y \in Y\} \supset \epsilon$ iff $A \models^{t} \wedge \Gamma \supset \epsilon$. □

The above propositions give sufficient tools to show that each non-positive partial conditional specification may be reduced to a disjunctive non-strict specification.

Proposition 3.19. Let $\Sigma = (S, F)$ be a signature and $Sp = (\Sigma, Ax)$ be a non-positive partial specification. Using the notation of Proposition 3.18, let Sp' be the conditional specification $(\Sigma, Ax_{Str} \cup \cup_{\alpha \in Ax} tot(\alpha))$, where Ax_{Str} consists of $D(f(y_1, \ldots, y_n)) \supset D(y_i)$ for all $f \in F_{s_1 \ldots s_n, s}$. The partial model class of Sp coincides with the class of all non-strict models of Sp', that is, $PMod(Sp) = \mathrm{Mod}(Sp')$.

Proof. It is immediate to check that a non-strict algebra over Σ is also a partial algebra over Σ iff it satisfies all axioms in Ax_{Str}. Because of Proposition 3.18, for each algebra A, $A \models^{t} \alpha$ iff $A \models \theta$ for all $\theta \in tot(\alpha)$, and hence $A \in \mathrm{Mod}(Sp')$ iff A is a partial algebra and $A \models^{t} \alpha$ for all $\alpha \in Ax$. Therefore $\mathrm{Mod}(Sp') = PMod(Sp)$. □

Let us show now that the necessary and sufficient conditions for the existence of an initial model in the case of (non-positive) partial conditional specifications, given in Astesiano and Cerioli (1989; 1995), can be deduced by Condition (5) of Theorem 3.9.

Theorem 3.20. Let Sp be a non-positive conditional specification. There exists an initial model in $PMod(Sp)$ iff for each instantiation $\phi = \alpha[t_y/y \mid y \in Y]$ of an axiom α by defined closed terms t_y, that is, $A \models D(t_y)$ for all $A \in PMod(Sp)$ and all $y \in Var(\alpha)$, at least one of the following conditions holds:

1 $A \models cons(\phi)$ for all $A \in PMod(Sp)$;
2 there exists $t =_e t' \in prem(\phi)$ such that $A \not\models t =_e t'$ for some $A \in PMod(Sp)$;
3 there exists $t = t' \in prem(\phi)$ such that $B \not\models t =_e t'$ for some $B \in PMod(Sp)$, and $A \models D(t)$ for all $A \in PMod(Sp)$, or $A \models D(t')$ for all $A \in PMod(Sp)$.

Proof. Using the notation of Proposition 3.19, $PMod(Sp) = \mathrm{Mod}(Sp')$, where $Sp' = (\Sigma, Ax')$, $Ax' = Ax_{Str} \cup \cup_{\alpha \in Ax} tot(\alpha)$. Hence there exists an initial model in $PMod(Sp)$ iff there exists an initial model in $\mathrm{Mod}(Sp')$. Thus, because of Theorem 3.9, there exists an initial model in $PMod(Sp)$ iff the following conditions are satisfied.

6(a) for all $f \in F_{s_1 \ldots s_n, s}$, if $A \models D(f(t_1, \ldots, t_n))$ for all $A \in \mathrm{Mod}(Sp')$, then one of the following conditions hold:
— $A \models D(t_i)$ for all $A \in \mathrm{Mod}(Sp')$ or
— $A \models f(t_1, \ldots, t_n) =_e f(t_1, \ldots, t_{i-1}, x, t_{i+1}, \ldots, t_n)$ for all $A \in \mathrm{Mod}(Sp')$.

6(b) for all $\vee\Delta \in Ax$ and all substitutions $U: T_\Sigma(Var(\vee\Delta)) \to T_\Sigma(X)$, if $A \models t =_e t'$ for all $A \in \mathrm{Mod}(Sp')$ and all $\neg t =_e t' \in U(\Delta)$, then there exists $\delta \in U(\Delta) \cap Eq(\Sigma, X)$ such that $A \models \delta$ for all $A \in \mathrm{Mod}(Sp')$.

Condition 6(a) follows from Lemma 3.11. Now we show that Condition 6(b) is equivalent to Conditions (1), (2) and (3).
Let us assume that Condition 6(b) holds and show that Conditions (1), (2) and (3) hold

also. To do this let us assume that there exist $\alpha = (\wedge\Gamma \supset \epsilon) \in Ax$ and $U: T_\Sigma(Var(\alpha)) \to T_\Sigma$ such that $A \models D(U(y))$ for all $A \in PMod(Sp)$ and all $y \in Var(\alpha)$, and $U(\alpha)$ does not satisfy Condition (1), or (2), that is, that

i there exists $B \in PMod(Sp)$ such that $B \not\models_U (\epsilon)$,
ii for all $t =_e t' \in U(\Gamma)$ and all $A \in PMod(Sp)$, $A \models t =_e t'$,

and show that it satisfies Condition (3).
Let Y be the family of variables of α and consider the formula $\vee\Delta$, where Δ is the union of the following 5 sets:

— $\{\epsilon\}$
— $\{D(t), D(t') \mid t = t' \in \Delta_2\}$
— $\{\neg t =_e t' \mid t = t' \in \Delta_1\}$
— $\{\neg\gamma \mid \gamma \in \Gamma \cap Eq(\Sigma, Y)\}$
— $\{\neg D(y) \mid y \in Y\})$

for $\Delta_1 = \{t = t' \mid t = t' \in \Gamma, A \models U(t =_e t') \text{ for all } A \in PMod(Sp)\}$ and $\Delta_2 = [prem(\phi) - Eq(\Sigma, X)] - \Delta_1$.
Then $\vee\Delta \in tot(\alpha)$, and hence, by definition of Sp', $\vee\Delta \in Ax'$. Since for all $A \in PMod(Sp)$, $A \models D(U(y))$, by the assumption over U, $A \models U(\gamma)$ for all $\gamma \in \Gamma \cap Eq(\Sigma, Y)$, because of Condition (ii), and $A \models U(t =_e t')$ for all $t =_e t' \in \Delta_1$, by definition of Δ_1, so we have that $A \models U(t =_e t')$ for all $\neg t =_e t' \in \Delta$. Thus, because of Condition 6(b), there exists $\delta \in \Delta \cap Eq(\Sigma, Y)$ such that $A \models U(\delta)$ for all $A \in PMod(Sp)$. Since $B \not\models U(\epsilon)$ for some $B \in PMod(Sp)$, because of Condition (i), there exists $t = t' \in \Delta_2$, so $B \not\models U(t =_e t')$ for some $B \in PMod(Sp)$, such that $A \models U(D(t))$ for all $A \in PMod(Sp)$, or $A \models U(D(t'))$ for all $A \in PMod(Sp)$, that is, Condition (3) holds for $U(\alpha)$.
Conversely, let us assume that Conditions (1), (2) and (3) hold for each instantiation $\phi = \alpha[t_y/y \mid y \in Y]$ of an axiom α by defined closed terms t_y, and show that Condition 6(b) holds too.
Let us assume that for some $\vee\Delta \in Ax'$ and substitution $U: T_\Sigma(Var(\vee\Delta)) \to T_\Sigma(X)$, $A \models t =_e t'$ for all $A \in \mathrm{Mod}(Sp')$ and all $\neg t =_e t' \in U(\Delta)$, and show that there exists $\delta \in U(\Delta) \cap Eq(\Sigma, X)$ such that $A \models \delta$ for all $A \in \mathrm{Mod}(Sp')$.
If $\vee\Delta \in Ax_{Str}$, then the cardinality of $U(\Delta) \cap Eq(\Sigma, X)$ is 1, because all the axioms in Ax_{Str} are positive Horn clauses, and hence Condition 6(b) holds. Thus let us assume that $\vee\Delta$ belongs to $tot(\alpha)$ for some $\alpha \in Ax$. Then Δ is of the form

$$\begin{array}{ll}\vee & (\{\epsilon\} \cup \{D(t), D(t') \mid t = t' \in \Delta_2\} \cup \{\neg t =_e t' \mid t = t' \in \Delta_1\} \\ & \cup\{\neg\gamma \mid \gamma \in \Gamma \cap Eq(\Sigma, Y)\} \cup \{\neg D(y) \mid y \in Y\})\end{array}$$

for some Δ_1, Δ_2 such that $\Delta_1 \cup \Delta_2 = \Gamma - Eq(\Sigma, Y)$, $(\wedge\Gamma \supset \epsilon) \in Ax$ and $Y = Var(\wedge\Gamma \supset \epsilon)$. Then, in particular, from $A \models t =_e t'$ for all $A \in \mathrm{Mod}(Sp')$ and all $\neg t =_e t' \in U(\Delta)$, we get $A \models D(U(y))$ for all $A \in \mathrm{Mod}(Sp')$, and hence, since strictness implies that all subterms of $U(y)$ are defined for all valuations so that none of them is a variable, $U(y)$ are closed terms such that $A \models D(U(y))$ for all $A \in \mathrm{Mod}(Sp')$. Thus Conditions (1), (2) and (3) apply to $U(\wedge\Gamma \supset \epsilon)$. Moreover, from $A \models t =_e t'$ for all $A \in \mathrm{Mod}(Sp')$ and all $\neg t =_e t' \in U(\Delta)$, we get $A \models U(t =_e t')$ for all $t =_e t' \in \Gamma$, that is, Condition (2) does not hold, and hence

1 $A \models U(\epsilon)$ for all $A \in PMod(Sp)$, or

3 there exists $t = t' \in \Gamma$ such that $B \not\models U(t =_e t')$ for some $B \in PMod(Sp)$ and $A \models U(D(t))$ for all $A \in PMod(Sp)$, or $A \models U(D(t'))$ for all $A \in PMod(Sp)$.

If Condition (1) holds, then Condition 6(b) holds also. Thus assume that (3) holds and show that 6(b) holds also. Since we have assumed that $A \models U(t =_e t')$ for all $\neg t =_e t' \in \Delta_1$ and $B \not\models U(t =_e t')$ for some $B \in PMod(Sp)$, $t = t' \in \Delta_2$, and hence Condition (3) implies that there exists $t = t' \in \Delta_2$ such that for all $A \in PMod(Sp)$ $A \models U(D(t))$, or for all $A \in PMod(Sp)$ $A \models U(D(t'))$, that is, Condition 6(b) holds. □

Remark. Although the results on the characterization and existence of initial models for partial and total conditional specifications can be deduced by the more general results for non-strict specifications, the theory of inference systems for non-strict specifications is still unexplored, and hence the results on logical deduction in both partial and total frames (see, for example, Astesiano and Cerioli (1989; 1995), Meseguer and Goguen (1985) and Cerioli (1989), where *conditional* systems for these frames are discussed) are not encompassed. Moreover, as we will see in the next section, non-strict disjunctive specifications inherit, via simulation, the inference systems from the $(\vee, \neg)$-fragment of first-order logic and, since non-strict conditional specifications have the full power of this fragment, it is impossible to found simpler systems tailored to the conditional fragment, as in the usual (total and partial) cases.

4. Relating total and non-strict algebras

Let us now relate the non-strict frame to the more usual total one. To formalize the concept of frame, we adopt the definition of an *institution* (see, for example, Goguen and Burstall (1984)).

Definition 4.1. (Goguen and Burstall 1984, Definition 14) An *institution* $\mathcal{I}$ consists of

— a category **Sign** of *signatures*;
— a functor Sen: **Sign** $\to$ **Set** giving the set of *sentences* over a given signature;
— a functor Mod: **Sign** $\to$ $\mathbf{Cat}^{op}$ giving the category (sometimes called the variety) of *models* of a given signature (the arrows in $Mod(\Sigma)$ are called the *model morphisms*);
— a satisfaction relation†

$$\models \subseteq |Mod(\Sigma)| \times Sen(\Sigma)$$

for each Σ in **Sign**, sometimes denoted $\models_\Sigma$, such that for each morphism $\phi: \Sigma_1 \to \Sigma_2$ in **Sign**, the *Satisfaction Condition*

$$M' \models Sen(\phi)(\xi) \iff Mod(\phi)(M') \models \xi$$

holds for each M' in $|Mod(\Sigma_2)|$ and each ξ in $Sen(\Sigma_1)$.

To relate two frames, the concept of a *simulation* (see, for example, Astesiano and Cerioli (1990; 1994) and Cerioli (1993)) is used. The intuition behind simulations is to code signatures and sentences from a 'new' institution into equivalent ones of an 'old'

† for each category **C** the class of the objects of **C** is denoted by $|\mathbf{C}|$.

institution in such a way that each 'new' model is represented by at least one 'old' model that satisfies exactly (the translation of) the same sentences. Thus models are partially translated from the old into the new institution by a surjective map.

To define simulations, the concept of a *partially* natural transformation is needed. Informally the basic idea is that a partially natural transformation $\alpha: F \rightarrow G$ for some $F, G: \mathbf{C} \rightarrow \mathbf{Cat}^{op}$ is a (usual) natural transformation from a *subfunctor* of F into G, where a subfunctor F' of F is a functor such that $F'(c)$ is a (possibly non-full) subcategory of $F(c)$ for each object c of $\mathbf{C}$, and $F'(h)$ is the restriction of $F(h)$ to $F'(c)$ for each arrow $h \in \mathbf{C}(c, c')$.

Definition 4.2. (Astesiano and Cerioli 1993, Definition 2.3) Given institutions $\mathcal{I} = (\mathbf{Sign}, Sen, Mod, \models)$ and $\mathcal{I}' = (\mathbf{Sign}', Sen', Mod', \models')$, a *simulation* $\mu: \mathcal{I} \rightarrow \mathcal{I}'$ consists of

— a functor $\mu_{Sign}: \mathbf{Sign} \rightarrow \mathbf{Sign}'$,

— a natural transformation $\mu_{Sen}: Sen \rightarrow Sen' \cdot \mu_{Sign}$, that is, a natural family of functions $\mu_{Sen_\Sigma}: Sen(\Sigma) \rightarrow Sen'(\mu_{Sign}(\Sigma))$, and

— a surjective *partially-natural* transformation $\mu_{Mod}: Mod' \cdot \mu_{Sign} \rightarrow Mod$, that is, a family of functors $\mu_{Mod_\Sigma}: dom(\mu)_\Sigma \rightarrow Mod(\Sigma)$, where $dom(\mu)_\Sigma$ is a (not necessarily full) subcategory of $Mod'(\mu_{Sign}(\Sigma))$ such that

  – μ_{Mod_Σ} is surjective on $|Mod(\Sigma)|$,

  – the family is partially-natural, *i.e.*, for each signature morphism $\sigma \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$

$$Mod(\sigma) \cdot \mu_{Mod_{\Sigma_2}} = [\mu_{Mod_{\Sigma_1}} \cdot Mod'(\mu_{Sign}(\sigma))]_{|dom(\mu)_{\Sigma_2}},$$

such that the following *satisfaction condition* holds:

$$A \models \mu_{Sen_\Sigma}(\xi) \iff \mu_{Mod_\Sigma}(A) \models \xi$$

for all $\Sigma \in |\mathbf{Sign}|$, all $A \in |dom(\mu)_\Sigma|$ and all $\xi \in Sen(\Sigma)$.

Whenever it is possible, we will drop the decorations of the simulations, provided that no ambiguity arises.

4.1. *Relationship between models*

Let us first deal with the semantic side of the problem, *i.e.*, the relationship between non-strict and total algebras, disregarding the logics that are used to work on the two frames. Formally this corresponds to considering institutions where the sentence functor and the satisfaction relation are empty.

Definition 4.3. The *institution of non-strict algebras* without sentences is the quadruple $\mathcal{NS} = (\mathbf{Sign}_{\mathcal{NS}}, \emptyset, Mod_{\mathcal{NS}}, \emptyset)$, where:

— $\mathbf{Sign}_{\mathcal{NS}}$ is the category whose objects are *non-strict signatures* defined in Definition 2.2 and whose arrows $\mathbf{Sign}_{\mathcal{NS}}(\Sigma_1, \Sigma_2)$ are pairs (σ, ϕ), where $\sigma: \Sigma_1 \rightarrow \Sigma_2$ is a sort renaming and ϕ is a function symbol renaming consistent with the sort renaming.

— $Mod_{\mathcal{NS}}: \mathbf{Sign}_{\mathcal{NS}} \rightarrow \mathbf{Cat}^{op}$ is the functor that yields for each signature Σ the category $\mathbf{NSAlg}(\Sigma)$ of non-strict algebras and for each signature morphism $(\sigma, \phi) \in$

$\mathbf{Sign}_{\mathcal{NS}}(\Sigma_1, \Sigma_2)$ the *reduct* functor $Mod_{\mathcal{NS}}(\sigma, \phi)\colon \mathbf{NSAlg}(\Sigma_2) \to \mathbf{NSAlg}(\Sigma_1)$, defined by $Mod_{\mathcal{NS}}(\sigma, \phi)(A_2) = (\{\sigma(s)^{A_2}\}_{s \in S}, \{\phi(f)^{A_2}\}_{f \in F})$ and $Mod_{\mathcal{NS}}(\sigma, \phi)(h_2) = \{h_2\sigma(s)\}_{s \in S}$.

The *institution of total algebras* without sentences is the quadruple $\mathcal{MS} = (\mathbf{Sign}_{\mathcal{NS}}, \emptyset, Mod_{\mathcal{MS}}, \emptyset)$, where $Mod_{\mathcal{MS}}\colon \mathbf{Sign}_{\mathcal{NS}} \to \mathbf{Cat}^{op}$ yields for each signature Σ the category of total many-sorted algebras and for each signature morphism $(\sigma, \phi) \in \mathbf{Sign}_{\mathcal{NS}}(\Sigma, \Sigma')$ the *reduct* functor $Mod_{\mathcal{MS}}(\sigma, \phi)\colon Mod_{\mathcal{MS}}(\Sigma') \to Mod_{\mathcal{MS}}(\Sigma)$, defined by $Mod_{\mathcal{MS}}(\sigma, \phi)(A) = (\{\sigma(s)^A\}_{s \in S}, \{\phi(f)^A\}_{f \in F})$ and $Mod_{\mathcal{MS}}(\sigma, \phi)(h) = \{h_{\sigma(s)}\}_{s \in S}$.

Following the intuition that a simulation codes a new into an old frame, we want to define a simulation of non-strict by total algebras.

Since the partial product of $s_1^A, \ldots, s_n^A$ is isomorphic, from a set-theoretical point of view, to the (usual) product of $s_1^A \cup \{\perp_{s_1}\}, \ldots, s_n^A \cup \{\perp_{s_n}\}$, where the symbol $\cup$ denotes the disjoint union, each non-strict algebra A is in some sense equivalent to the total algebra $A_\perp$, defined by

Algebra $A_\perp =$
$s^{A_\perp} = s^A \cup \{\perp_s\}$
for each $a_i \in s_i^{A_\perp}$ for $i = 1 \ldots n$, let $\underline{a}$ be defined by $\underline{a}(i) = a_i$ if $a_i \in s_i^A$ and $\underline{a}(i)$ is undefined if $a_i = \perp_{s_i}$
$f^{A_\perp}(a_1, \ldots, a_n) = f^A(\underline{a})$ if $f^A(\underline{a})$ is defined, otherwise $f^{A_\perp}(a_1, \ldots, a_n) = \perp_s$
for all $f\colon s_1 \times \ldots \times s_n \to s$.

However, this equivalence disregards the homomorphisms. Indeed, some homomorphism h between the trivial totalizations cannot be translated into the non-strict frame, because h maps 'defined' into 'undefined' elements, that is, $h(a) = \perp_s$ for some $a \neq \perp_s$, while the non-strict homomorphisms are total functions. Moreover, some non-strict homomorphisms have no total correspondent, because the introduction of *one* element to represent *all* the undefined terms may cause a lack of existence of homomorphisms, as is shown by the following example.

Example 4.4. Let Σ be the one-sorted signature consisting of just three constant symbols a, b, c, and A, B be the non-strict algebras over Σ, defined by

Algebra $A =$
$s^A = \{1\}$ $\quad a^A = 1$; b^A, c^A are undefined

Algebra $B =$
$s^B = \{1\}$ $\quad a^B = 1 = b^B$; c^B is undefined.

Then there is a non-strict homomorphism $h\colon A \to B$, defined by $h(1) = 1$. Now consider the trivial totalizations of A and B.

Algebra $A_\perp =$
$s^{A_\perp} = \{1, \perp\}$ $\quad a^{A_\perp} = 1$; $b^{A_\perp} = \perp = c^{A_\perp}$

Algebra $B_\perp =$
$s^{B_\perp} = \{1, \perp\}$ $\quad a^{B_\perp} = 1 = b^{B_\perp}$; $c^{B_\perp} = \perp$.

Then there does not exist any total homomorphism from $A_\perp$ into $B_\perp$, because $b^{A_\perp} = c^{A_\perp}$, while $b^{B_\perp} \neq c^{B_\perp}$.

Summarizing the above discussion, we define a simulation of non-strict by total algebras, which is a rigorous formalization of the usual totalization by $\perp$.

Definition 4.5. The simulation $\mu_0^{\perp}: \mathcal{NS} \to \mathcal{MS}$ is defined by:

— $\mu_0^{\perp}(S, F) = (S, F \cup \{\perp_s\}_{s \in S})$ and $\mu_0^{\perp}(\sigma, \phi) = (\sigma, \phi')$, where $\phi'(f) = \phi(f)$ for all $f \in F$ and $\phi'(\perp_s) = \perp_{\sigma(s)}$.

— $dom(\mu_0^{\perp})$ is the category whose objects are the total algebras A' where the interpretation of function symbols are *regular* function, that is,

$$f^{A'}(a_1, \ldots, a_{i-1}, \perp_{s_i}, a_{i+1}, \ldots, a_n) \neq \perp_s^{A'}$$

implies

$$f^{A'}(a_1, \ldots, a_{i-1}, \perp_{s_i}, a_{i+1}, \ldots, a_n) = f^{A'}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n)$$

for each a_i, and whose morphisms h *preserve definedness*, that is, $a \neq \perp_s^{A'}$ implies $h(a) \neq \perp_s^{B'}$.

— For each A' in the objects of $dom(\mu_0^{\perp})$ the translation $A = \mu_0^{\perp}(A')$ is the non-strict algebra that consists of $s^A = s^{A'} - \{\perp_s^{A'}\}$ for each $s \in S$, and for each $f \in F_{s_1 \ldots s_n, s}$ the function f^A is defined by:
if $f^{A'}(a_1, \ldots, a_n) \neq \perp_{s_i}$, then $f^A(\underline{a}) = f^{A'}(a_1, \ldots, a_n)$, otherwise $f^A(\underline{a})$ is undefined, where
for each $\underline{a} \in s_1^A \times_p \ldots \times_p s_n^A$ let a_i be $\underline{a}(i)$, if $\underline{a}(i)$ is defined, $\perp_{s_i}$ otherwise.
For each arrow h' in $dom(\mu_0^{\perp})$ the translation $h = \mu_0^{\perp}(h')$ is the restriction of h' to $\mu_0^{\perp}(A')$.

It is easy to check that the components of $\mu_0^{\perp}$ with respect to the models are partially natural and hence that $\mu_0^{\perp}$ is a simulation.

Since $\mu_0^{\perp}$ does not take into account the categorical structure, the initiality in the total and in the non-strict frames are unrelated. Indeed the trivial totalization of an initial model satisfies a lot of equalities between 'undefined' terms, which are not satisfied by other models in the class, so the no-confusion condition in the total frame is not satisfied and hence the trivial totalization of an initial model is in general not initial. Conversely, if the trivial totalization of a non-strict algebra is initial, the algebra is *maximally* defined and hence it is not initial in the non-strict frame.

In order to have a representation of the category of non-strict algebras, we need a definition of (total) homomorphism that does not involve the 'undefined' part. To do this it is useful, not to say necessary, to have a tool to individuate the 'undefined' elements, for example a family of unary predicates, one for each sort, dividing the carriers into 'defined' and 'undefined'. Following a similar idea, both Broy and Wirsing (1984) and Poigné (1987) define homomorphisms that are partial functions, having as domain the 'defined' part. This approach can be generalized to include non-strictness. To fix the notation, let us introduce the *total algebras with predicates*, or *first-order structures*.

Definition 4.6. The *institution of first-order structures* without sentences is the quadruple $\mathcal{TL} = (\mathbf{Sign}_{\mathcal{TL}}, \emptyset, Mod_{\mathcal{TL}}, \emptyset)$, where:

— $\mathbf{Sign}_{\mathcal{TL}}$ is the category whose objects are *signatures with predicates*, that is, triples (S, F, P), where S is a set of *sorts*, F is an $S^* \times S$-sorted family of *function* symbols and P is a S^+-sorted family of *predicate* symbols, and whose arrows $\mathbf{Sign}_{\mathcal{TL}}(\Sigma_1, \Sigma_2)$

are triples (σ, ϕ, π), where σ, ϕ and π are consistent renamings of, respectively, sorts, function and predicate symbols.

— $Mod_{\mathcal{MS}}: \mathbf{Sign}_{\mathcal{TL}} \to \mathbf{Cat}^{op}$ is the functor that yields for each signature with predicates $\Sigma = (S, F, P)$ the category of algebras with predicates, or first-order structures, whose objects are triples $(\{s^A\}_{s \in S}, \{f^A\}_{f \in F}, \{p^A\}_{p \in P})$, where s^A are arbitrary sets, f^A are total functions and $p^A \subseteq s_1^A \times \ldots \times s_n^A$ are the truth sets, and whose arrows $h: A \to B$ are families $h = \{h_s: s^A \to s^B\}_{s \in S}$ of total functions such that $h_s(f^A(a_1, \ldots, a_n)) = f^B(h_{s_1}(a_1), \ldots, h_{s_n}(a_n))$ and $(a_1, \ldots, a_n) \in p^A$ implies $(h_{s_1}(a_1), \ldots, h_{s_n}(a_n)) \in p^B$.
For each signature morphism $(\sigma, \phi, \pi) \in \mathbf{Sign}_{\mathcal{TL}}(\Sigma_1, \Sigma_2)$, the *reduct* functor $Mod_{\mathcal{TL}}(\sigma, \phi, \pi): Mod_{\mathcal{TL}}(\Sigma_2) \to Mod_{\mathcal{TL}}(\Sigma_1)$ is defined by
  – $Mod_{\mathcal{TL}}(\sigma, \phi, \pi)(A_2) = (\{\sigma(s)^{A_2}\}_{s \in S}, \{\phi(f)^{A_2}\}_{f \in F}, \{\pi(p)^{A_2}\}_{p \in P})$,
  – $Mod_{\mathcal{TL}}(\sigma, \phi)(h_2) = \{h_{2\sigma(s)}\}_{s \in S}$.

The basic idea of the following simulation of non-strict by first-order structures is to split the carriers of a first-order structure into defined and undefined elements, provided that at least one undefined element, denoted by $\perp$, exists, by means of unary *definedness* predicates. Thus the simulation is defined on each first-order structure satisfying the monotonicity condition and where $\perp$ is undefined; it yields the non-strict algebra where the undefined part of the carriers has been dropped. Since the homomorphisms in the first-order frame preserve the truth of predicates, each homomorphism between two such first-order structures can also be translated into a non-strict homomorphism. Thus the domain of this simulation is a full subcategory.

Definition 4.7. The simulation $\mu_0^P: \mathcal{NS} \to \mathcal{TL}$ is defined by:

— $\mu_0^P(S, F) = (S, F \cup \{\perp_s\}_{s \in S}, \{D_s : s, eq_s : s \times s\}_{s \in S})$ and $\mu_0^P(\sigma, \phi) = (\sigma, \phi', \pi)$, where $\phi'(f) = \phi(f)$ for all $f \in F$, $\phi'(\perp_s) = \perp_{\sigma(s)}$, $\pi(D_s) = D_{\sigma(s)}$ and $\pi(eq_s) = eq_{\sigma(s)}$.

— $dom(\mu_0^P)$ is the full sub-category whose objects are the first-order structures A' such that
  1 $D_s^{A'}(f^{A'}(a_1, \ldots, a_n))$ implies $D_{s_i}^{A'}(a_i)$ or $f^{A'}(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) = f^{A'}(a_1, \ldots, a_n)$, for all a,
  2 $eq_s^{A'}(a, a')$ iff $D_s^{A'}(a)$, $D_s^{A'}(a')$ and $a = a'$,
  3 $\neg D_s^{A'}(\perp_s^{A'})$.

For each A' in the objects of $dom(\mu_0^P)$ the translation $A = \mu_0^P(A')$ is the non-strict algebra, which consists of $s^A = D_s^{A'}$ for each $s \in S$ and for each $f \in F$ the function f^A is defined by:
if $D_s^{A'}(f^{A'}(a_1, \ldots, a_n))$, then $f^A(\underline{a}) = f^{A'}(a_1, \ldots, a_n)$, else $f^A(\underline{a})$ is undefined,
where
for each $\underline{a} \in s_1^A \times_p \ldots \times_p s_n^A$ let a_i be $\underline{a}(i)$, if $\underline{a}(i)$ is defined, $\perp_{s_i}$ otherwise.
For each arrow h' in $dom(\mu_0^P)$ the translation $h = \mu_0^P(h')$ is the restriction of h' to $\mu_0^P(A')$.

By applying some results from Astesiano and Cerioli (1993), we get that the initial model is preserved by μ_0^P.

Definition 4.8. Let $\mu: \mathcal{I} \to \mathcal{I}'$ be a simulation. Then μ is called *categorical* iff

1 $dom(\mu)_\Sigma$ is a full subcategory of $Mod'(\mu(\Sigma))$ for each $\Sigma \in |\mathbf{Sign}|$;

2 μ preserves inductive objects, *i.e.*, for each $I' \in |dom(\mu)|$ if $Mod'(\mu(\Sigma))(I', A')$ has cardinality at most 1 for each $A' \in |Mod'(\mu(\Sigma))|$, then $Mod(\Sigma)(\mu(I'), A)$ has cardinality at most 1 for each $A \in |Mod(\Sigma)|$.

Lemma 4.9. If $\mathcal{I}'$ has equalizers (that is, $Mod'(\Sigma')$ has equalizers for each Σ') and $C' \subseteq dom(\mu)$ is closed with respect to regular subobjects, then each categorical simulation μ preserves initial models, that is, I' initial in C' implies $\mu(I')$ initial in $\mu(C')$.

Proof. See Proposition 2.9 of Astesiano and Cerioli (1993). □

Proposition 4.10. The simulation $\mu_0^P : \mathcal{NS} \to \mathcal{TL}$ defined in Definition 4.7 is categorical. Moreover, for each class $C' \subseteq dom(\mu_0^P)$ of first-order structures closed with respect to subalgebras, if I' is initial in C', then $\mu_0^P(I')$ is initial in $\mu_0^P(C')$.

Proof. By definition, $dom(\mu_0^P)$ is a full subcategory. Moreover, in both frames inductive objects coincide with term-generated algebras and it is easy to check that term-generated first-order structures are translated via μ_0^P into term-generated non-strict algebras. Therefore μ_0^P is categorical, and hence Lemma 3.1.7 applies, because the category of first-order structures has equalizers, which coincide with subalgebras. □

The relationship between non-strict algebras and first-order structures described by the categorical simulation μ_0^P is strengthened by the existence of left adjoints of the model components of μ_0^P (from now on denoted by *Tot*), corresponding, as usual, to free constructions. Indeed, *Tot* preserves initiality, because left adjoints do, and, moreover, because of Proposition 4.10, μ_0^P preserves initiality, too, and hence the existence of the initial model in the non-strict and in the total frame are completely equivalent.

To build such *Tot*, we need some preliminary technical results.

Lemma 4.11. Let $\Sigma = (S, F)$ be a non-strict signature, A be a non-strict algebra over Σ, X^A be $\{X_s\}_{s\in S}$, with $X_s = s^A \cup \{\perp\}$ for all $s \in S$, and $V_A : X^A \to A$ be the valuation defined by $V_A(a) = a$ if $a \in s^A$, $V_A(\perp)$ undefined. Let $\equiv^A$ denote the total congruence over $T_\Sigma(X^A)$ generated by $\{(t, t') \mid t \in T_\Sigma(X^A), A \models_{V_A} t =_e t'\}$ and $Tot(A)$ denote the algebra $(T_\Sigma(X^A)/\equiv^A, \{\perp_s^{Tot(A)}\}_{s\in S}, \{D_s^{Tot(A)}, eq_s^{Tot(A)}\}_{s\in S})$, where $\perp_s^{Tot(A)} = [\perp]_{\equiv^A}$, $D_s^{Tot(A)}([t]_{\equiv^A})$ iff $A \models_{V_A} D(t)$ and $eq_s^{Tot(A)}([t]_{\equiv^A}, [t']_{\equiv^A})$ iff $A \models_{V_A} t =_e t'$.
The following facts hold:

1 $t \equiv^A t'$ and $A \models_{V_A} D(t)$ or $A \models_{V_A} D(t')$ imply $A \models_{V_A} t =_e t'$;

2 $Tot(A)$ belongs to $dom(\mu_0^P)$;

3 $\epsilon^A : A \to \mu_0^P(Tot(A))$, defined by $\epsilon^A(a) = [a]_{\equiv^A}$, is an isomorphism.

Proof.

1 By induction over the definition of $\equiv^A$.
The only non-trivial step is for the functional closure. Let us assume that $t_i \equiv^A t'_i$ and that the property holds for each pair t_i, t'_i, and show that the property also holds for $f(t_1, \ldots, t_n), f(t'_1, \ldots, t'_n)$. Since $A \models_{V_A} D(t_i)$ or $A \models_{V_A} D(t'_i)$ imply $A \models_{V_A} t_i =_e t'_i$ for $i = 1 \ldots n$, $t_i^{A,V_A} = {t'}_i^{A,V_A}$ for $i = 1 \ldots n$, and hence $f(t_1, \ldots, t_n)^{A,V_A} = f(t'_1, \ldots, t'_n)^{A,V_A}$. Thus we have the thesis.

2 Because of (1) and of the definition of both $D_s^{Tot(A)}$ and $eq_s^{Tot(A)}$, Conditions (2) and (3) of Definition 4.7 are satisfied. Let us assume that $[f(t_1, \ldots, t_n)]_{\equiv^A} \in D_s^{Tot(A)}$ and

$[t_i]_{\equiv^A} \notin D_{s_i}^{Tot(A)}$. Then, because of (1), $f(t_1,\ldots,t_n)^{A,V_A} \in s^A$ and $t_i^{A,V_A} \notin s_i^A$, that is, $f^A(t_1^{A,V_A},\ldots,t_{i-1}^{A,V_A},?,t_{i+1}^{A,V_A},\ldots,t_n^{A,V_A}) \in s^A$ and hence, by the monotonicity of f^A,

$$f^A(t_1^{A,V_A},\ldots,t_{i-1}^{A,V_A},?,t_{i+1}^{A,V_A},\ldots,t_n^{A,V_A}) =_e f^A(t_1^{A,V_A},\ldots,t_{i-1}^{A,V_A},t^{A,V_A},t_{i+1}^{A,V_A},\ldots,t_n^{A,V_A})$$

for all $t \in T_{\mu_0^P(\Sigma)}(X^A)$, so $f(t_1,\ldots,t_n) \equiv^A f(t_1,\ldots,t_{i-1},t,t_{i+1},\ldots,t_n)$, and hence Condition (1) is satisfied also.

3 This is obvious. □

Let us consider a non-strict homomorphism $h\colon A \to B$. In order to define its image along Tot, we use h as a valuation from X^A into X^B and then show that $t \equiv^A t'$ implies $h(t) \equiv^B h(t')$, so that $Tot(h)([t]_{\equiv^A}) = [h(t)]_{\equiv^B}$ is well defined.

$$\begin{array}{ccccc} A & & X^A & \longrightarrow & T_\Sigma(X^A)/\equiv^A \\ h\big\downarrow & & h\big\downarrow & & \big\downarrow Tot(h) \\ B & & X^B & \longrightarrow & T_\Sigma(X^B)/\equiv^B \end{array}$$

Lemma 4.12. Let A and B be non-strict algebras over Σ, and $h\colon A \to B$ be a non-strict homomorphism. Using the notation of Lemma 4.11:

1 For each $t \in T_\Sigma(X^A)$ let $h(t)$ denote the term $t[h(a)/a \mid a \in s^A] \in T_\Sigma(X^B)$. Then for all $t,t' \in T_\Sigma(X^A)$ $t \equiv^A t'$ implies $h(t) \equiv^B h(t')$.

2 $Tot(h)\colon Tot(A) \to Tot(B)$, defined by $Tot(h)([t]_{\equiv^A}) = [h(t)]_{\equiv^B}$, is a homomorphism of first-order structures.

Proof.

1 It is easy to check that, by definition of congruence, $\{(h(t),h(t')) \mid t \equiv^A t'\} \subseteq \approx$, where $\approx$ is the congruence generated by $\{(h(t),h(t')) \mid A \models_{V_A} t =_e t'\}$, because $\equiv^A$ is generated by $\{(t,t') \mid A \models_{V_A} t =_e t'\}$. Thus it is sufficient to show that $\approx \subseteq \equiv^B$. To do this let us assume that $A \models_{V_A} t =_e t'$ for some $t,t' \in T_\Sigma(X^A)$ and show that $B \models_{V_B} h(t) =_e h(t')$. Because of Corollary 2.9, $B \models_{h \cdot V_A} t =_e t'$, and hence, $B \models_{V_B} h(t) =_e h(t')$, since $t^{B,h\cdot V_A} = h(t)^{B,V_B}$ by definition of V_A and V_B.

2 Because of (1), $Tot(h)$ is a well-defined Σ-homomorphism. Thus we only have to show that it preserves the operations $\perp_s$ and the truth of the predicates. By definition, $Tot(h)([\perp]_{\equiv^A}) = [h(\perp)]_{\equiv^B} = [\perp]_{\equiv^B}$.
Because of Lemma 4.11, $[t]_{\equiv^A} \in D_s^{Tot(A)}$ implies $A \models_{V_A} D(t)$, and hence, because of Corollary 2.9, $B \models_{h\cdot V_A} D(t)$, that is, $B \models_{V_B} D(h(t))$, so $Tot(h)([t]_{\equiv^A}) = [h(t)]_{\equiv^B} \in D_s^{Tot(B)}$.
Analogously, $([t]_{\equiv^A},[t']_{\equiv^A}) =_e^{Tot(A)}$ implies $A \models_{V_A} t =_e t'$, so $B \models_{V_B} h(t) =_e h(t')$, and hence $([h(t)]_{\equiv^B},[h(t')]_{\equiv^B}) =_e^{Tot(B)}$. Therefore, $Tot(h)$ is a homomorphism of first-order structures. □

And, finally, we can put Lemma 4.12 and Proposition 4.13 together to define the functor Tot.

Theorem 4.13. Let $\Sigma = (S, F)$ be a non-strict signature. Now, using the notation of Lemma 4.11 and Lemma 4.12, *Tot* is a functor and is the left adjoint and left inverse of μ_0^P. Moreover, if I is initial in a class C of non-strict algebras closed with respect to isomorphisms, then $Tot(I)$ is initial in $\mu_0^{P^{-1}}(C)$.

Proof. It is just a trivial check to prove that *Tot* is a functor.
Let us show that *Tot* is the left adjoint of μ_0^P and that the family of the isomorphisms ϵ^A, defined in Lemma 4.11, is the counit of the adjunction. Let A be a non-strict algebra, B' a first-order structure belonging to $dom(\mu_0^P)$ and $h: A \to \mu_0^P(B')$ a non-strict homomorphism. We denote by $k^{B'}: Tot(\mu_0^P(B')) \to B'$ the homomorphism defined by $k^{B'}([t]_{\equiv_{B'}}) = t^{V,B'}$ for $V: \mu_0^P(B') \to B'$ the identical valuation, and show that $h^\sharp = k^{B'} \cdot Tot(h)$ is the unique homomorphism from $Tot(A)$ into B' such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\epsilon^A"] \arrow[dr, "h"'] & \mu_0^P(Tot(A)) \arrow[d, "\mu_0^P(h^\sharp)"] & Tot(A) \arrow[d, "h^\sharp"] \\ & \mu_0^P(B') & B' \end{tikzcd}$$

By definition of ϵ^A and $h^\sharp$,

$$\mu_0^P(h^\sharp) \cdot \epsilon^A(a) = \mu_0^P(h^\sharp)([a]_{\equiv_A}) = \mu_0^P(k^{B'} \cdot Tot(h))([a]_{\equiv_A})$$

and

$$\mu_0^P(k^{B'} \cdot Tot(h))([a]_{\equiv_A}) = k^{B'} \cdot Tot(h)([a]_{\equiv_A}) = k^{B'}([h(a)]_{\equiv_{B'}}),$$

since μ_0^P is the restriction. Finally, $k^{B'}([h(a)]_{\equiv_{B'}}) = h(a)$, by definition of $k^{B'}$, so the diagram commutes.
Moreover, $h^\sharp$ is the unique arrow that makes the diagram commute. Indeed let k be such that $\mu_0^P(k) \cdot \epsilon^A = h$. Then, by definition of μ_0^P and ϵ^A, $\mu_0^P(k) \cdot \epsilon^A = k([a]_{\equiv_A}) = h(a)$ for each $a \in s^A$, and hence k and $h^\sharp$ coincide on the (equivalence classes of) variables, and hence, by induction, on $Tot(A)$.
Finally, since left adjoints preserve initiality and *Tot* is the left adjoint of μ_0^P, if I is initial in C, then $Tot(I)$ is initial in any class C' such that both $Tot: C \to C'$ and $\mu_0^P: C' \to C$. In particular, if C is closed with respect to isomorphisms, $\mu_0^{P^{-1}}(C)$ is closed with respect to isomorphisms also, and hence, A being isomorphic to $\mu_0^P(Tot(A))$ by Lemma 4.11, $Tot: C \to \mu_0^{P^{-1}}(C)$, and, obviously, $\mu_0^P: \mu_0^{P^{-1}}(C) \to C$, so $\mu_0^P(I)$ is initial in $\mu_0^{P^{-1}}(C)$. □

4.2. *Preserving logics*

Let us now investigate the logical aspects of the relationships between non-strict and total algebras. To this aim we consider the institutions $\mathcal{CNS}$ and $\mathcal{DNS}$ of non-strict algebras with, respectively, conditional and disjunctive axioms as sentences, and the institutions $\mathcal{TL}$ and $\mathcal{DTL}$ of first-order structures with, respectively, conditional and disjunctive axioms built on atomic formulas of the form $p(t_1, \ldots, t_k)$ as sentences. The validity in the total frame is defined analogously to the non-strict frame case, starting from $A \models_V p(t_1, \ldots, t_k)$ iff $(t_1^{A,V}, \ldots, t_k^{A,V}) \in p^A$, but now the valuations of variables are total functions (see the validity $\stackrel{t}{\models}$).

We consider first the trivial totalization $\mu_0^{\perp}$. Let A' belong to $dom(\mu_0^{\perp})$ and consider a ground existential equality $t =_e t'$. Then $A = \mu_0^{\perp}(A')$ satisfies $t =_e t'$ iff both t and t' denote the same element of $s^A = s^{A'} - \{\perp_s^{A'}\}$, that is, iff $t^{A'} = t'^{A'} \neq \perp_s^{A'}$. Thus, to extend $\mu_0^{\perp}$ to a simulation working on equations of the non-strict frame, inequalities are needed in the total frame. This is another inadequacy of the trivial totalization, which has already been proved incapable of dealing with the categorical structure of the non-strict frame.

Let us consider now the simulation μ_0^P, defined in Definition 4.7. It is easy to extend μ_0^P to work on conditional (disjunctive) formulas, *i.e.*, to define two simulations $\mu_C : \mathcal{CNS} \to \mathcal{TL}$ and $\mu_D : \mathcal{DNS} \to \mathcal{DTL}$ coinciding with μ_0^P on signatures and models. Indeed, any conditional (disjunctive) formula can be naturally translated from the non-strict into the first-order frame, by just replacing the existential equalities by the eq_s predicates, which were indeed introduced to represent existential equality.

Definition 4.14. The simulation $\mu_C : \mathcal{CNS} \to \mathcal{TL}$ is the extension of $\mu_0^P : \mathcal{NS} \to \mathcal{TL}$, that on $\wedge\{t_i =_e t'_i \mid i \in I\} \supset t =_e t'$ yields $\wedge\{eq_{s_i}(t_i, t'_i) \mid i \in I\} \supset eq_s(t, t')$.
Similarly, the simulation $\mu_D : \mathcal{DNS} \to \mathcal{DTL}$ is the extension of $\mu_0^P : \mathcal{NS} \to \mathcal{TL}$ yielding $\vee\{eq_{s_i}(t_i, t'_i) \mid i \in I\} \vee \{\neg eq_{s_j}(t_j, t'_j) \mid j \in J\}$ on $\vee\{t_i =_e t'_i \mid i \in I\} \vee \{\neg t_j =_e t'_j \mid j \in J\}$.

Since the domain of μ_D is the model class of the set Ax_{DC} consisting of the following disjunctive axioms:

— $\neg D_s(f(x_1, \ldots, x_n)) \vee D_{s_i}(x_i) \vee eq_s(f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n), f(x_1, \ldots, x_n))$;
— $eq_s(x, x') \vee \neg D_s(x) \vee \neg D_s(x') \vee \neg x = x'$;
— $\neg eq_s(x, x') \vee D_s(x)$;
— $\neg eq_s(x, x') \vee D_s(x')$;
— $\neg eq_s(x, x') \vee x = x'$;
— $\neg D_s(\perp_s)$;

the non-strict model class of a set Ax of disjunctive formulas is simulated by the total model class of the set $\mu_D(Ax) \cup Ax_{DC}$ of disjunctive formulas in the first-order frame. Therefore μ_D induces a correspondence between the specifications in the two formalisms, and hence both frames have the same expressive power.

Instead, the simulation μ_C does not relate conditional to conditional specifications, because proper disjunctive specifications are required to describe the domain of the simulation; indeed there does not exist a total conditional specification whose model class is $dom(\mu_0^P)$, because the trivial total algebra Tr over $\mu_0^P(\Sigma)$, having singleton sets as carriers, the unique obvious interpretation of function symbols and the totally true predicates (that is, $D_s^{Tr} = s^{Tr}$ and $eq_s^{Tr} = s^{Tr} \times s^{Tr}$) is a model of each conditional specification but does not belong to the domain. Therefore, in general it is impossible to translate a conditional (equational) non-strict specification into a conditional first-order one, because it is impossible, at least, for the specification without axioms $(\Sigma, \emptyset)$.

Since the first-order structures representing the models of a non-strict disjunctive specifications are the models of a (total) disjunctive specification, and initiality is both preserved, because μ_0^P is categorical, and reflected, because of the existence of the left adjoint *Tot*, by μ_0^P, the existence of an initial model for a non-strict disjunctive specification is equivalent to the existence of an initial model for the first-order disjunctive specification that is simulating it.

Corollary 4.15. Let $Sp = (\Sigma, Ax)$ be a non-strict disjunctive specification and $\mu_0^P(Sp)$ denote the first-order disjunctive specification $(\mu_0^P(\Sigma), \mu_0^P(Ax) \cup Ax_{DC})$. Then I is initial for Sp iff $Tot(I)$ is initial for $\mu_0^P(Sp)$.

Proof. Since the model classes of disjunctive specifications in both frames are closed with respect to isomorphisms and regular subobjects, Proposition 4.10 and Theorem 4.13 apply.
If I is initial for Sp, then $Tot(I)$ is initial for $\mu_0^P(Sp)$, because of Theorem 4.13, and if $Tot(I)$ is initial for $\mu_0^P(Sp)$, then $I \approx \mu_0^P(Tot(I))$ is initial for Sp, because of Proposition 4.10. □

Since the domain of the simulation μ_D is the model class of a set of axioms, by applying a result from Astesiano and Cerioli (1993), it is possible to translate inference systems from the total disjunctive frame (which is a fragment of first-order logic) into the non-strict frame in such a way that soundness and completeness are preserved.

Proposition 4.16. Let $\vdash$ be an inference system for the disjunctive sentences built on (possibly negated) atoms in the total frame. Then $\vdash_{\mu_D}$ is the inference system for disjunctive non-strict specifications defined by $Ax \vdash_{\mu_D} \xi$ iff $\mu_0^P(Ax) \cup Ax_{DC} \vdash \mu_0^P(\xi)$.
If $\vdash$ is sound (complete), then $\vdash_{\mu_D}$ is sound (complete) also.

Proof. See Corollary 3.5 of Astesiano and Cerioli (1993). □

Let us now summarize our previous investigation of the relationships between total and non-strict algebras. Each non-strict algebra may be represented by a total algebra, where a special element $\perp$ has been added to the carriers to denote the 'undefined' elements, but this trivial totalization cannot be lifted to a categorical correspondence, since it cannot be expressed by a functor.

For a satisfactory categorical translation of non-strict into total algebras we need some more algebraic tools (in particular, definedness predicates), and get that each non-strict algebra may be represented by many different first-order structures, which we can think of as its *implementations*, satisfying the same formulas. However, the class of first-order structures representing the models of a non-strict equational specification cannot be described by conditional axioms, so there is no correspondence between (equational) conditional non-strict specifications and conditional first-order specifications.

There exists a disjunctive specification having as models the first-order structures representing non-strict algebras. Thus each disjunctive non-strict specification is represented by a disjunctive first-order specification.

5. Conclusion

This paper has presented two main results, clarifying, we believe, two basic issues.

The central result is a *theory of initial models of non-strict disjunctive specifications*, which encompasses not only the well-known initial theories of total conditional specifications and partial positive-conditional specifications, but also the recently explored partial non-positive conditional specifications.

Moreover, *the relationship between non-strict and totalized algebras* is analyzed, emphasizing that the relationship has been dealt with at three different levels: models, categories and specifications. Only at the first level, which is the only one considered in basic denotational semantics, the correspondence is trivial.

We see a main direction for further research to be allowing don't care conditions and error-handling in the same paradigm: a promising approach seems to be to merge the present theory with the development of error-handling in Poigné (1987).

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