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Investment Basics XLI: Duration and convexity

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1. Introduction

This is the second in a series of two articles dealing with measuring and managing the impact of interest rate changes on bond prices. Whereas the first article, which appeared in the previous issue, discussed bond price volatility; this article discusses the concepts of duration and convexity.

2. Duration

The previous article identifies three key variables as determinants of bond price volatility, namely a bond's term to maturity, initial required market yield, and coupon rate. Whereas the term to maturity is positively related to a bond's price volatility, the other two variables are inversely related to a bond's price volatility. Measures of bond price volatility discussed are the price value of a basis point and the change in the required market yield for a specified price change. However, these measures are inadequate, inaccurate, and cumbersome to gauge. A better measure of bond price volatility is derived using calculus and is called duration. This measure combines the attributes of both a bond's term to maturity and coupon. The two most common variants of duration are Macaulay's duration and modified duration.

2.1 Macaulay's Duration

The concept of duration was first introduced by Frederick Macaulay in 1938 while trying to define the correct measure of the life of a fixed income investment. Because term to maturity ignores the amount and timing of all but the final cash flow, Macaulay standardized the life of a fixed income investment by viewing each payment as a zero-coupon (pure discount) bond. This way, a bond's term to maturity and coupon payments are combined into a single measure of its life, which is its effective maturity (or weighted average maturity) of its cash flows on a present value (PV) basis.

Since a change in interest rates leads to a proportional change in a bond's price that can be related to changes in the yield to maturity (YTM), it is appropriate to use YTM (denoted as y) in calculating the bond's duration. The PV of each cash flow from the bond is firstly divided by the PV of the bond, which is also its current price (P0). The numerator on the right hand side of equation (1) below is the PV of a cash flow occurring at time t, whereas the denominator is the PV of all payments that the bondholder will receive from the bond until it matures. Therefore, the sum of all the weights is necessarily one. The weight w assigned to a specific cash flow (F) from the bond receivable at time t is therefore:

$$w_t = \frac{F_t / (1+y)^t}{P_0} \quad \text{...(1)}$$

In order to calculate duration, each weighting factor is then multiplied by the corresponding time in years (t) when the cash flow is receivable. The sum of these products represents the weighted average maturity of the bond, or Macaulay's duration (D), as shown in equation (2) below:

$$D = \sum_{t=1}^{T} t\,(w_t) \quad \text{...(2)}$$

2.2 Properties of Macaulay's Duration

Macaulay's duration can be used to determine and compare the YTM-price volatility of bonds with different maturities and coupons. Several properties of Macaulay's duration can be distinguished from the preceding discussion. These properties show how duration is related to the three key determinants of bond price volatility, namely term to maturity, coupon rate, and YTM. These properties can be summarized as follows:

(i) Because coupon payments are given weight in calculating duration, Macaulay's duration of a coupon bond is always less than the bond's term to maturity.

(ii) The duration of a zero-coupon bond is exactly equal to its term to maturity. This is because the single payment of the bond is made only when the bond matures.

(iii) If maturity is held constant, a bond with a lower coupon rate has a higher duration than a bond with a higher coupon rate.

(iv) Given a specific coupon rate, a bond with a longer maturity has a higher duration than a bond with a shorter term to maturity. Additionally, duration increases with maturity for bonds selling at par or at a premium. For bonds selling at high discounts, duration may decrease with increasing maturity. This is because bonds selling at a deep discount tend to increase in price as maturity approaches.`

(v) Other things held constant, the duration of a coupon bond is higher if the bond's YTM is lower.

2.3 Modified Duration

Macaulay's duration needs to be modified in order to determine the amount by which a bond's price will change for small changes in interest rates. The term modified duration (D*) thus refers to Macaulay's duration in equation (2) above, divided by (1+y); and is given by:

$$D^* = \frac{D}{(1 + y)} \quad ...(3)$$

Denoting the change in price as DP and the initial price as P, modified duration is a proxy for the change in the bond's price as YTM changes. Since duration and YTM are inversely related, when interest rates change, the corresponding change in a bond's price can be related to a change in YTM as follows:

$$\frac{\Delta P}{P} = -D \left[\frac{\Delta(1+y)}{(1+y)}\right] \quad ...(4)$$

Recalling equation (3) and noting that [D(1+y)] is equal to Dy; equation (4) can be rewritten as:

$$\Delta P / P = -D^* (\Delta y) \quad ...(5)$$

Equation (5) shows that a bond's percentage price change is the negative product of modified duration and the change in YTM. Thus there is a negative relationship between modified duration and the approximate percentage change in a bond's price for a given change in YTM, which is consistent with the principle that bond prices move in opposite direction of interest rates. Investors also like to know the rand price volatility of a bond. A bond's approximate percentage price change (in rands) for a given change in interest rate is given by equation (6) below, where Db is the change in interest rate (measured in basis points divided by 100):

$$(\Delta P / P)\ (100) = -D^* (\Delta b) \quad ...(6)$$

3. CONVEXITY

Duration is thus a measure of sensitivity of the price of a bond (or other financial asset) to interest rate changes, and can therefore be used to measure interest rate exposure. However, Macaulay's and modified duration can provide fairly accurate estimates of changes in a bond's price only for small changes in YTM. As changes in YTM become larger, the estimation of a bond's price change using the duration measure becomes more inaccurate. This is because duration attempts to estimate a convex relationship with a straight line (the tangent to the convex function of bond price-YTM). To be more accurate, the estimation of a bond's price changes should properly reflect this convexity.

To improve on accuracy, duration has to be augmented with an additional measure in order to capture the curvature or convexity of a bond's price-YTM relationship. Convexity here denotes the degree to which duration (the slope of the tangent line) changes as YTM changes. For an option-free bond, as YTM increases, convexity decreases, and vice versa. In Figure 1 below, the convex curve represents the bond price-YTM relationship for Bond 1 with a coupon rate of 10% (paid semiannually), a term to maturity of four years, an initial YTM of 10%, and therefore a current price of R100. The tangent line (DD) represents the modified duration measure (D*) estimated basing on the initial YTM = y* = 10%. If a vertical line is drawn between any other YTM (on the horizontal axis) and the tangent line, the distance between the horizontal axis and the tangent line will represent the bond's new price estimated using duration. It can be seen from Figure 1 that for small changes in YTM (around y*), duration can estimate the change in the bond's price fairly well. For a large rise in YTM, such as from y* to y^{R}, duration overestimates the bond's price decrease by $P^{R'} - P^{R}$. Conversely, for a large drop in YTM, such as from y* to y^{F}, duration underestimates the bond's price appreciation by $P^{F*} - P^{F}$. Furthermore, if we draw successive tangent lines to the bond price-YTM curve as YTM changes, the slope of the tangent line would get flatter as YTM rises; implying that duration decreases. In contrast, the slope of the tangent line would get steeper as YTM falls, implying that duration increases.

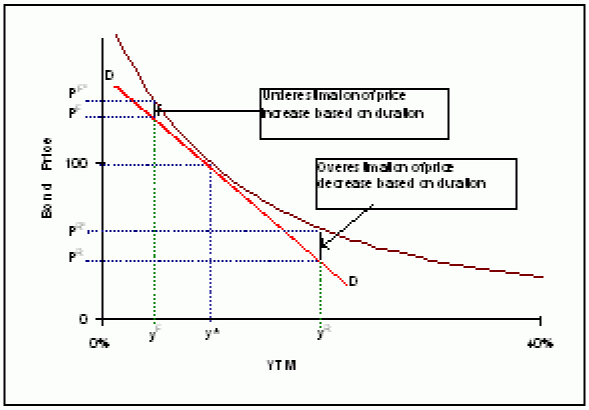

Figure 1: Duration and Convexity

In order to accurately estimate a bond's price change for large changes in YTM, both the first derivative (duration-related) and second derivative (convexity-related) of the price-YTM curve must be considered. Convexity (V), multiplied by the change in YTM squared, is added to the right hand side of the basic duration-YTM sensitivity formula (equation 4). The added term captures the error in estimating the bond's price change using duration alone, and the approximate percentage price change is estimated by:

$$\frac{\Delta P}{P} = -D^{*}\left[\frac{\Delta y}{(1+y)}\right] + V\left[\frac{\Delta y}{(1+y)}\right]^{2} \qquad \text{...(7)}$$

The first term on the right hand side of equation (7) is linear and represents the slope of the tangent line and thus provides the first-order effects of a change in YTM. The second term is quadratic and convexity-related. Convexity is obtained by:

$$V = \left(\frac{1}{2}\right) \frac{\sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^t}}{P_0} \qquad \text{...(8)}$$

As with duration, deriving convexity (V) involves the weighting of cash flows (coupons and par value) by a time factor t(t+1)) as shown in the numerator of the right hand side of equation (8). After that, the resulting number is divided by the bond's current price and then standardized by multiplying it with a constant of one-half.

Apart from improving on the estimation of a bond's percentage price change of an option-free bond, convexity has other important investment applications. Although two bonds may have equal modified duration, they may experience different price changes for large changes in YTM depending on each bond's convexity. Thus for a bond with a higher convexity than another, the implication is that if YTM rises, the bond's convexity will entail a lower capital loss. If YTM drops, that bond will experience a higher price appreciation than a bond with a lower convexity. Therefore, an efficient bond market will take convexity into account when pricing bonds; that is, the market will price convexity.

4. USES OF DURATION

Investors of bond portfolios[1] can use duration and convexity to develop some trading strategies for maximizing return or minimizing losses when interest rates change. If the bond investor expects a major decline in interest rates, investing in a portfolio of bonds with high interest rate sensitivity will lead to substantive price appreciation (capital gains) when interest rates actually fall. This implies investing in bonds with longer terms to maturity and/or low coupons (ideally, this calls for investment in long-term, zero-coupon bonds). In contrast, if the investor expects a major increase in interest rates, investing in a portfolio of bonds with low interest rate sensitivity will lead to minimum capital losses when interest rates actually rise. This implies investing in bonds with shorter terms to maturity and/or high coupons. Benefiting from such strategies depends on the forecasting accuracy of the bond portfolio investor.

Furthermore, investors in bond portfolios often want to insulate the maturity structure of their portfolios against interest rate risk through immunization strategies. Immunization is the investment of assets in such a way that their total value is unaffected by changes in interest rates. Immunization[2] techniques may be aimed at either attaining a certain portfolio value at a target date in the future (cash flow matching and dedication strategies), or at protecting one's portfolio against the effects of changing interest rates, thus allowing the investor to lock in a specific rate of return over a specified investment horizon. A portfolio of bonds is immunized from interest rate risk (price risk and reinvestment rate risk) if the duration of the portfolio is kept equal to the desired investment horizon. Subsequently, following an interest payment, the immunized portfolio is then rebalanced by buying or selling bonds (and investing any interest payment cash flows in selected securities) so as to keep the weighted average duration of the bond portfolio equal to the remaining investment horizon.

Despite having important uses in bond investments, duration and convexity measures have some serious limitations. Firstly, the interest rate sensitivity of a bond portfolio can only be estimated if there is a change in interest rates that leads to a parallel shift in the yield curve (bonds of different maturities experiencing the same change in YTM). In practice however, changes in interest rates do not lead to an equal change in YTM across bonds of different maturities because the yield curve rarely experiences a parallel shift. Thus two bond portfolios that may have the same duration at the beginning of the investment horizon may end up being affected differently by interest rate changes one period later, depending on how the yield curve has shifted. This problem constrains immunization strategies because it cannot be reflected by duration and convexity measures. Another limitation of duration and convexity is that these measures cannot be used to estimate interest rate sensitivity of bonds with embedded options, such as callable bonds and convertibles. With such bonds, changes in YTM (arising from changes in interest rates) may affect not only the prices of the bonds; but the realization of the cash flows from the bonds as well. This is because the bonds may be called or converted as a result of changes in interest rates.

[1]The duration of a portfolio of bonds is the weighted average of the durations of the bonds in the portfolio, with weights assigned to each bond category according to its proportionate value.

[2]For a more thorough discussion on immunization, see: Bodie Z., Kane A. and Marcus, AJ. 1996; chapter 15.

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