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The Geometry of Complex Domains

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To our wives,
Paige, Sung-Ock, and Randi

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Preface

Grand visions in mathematics can begin with simple observations. It is hardly more than a homework exercise to prove that what we nowadays call the Poincaré metric on the unit disc is invariant under the biholomorphic maps of the unit disc to itself. But this easily established fact, when combined with the (profound) uniformization theorem of Poincaré and Koebe, yields the striking conclusion that, with a small number of exceptions, every Riemann surface has a canonical complete Hermitian metric of constant Gauss curvature -1. This result became a basic tool for the study of Riemann surfaces. From this result also grew the whole subject of canonical metrics, an area which has become central in transcendental algebraic geometry and in the topology of low-dimensional manifolds.

It is natural to ask what analogue there might be in higher complex dimensions of the Poincaré metric on the unit disc. Indeed, this was asked not long after the era in the early 1900s of the uniformization theorem (Theorem 2.5.1) and the canonical metric idea for Riemann surfaces. The higher dimensional situation is inevitably different from the situation in complex dimension 1 because the Riemann mapping theorem fails in higher dimensions. It was Poincaré again who showed that the unit ball in $\mathbb{C}^2$ was not biholomorphic to the product of the unit disc with itself. In a similar vein, it was understood around the same time that uniformization of algebraic surfaces was not possible in the same form as the Riemann surface result: there is no single simply connected cover for all the algebraic surfaces with only a few exceptions, no analogue to the unit disc being the universal cover of all but a few Riemann surfaces. But quite early on, in the 1920s, Stefan Bergman showed how to attach to each bounded domain in $\mathbb{C}^n$, $n \geq 1$, a canonical metric with the biholomorphic invariance properties of the Poincaré metric on the unit disc: each biholomorphic mapping of a bounded domain to itself was an isometry of the metric, and moreover, any biholomorphic mapping of one bounded domain to another was an isometry of their respective metrics. Uniformization was a failure, but invariant metrics were successful indeed.

The Bergman metric is only numerically computable in most instances, not given by formulas, and for some time it remained primarily an intriguing general idea rather than a specifically useful one. But the development of the detailed theory of the $\overline{\partial}$ operator by Hörmander, Andreotti-Vesentini, Kohn, and many others made accessible information about the behavior of the Bergman kernel and metric, especially on strongly pseudoconvex bounded domains with smooth boundary. The Bergman kernel is expressible directly in terms of the solutions of $\overline{\partial}$ that are orthogonal to holomorphic functions, and this expression means that the kernel and hence the metric can be analyzed in $\overline{\partial}$ terms. In particular, Fefferman's asymptotic expansion of the Bergman kernel (1974) near the boundary of a C^∞ bounded, strongly pseudoconvex domain opened up the possibility of realizing the grand vision of unifying complex function theory and geometry in this case.

This unification of function theory and geometry for domains in $\mathbb{C}^n$ is the subject of this book—hence its title. In particular, the use of geometric methods yields many results about biholomorphic mappings in general and especially about automorphisms, that is, biholomorphic maps of a domain to itself. The fact that a biholomorphic map is an isometry means that the curvature invariants of differential geometry are preserved by biholomorphic maps, and this provides a powerful method of studying the biholomorphic mappings themselves.

While the Bergman metric has become over the years a familiar item in several complex variables that occurs in many texts on the subject, the study via curvature of the geometry of the Bergman metric has been largely confined to research papers up to now. Thus it seemed to the authors that the body of information on this and related topics was both large enough and coherent enough to justify its treatment in a book. That it was large enough is clear from the length of this book. The question of being coherent we leave to the reader, with hope for the best.

This book is not self-contained: on occasion we use, without apology and sometimes without proof, standard results of several complex variables and in particular of the theory of the $\overline{\partial}$ operator. Even so, we have tried to make the book as accessible as possible to the nonspecialist. Most of the arguments can be followed convincingly by simply taking the unproved background results on faith, these being usually very specific and easily stated, if not easily proven. In this sense, the book will be accessible, we hope, to anyone with a basic background in complex analysis and differential geometry. We have also separated out the more technical aspects of the differential geometry so that the complex analyst can most appreciate the shape of the arguments involving curvature by simply knowing that somehow curvature attaches differential invariants to each point that must be preserved under isometries and hence preserved under biholomorphic maps. Really detailed information on differential geometry is rather seldom needed. Geodesics, for example, hardly occur in the book at all. We have tried, in short, to make almost everything accessible to as many readers as possible without short-changing the readers with more specific expertise. Brave words, but we did try.

This book is wide-ranging, though all the topics are related. And a description of the mathematical prerequisites of the book as a whole and of the various chapters specifically may be useful. All of the book presumes basic knowledge of complex analysis in several variables, with the exception of Chapter 2, which concerns one variable only. Especially important is some working knowledge of normal families. A quick summary of what is needed is given in Chapter 1. Chapter 1 also provides a summary of what is known and needed about automorphism groups being Lie groups. These results can be taken on faith if need be. Chapter 1 also begins to talk about Riemannian metrics. Not much depth is needed here nor will be needed later about Riemannian geometry, but the reader is presumed to have in mind what a Riemannian metric is, at least. Chapter 2 is about automorphisms of Riemann surfaces. The results there provide motivation for later developments, but as it happens, the contents of this chapter are not explicitly used anywhere else in the book. Again, metric concepts are used but at a quite elementary level—Gauss curvature and some ideas about geodesics suffice. In Chapter 3, the idea of the Bergman metric is introduced, and the geometry of the Bergman metric is systematically exploited. The Bergman metric is by nature a Kähler metric, but rather little is needed here about Kähler geometry in detail. Indeed, it is not really necessary to know what a Kähler metric is. What is needed is the realization that attached to a metric structure, a Riemannian metric in general, are some second-order differential invariants which are preserved by mappings that preserve the metric itself. Of course, the deeper meaning of these curvature invariants, if known, will enhance the reader's appreciation of the power and elegance of their application to complex analysis. But in the strictly logical sense, one could think of them as simply formulas, which happened to have certain important invariance properties. The same remarks apply to the continuation of these developments in Chapter 4.

Chapter 5 involves some considerable background in Lie group theory, especially in its second half, on the Bedford–Dadok argument. But Chapter 5 is not needed for the later parts of the book, and the reader who is so inclined can simply take as answered the question of which compact Lie groups occur as the automorphism group of a smoothly bounded strongly pseudoconvex domain in Euclidean space—first all of them do—and skip this chapter altogether.

Chapter 6 is similarly not needed for subsequent developments. It answers a natural and interesting (in the authors' view) question, and the argument in the noncompact case is not far outside the usual ways of thinking in several complex variables. The compact case involves some ideas from further afield, in algebraic geometry, and can be omitted without penalty if desired.

Chapter 7 reviews some metric ideas more general than the smooth Riemannian metrics that were used earlier. These more general metrics are of fundamental importance in several complex variables and are likely somewhat familiar to complex analysts in any case. References are given to further details about these metrics. This material is of central importance to the whole subject, though it is not needed in subsequent chapters as such. Automorphisms of Reinhardt domains, the subject of Chapter 8, require some information about Lie algebras if they are to be studied in detail, but the reader can gain a good impression without this.

Chapters 9 and 10 are in fact the natural continuation of Chapters 3 and 4 and can be read effectively immediately after Chapter 4, with the intervening chapters skipped. Chapters 9 and 10 introduce what is known as the scaling method at a rather more leisurely pace than is followed in the rest of the book, since this material is both very important and not so widely available in systematic form. Indeed some of the material here is new. Chapter 11 looks back on the whole book and discusses where the results could have been stated and proved more generally. For ease of reading, many of the results in the earlier parts of the book were stated in special cases—e.g., for domains in Euclidean spaces rather than complex manifolds—and Chapter 11 clarifies what additional generality holds without the introduction of fundamentally new arguments.

This book has been under construction for some considerable time. The authors have benefited during this effort from interactions with many colleagues. We thank them all. In particular, the third named author (Krantz) thanks Alexander Isaev for his collaboration and for many helpful ideas over the years. Several institutions have offered us mathematical hospitality during the writing. In addition to our home institutions, we thank MSRI, the Technical University of Denmark, the American Institute of Mathematics, and l'École Polytechnique de France (Palaiseau). We thank Ms. Ae-Ryoung Seo of POSTECH and Mr. Felipe Garcia Hernandez of UCLA, who each read the whole manuscript and made helpful suggestions. It goes without saying that any remaining errors are the authors' sole responsibility.

Some mathematical subjects begin slowly, by accumulation of many small contributions, like a river forming from many small streams. The general idea of the deep relationship between function theory and geometry does indeed have many historical sources in the nineteenth century, as indicated briefly in the opening paragraphs of this preface. But the specific subject of this book began definitely and quite suddenly with the work of Stefan Bergman. Without his work, this book would not have existed. We dedicate it to his memory.

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1

Preliminaries

1.1 Automorphism Groups

A subset $\Omega \subseteq \mathbb{C}^n$ will be called a *domain* if it is connected and open. The *automorphism group* $\mathrm{Aut}\,(\Omega)$ of Ω is by definition the set of all holomorphic mappings $f : \Omega \to \Omega$ with inverse map f^{-1} existing and also holomorphic. The group operation is the composition of mappings, and it is easy to check that this binary operation makes $\mathrm{Aut}\,(\Omega)$ into a group. When $n = 1$, it is well known and easy to prove that f^{-1} will be automatically holomorphic when it is defined. This follows from the argument principle because a locally injective holomorphic function has nowhere zero first derivative. This result is also true in several complex variables, but requires more effort to prove. One must show that a locally injective, equi-dimensional holomorphic mapping has nowhere vanishing holomorphic Jacobian determinant; from this it follows immediately that f^{-1} is holomorphic. This result is conceptually fundamental, but plays little explicit role in what follows and will not be discussed further. [See, e.g., [Narasimhan 1971] for a proof.]

The definition of automorphism group can obviously be extended to the case where Ω is replaced by a complex manifold M. The same observation applies to the redundancy of the hypothesis that f^{-1} be holomorphic since the proof of that result can be performed in local coordinates. Much of the theory of automorphism groups of domains in space can be transferred, without any extra work, directly to the complex manifold case; we shall often treat the two situations simultaneously. Other results are quite different for manifolds than for domains in $\mathbb{C}^n$, and we shall indicate some of these distinctions later.

Just as, in one complex variable, the study of Riemann surfaces can clarify basic function-theoretic questions, the study of manifolds in higher dimensions can clarify the situation for domains in space. However, little detailed knowledge of complex manifold theory will be needed for the reading of this book.

The subject of the geometry of open sets in $\mathbb{C}^n$ and of the geometry of open complex manifolds in general divides itself rather naturally into two

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parts. It is really two subjects. In one of these, the domains and manifolds are such that their automorphism groups are finite dimensional and indeed are Lie groups. In the other, the automorphism groups involve infinitely many parameters. The one-variable, Riemann surface situation (for example) is deceptively simple. The group $\mathrm{Aut}(M)$ when M is a Riemann surface is *always* a Lie group, as we shall prove in Chapter 2. By contrast, if one takes $\Omega = \mathbb{C}^2$, then the group $\mathrm{Aut}(\Omega)$ is *not* a Lie group but rather is infinite dimensional in a certain sense. For example, if $f : \mathbb{C} \to \mathbb{C}$ is any entire function, then $(z_1, z_2) \mapsto (z_1 + f(z_2), z_2)$ is an automorphism of $\mathbb{C}^2$.

The present book is primarily about the situations in which $\mathrm{Aut}(\Omega)$ is a (finite-dimensional) Lie group and satisfies an additional condition that the action is *proper* in the following sense: the action map $A : \mathrm{Aut}(\Omega) \times \Omega \to \Omega \times \Omega$ defined by $(\varphi, z) \mapsto (\varphi(z), z)$ is proper. That is, $A^{-1}(C)$ is compact for each compact subset C of $\Omega \times \Omega$. In particular, the isotropy group $I_p \times \{p\} := \{\varphi \in \mathrm{Aut}(\Omega) : \varphi(p) = p\}$ is compact for any $p \in \Omega$ since $I_p = A^{-1}(p, p)$. For a statement like this to make sense, we need to define a topology on $\mathrm{Aut}(\Omega)$. The appropriate topology, which will be used throughout, is the compact-open topology, equivalently the topology of uniform convergence on compact sets. [It should be noted that all the complex manifolds that we shall consider in the sequel will be paracompact; thus no topological pathologies will arise. In particular, the compact-open topology is metrizable in this case.]

If Ω is a bounded domain in $\mathbb{C}^n$, then $\mathrm{Aut}(\Omega)$ is necessarily a Lie group. This was proved specifically by H. Cartan ([Cartan 1935]). Our approach to this will be via normal families and the Bochner–Montgomery theorem (Theorem 1.3.11 below), which characterizes the subgroups of the diffeomorphism group which are Lie groups. Our approach will also yield the properness of the action of $\mathrm{Aut}(\Omega)$ on Ω (Theorem 1.3.12).

Any covering-space quotient of a manifold M with $\mathrm{Aut}(M)$ acting properly, and in particular any covering-space quotient of a bounded domain, also has its automorphism group acting properly. Also, any Riemann surface except the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and $\mathbb{C}$ itself has this proper-action property.[1]

In addition to bounded domains in $\mathbb{C}^n$ and their quotients, there are other classes of complex manifolds for which the automorphism group action is proper. Some aspects of this phenomenon will be considered in Chapter 7.

The role of proper action can be made explicit even at this early stage of our development. This condition is necessary for the existence of a (smooth) Riemannian metric for which all the elements of the automorphism group are isometries. Actually, the condition of proper action is also sufficient for the

[1]That the property holds for tori and for $\mathbb{C}$ with one point removed is, in a sense, accidental: for these Riemann surfaces are both covered by $\mathbb{C}$, which itself does *not* have the desired property that the action of the automorphism group is proper. But all other Riemann surfaces (except the sphere and the cylinder) are quotients of the unit disc $D = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, and for these the general principle applies.

existence of such an "invariant metric" [Palais 1961].[2] This will be discussed in more detail in Section 1.3.

Thus, for the domains and manifolds that we shall consider, the automorphism group, which is at first sight a function-theoretic object, will turn out to be also a geometric one via the existence of an invariant metric. These matters will usually be treated here by constructing explicitly an invariant metric rather than by appealing to the general results of Lie group theory.

In Riemann surface theory, this idea of relating function theory to geometry goes back at least to Poincaré and even Riemann. In higher dimensions, some aspects of the idea also have a long history, but many developments have occurred in recent times as well. It is this interaction between function theory and geometry that makes the whole subject so varied and interesting. And while we begin with the function theory, geometry soon takes center stage and plays a major role thereafter.

1.2 Some Fundamentals from Complex Analysis of Several Variables

We shall use systematically the standard notational conventions for coordinates in $\mathbb{C}^n$, first

$$z = (z_1, \ldots, z_n) \qquad \text{and} \qquad w = (w_1, \ldots, w_n).$$

We shall also write

$$|z| = \left(\sum_{j=1}^{n} |z_j|^2 \right)^{\frac{1}{2}} .$$

Thus a mapping from an open subset of $\mathbb{C}^n$ into $\mathbb{C}^m$ is given by an m-tuple of complex-valued functions of n complex variables:

$$w = (w_1, \ldots, w_n) = f(z) = (f_1(z_1, \ldots, z_n), \ldots, f_m(z_1, \ldots, z_n)).$$

Such a map is, by definition, holomorphic if each of the functions f_j, $j = 1, \ldots, m$, is holomorphic in one and hence any of the various equivalent senses of the word "holomorphic."

Here and elsewhere we take for granted basic elements of the theory of functions of several complex variables, for which see [Grauert/Fritzsche 1976], [Hörmander 1990], or [Krantz 2001] for instance. In particular, we assume that

[2]It is a familiar fact that the group of isometries of a (smooth) Riemannian manifold acts properly. But the partial converse, that a properly-acting subgroup of the group of diffeomorphisms acts as isometries for some smooth metric, is not obvious.

the reader is aware that, for $\mathbb{C}$-valued functions $f(z_1, \ldots, z_n)$ defined on an open subset of $\mathbb{C}^n$, the following ideas are equivalent:

- The function f is holomorphic in each variable separately; [3]
- The function f is real-continuously differentiable (C^1) and satisfies the Cauchy–Riemann equations in each variable separately;
- The function f has at each point $p = (p_1, p_2, \ldots, p_n)$ of its domain a power series expansion

$$f(z) = \sum_{i_1, i_2, \ldots, i_n \geq 0} a_{i_1 i_2 \cdots i_n} (z_1 - p_1)^{i_1} (z_2 - p_2)^{i_2} \cdots (z_n - p_n)^{i_n}$$

which converges absolutely to f for all $(z_1, z_2, \ldots, z_n)$ in some open neighborhood of p.

As will be taken for granted here, many of the ideas of one complex variable have more or less automatic extensions to several variables. These include the Cauchy integral formula in several variables: recall that the polydisc $D^n(p, r)$ of polyradius $r = (r_1, \ldots, r_n)$ with $r_j > 0$ for every j is defined to be

$$D^n(p, r) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j - p_j| < r_j \text{ for every } j\}.$$

If the closure $\mathrm{cl}(D^n(p, r))$ of this polydisc is contained in the (open) domain of definition of a holomorphic function f then, for each $(z_1, \ldots, z_n)$ in the open polydisc,

$$\begin{aligned} &f(z_1, \ldots, z_n) \\ &\quad = \frac{1}{(2\pi i)^n} \oint_{|\zeta_1 - p_1| = r_1} \cdots \oint_{|\zeta_n - p_n| = r_n} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_n \cdots d\zeta_1, \end{aligned}$$

where the integral is an iterated line integral. This reconstructs the power series expansion of f around $(p_1, \ldots, p_n)$, by expansion of the integrand and integration term-by-term. Differentiation of this formula under the integral sign together with obvious estimates also yields the following, which we shall apply repeatedly: if a sequence $\{f_j\}$ of $\mathbb{C}$-valued holomorphic functions on an open subset U of $\mathbb{C}^n$ converges uniformly on each compact subset of U, then every derivative (of any order) of the sequence also converges uniformly on each compact subset, and the derivative of the limit is equal to the limit of the derivative.

This last result, which is a direct analogue of a familiar fact about one-variable theory, will be especially important to us since, in effect, it says that the compact-open topology for holomorphic functions is the same as the C^∞ topology. Thus sets or groups of holomorphic mappings have a natural, unique topology. This means that the subtle questions associated to the phrase

[3] In the background here is the famous theorem of Hartogs that a function holomorphic in each variable separately is automatically continuous, indeed real analytic.

"Hilbert's fifth problem" play no role here; such matters are automatically straightforward.

Hurwitz's theorem in one variable on limits of zero-free functions has a direct generalization to several variables: first, if $f_j \colon \Omega \to \mathbb{C}$, $j = 1, 2, 3, \ldots$, are holomorphic functions from a domain (i.e., a connected open set) in $\mathbb{C}^n$ with $0 \notin f_j(\Omega)$, and if the sequence $\{f_j\}$ converges uniformly on compact subsets of Ω to a (necessarily holomorphic) limit $f_0 \colon \Omega \to \mathbb{C}$, then either $f_0(\Omega) = \{0\}$, i.e., $f_0 \equiv 0$, or $0 \notin f_0(\Omega)$, i.e., f_0 is nowhere zero. The proof is obtained by observing that, if $f_0(z_0) = 0$ for some $z_0 \in \Omega$, then, by the one-variable Hurwitz theorem, the function $\zeta \mapsto f_0(z_0 + a\zeta)$, for $\zeta \in \mathbb{C}$ with $|\zeta|$ small and for $a \in \mathbb{C}^n$ with $\|a\| = 1$, is defined and identically zero. Then that $f_0 \equiv 0$ follows by analytic continuation.

Since one of the main subjects of this book is self-mappings of domains in $\mathbb{C}^n$ or, on occasion, complex manifolds, we have some special interest in holomorphic mappings where domain and range have equal dimension; first, n-tuples $(f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))$ of holomorphic functions of n variables. Attached to this situation is the holomorphic Jacobian determinant $\mathcal{J}$, first, the ordinary determinant of the $n \times n$ complex matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_n} \end{pmatrix}.$$

A linear algebra calculation shows that the Jacobian determinant of the mapping considered as a real mapping from an open subset of $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$ is $|\mathcal{J}|^2$. This is a generalization of the familiar fact from one variable that the real differential of a holomorphic function is a rotation followed by dilation by a factor of $|f'|$, so that its action on the area element is multiplication by $|f'|^2$.

Returning to the $\mathbb{C}^n$ situation in general, we see that the holomorphic mapping from an open subset into $\mathbb{C}^n$ again is nonsingular as a real mapping at a given point if and only if its holomorphic Jacobian determinant $\mathcal{J}$ is nonzero at that point. Combining this observation with Hurwitz's theorem, we see that the limit (uniformly on compact sets) of everywhere nonsingular mappings of a connected open set in $\mathbb{C}^n$ to $\mathbb{C}^n$ is either everywhere nonsingular or everywhere singular. In the latter case, the limit mapping has image with empty interior (by Sard's theorem (Theorem 5.3.2)). This line of thought is associated to the idea that the limit of biholomorphic mappings is either biholomorphic or in some sense "degenerate." This point will be explored in detail in later sections.

It is of interest to characterize holomorphic mappings in terms of their real differentials. This is done in effect by way of the Cauchy–Riemann equations. Let $(f_1(z_1, \ldots, z_n), \ldots, f_m(z_1, \ldots, z_n))$ be a holomorphic mapping into $\mathbb{C}^m$ defined on an open subset of $\mathbb{C}^n$. Then we write $f_j = u_j + \sqrt{-1}v_j$, where u_j, v_j are real-valued. The Cauchy–Riemann equations are as usual

$$\frac{\partial u_j}{\partial x_\ell} = \frac{\partial v_j}{\partial y_\ell} \qquad \text{and} \qquad \frac{\partial u_j}{\partial y_\ell} = -\frac{\partial v_j}{\partial x_\ell}, \quad j = 1, \ldots, m, \quad \ell = 1, \ldots, n.$$

We write here, by convention, $z_\ell = x_\ell + \sqrt{-1}y_\ell$. This can be thought of in a less coordinate-dependent fashion as follows. Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by sending $(z_1, \ldots, z_n)$ to $(x_1, y_1, \ldots, x_n, y_n)$. Define an $\mathbb{R}$-linear map J_{2n} of $\mathbb{R}^{2n}$ to itself by sending $(x_1, y_1, \ldots, x_n, y_n)$ to $(-y_1, x_1, \ldots, -y_n, x_n)$. Then the Cauchy–Riemann equations for a map $F : U \to \mathbb{C}^m$, with U open in $\mathbb{C}^n$, are equivalent to

$$J_{2m} \circ dF = dF \circ J_{2n},$$

where dF is the real differential of F considered as a C^∞ function from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2m}$.

This characterization of holomorphicity has an immediate consequence that is important for the theory of complex manifolds. first, if two complex local coordinate systems $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ are holomorphically related, then the J operator determined from the z-coordinates is the same operator as the J operator determined from the w-coordinates. The meaning of this assertion is familiar in Riemann surface theory: J is rotation by 90° counterclockwise in the orientation determined by the Riemann surface structure. The meaning of this is the same in any holomorphic coordinate system because the real differential of the coordinate change is orientation-preserving and conformal. In higher dimensions, there is again a coordinate-invariant operator J on the real tangent space at each point of a complex manifold. This operator corresponds to the J operator in any coordinate system, and the observation in the previous paragraph shows that it is independent of coordinate choice.

The J operator thus obtained provides a way to connect real Riemannian geometry with complex behavior, since J is a real $(1,1)$ tensor but it completely determines which (locally defined) functions are holomorphic. This approach to the geometry of complex manifolds is presented systematically in, e.g., [Greene 1987], [Wells 1979]; see also [Kobayashi/Nomizu 1963].

1.3 Normal Families and Automorphisms

Let $D \subset \mathbb{C}$ denote the open unit disc $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Also $D(p, r) \subset \mathbb{C}$ denotes the open disc with radius r centered at p. For $r > 0$ we let

$$D^n(0, r) \equiv \underbrace{D(0, r) \times \cdots \times D(0, r)}_{n \text{ times}}.$$

Further, if $p = (p_1, \ldots, p_n) \in \mathbb{C}^n$ and $r > 0$, then

$$D^n(p, r) \equiv D(p_1, r) \times \cdots \times D(p_n, r).$$

If $f : D \to D \subset \mathbb{C}$ is a holomorphic function with $f(0) = 0$ and $|f'(0)| = 1$, then f has the form $f(z) = f'(0)z$. In particular, if $f \in \mathrm{Aut}\,(D)$ and if such an f has $f'(0) = 1$, then $f(z) = z$. This is part of the classical Schwarz lemma. The following result is a direct generalization to several variables, and to arbitrary bounded domains. There are many possible generalizations of the Schwarz lemma, some of which will be discussed later on in this book, but this one is the one that will play the most direct role in our investigations. For example, it will enable us to see that, if Ω is a bounded domain, then $\mathrm{Aut}\,(\Omega)$ has compact isotropy group at each point.

Theorem 1.3.1 (H. Cartan). *Suppose that Ω is a bounded domain in $\mathbb{C}^n$. Let $\phi : \Omega \to \Omega$ be holomorphic and suppose that, for some $p \in \Omega$, $\phi(p) = p$ and $d\phi(p) = id$. [Here $d\phi$ is the n-dimensional complex differential.] Then ϕ is the identity mapping from Ω to itself.*

Boundedness of Ω is an essential hypothesis: consider the automorphism of $\mathbb{C}^2$ given by $(z_1, z_2) \mapsto (z_1 + z_2^2, z_2)$.

Proof of Theorem 1.3.1. We may assume that $p = \mathbf{0}$ (the origin). For proof by contradiction, assume that ϕ does not coincide with the identity mapping. Expanding ϕ in a power series about $p = \mathbf{0}$ (and remembering that ϕ is vector-valued, hence so is the expansion) yields

$$\phi(z) = z + P_k(z) + O(|z|^{k+1}),$$

where P_k is the first nonvanishing homogeneous polynomial (of degree k) of order exceeding 1 in the Taylor expansion. Defining $\phi^j(z) = \phi \circ \cdots \circ \phi$ (j times); direct computation then gives that

$$\begin{aligned}
\phi^2(z) &= z + 2P_k(z) + O(|z|^{k+1}) \\
\phi^3(z) &= z + 3P_k(z) + O(|z|^{k+1}) \\
&\ \ \vdots \\
\phi^j(z) &= z + jP_k(z) + O(|z|^{k+1}).
\end{aligned}$$

Choose polydiscs $D^n(0, a) \subseteq \Omega \subseteq D^n(0, b)$. The Cauchy estimates imply then that, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| := \alpha_1 + \cdots + \alpha_n = k$,

$$j \cdot \left| \left(\frac{\partial}{\partial z} \right)^{\alpha} \phi \Big|_{\mathbf{0}} \right| = \left| \left(\frac{\partial}{\partial z} \right)^{\alpha} \phi^j \Big|_{\mathbf{0}} \right| \leq n \cdot \frac{b \cdot \alpha!}{a^k},$$

where

$$\left(\frac{\partial}{\partial z} \right)^{\alpha} = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}.$$

Note that the rightmost item in this estimate is independent of j. Hence, for each such multi-index α with $|\alpha| = k$, $(\partial/\partial z)^{\alpha} \phi\big|_{\mathbf{0}} = \mathbf{0}$. Thus $P_k = 0$, a contradiction. $\square$

This argument in particular applies when the dimension $n = 1$ and the domain Ω is the unit disc. There it gives a conceptually direct proof of the corresponding part of the classical Schwarz lemma.

Cartan's result has some further immediate but surprising consequences.

Corollary 1.3.2. *Suppose that Ω is a bounded, circular domain in $\mathbb{C}^n$, that is $(e^{i\theta}z_1, e^{i\theta}z_2, \ldots, e^{i\theta}z_n) \in \Omega$ whenever $(z_1, z_2, \ldots, z_n) \in \Omega$ for every $\theta \in \mathbb{R}$. If $\mathbf{0} \in \Omega$ and $f \in \operatorname{Aut}(\Omega)$ with $f(\mathbf{0}) = \mathbf{0}$, then f is a linear mapping.*

Proof. For $\theta \in \mathbb{R}$ and $z \in \Omega$, let $F(z) = e^{-i\theta} f(e^{i\theta} z)$. Then $F \in \operatorname{Aut}(\Omega)$, since Ω is circular. By the chain rule it follows that

$$d(f^{-1} \circ F)\big|_0 = \mathrm{id}.$$

Hence

$$f^{-1} \circ F = \mathrm{id}$$

on Ω, or equivalently $f = F$. If we write $f = (f_1, \ldots, f_n)$, $F = (F_1, \ldots, F_n)$, and

$$f_j(z) = \sum_{|N|=1}^{+\infty} a_N z^N$$

is the Taylor expansion of f_j, then the Taylor expansion of F_j is, by definition of F and by substitution,

$$F_j = \sum_{|N|=1}^{+\infty} e^{-i\theta} a_N e^{i|N|\theta} z^N.$$

But $F_j = f_j$. Therefore $e^{i(|N|-1)\theta} a_N = a_N$ for all multi-indices N and all $\theta \in \mathbb{R}$. This implies that $a_N = 0$ for $|N| \geq 2$.[4] Thus each f_j is linear. $\square$

It is easy to modify this argument to show that, if Ω_1, Ω_2 are two bounded, circular domains containing the origin $\mathbf{0}$ and if $F : \Omega_1 \to \Omega_2$ is biholomorphic with $F(\mathbf{0}) = \mathbf{0}$, then F is linear. This immediately implies that, when $n \geq 2$, the unit ball $\{(z_1, \ldots, z_n) \colon |z_1|^2 + \cdots + |z_n|^2 < 1\}$ and the unit polydisc $\{(z_1, \ldots, z_n) \colon |z_j| < 1, j = 1, \ldots, n\}$ are not biholomorphic: If there were a biholomorphic map between them, then applying suitable biholomorphic maps to each variable in the unit polydisc separately would produce a biholomorphic map that took $\mathbf{0}$ to $\mathbf{0}$. This would then have to be linear, which is not possible, since, e.g., the ball has smooth boundary and the polydisc does not (when $n \geq 2$). Thus the direct analogue of the Riemann mapping theorem fails in $\mathbb{C}^n$, $n \geq 2$: (bounded) domains can be homeomorphic to the ball without being biholomorphic to it. This failure, even for small perturbations of the ball, will be explained in much more detail in later chapters.

[4] Here $N = (n_1, \ldots, n_n)$ and $|N| = n_1 + \cdots + n_n$.

The second corollary will play an important role in what follows.

Corollary 1.3.3. *If Ω is a bounded domain in $\mathbb{C}^n$ and $p \in \Omega$, then the mapping*

$$f \longmapsto df\big|_p$$

is an injective homomorphism of the group

$$I_p \equiv \{f \in \operatorname{Aut}(\Omega) : f(p) = p\}$$

into $GL(n, \mathbb{C})$.

Proof. If $df\big|_p = dg\big|_p$ for $f, g \in I_p$, then the chain rule gives that $d(f^{-1} \circ g)\big|_p =$ id, where the identity map id is given by the $n \times n$ identity matrix $I_n \in GL(n, \mathbb{C})$. By Theorem 1.3.1, $f^{-1} \circ g : \Omega \to \Omega$ is the identity mapping. Hence $f \equiv g$. We conclude that $f \mapsto df\big|_p$ is injective on I_p. The homomorphism property is a special case of the chain rule. □

If a group G acts on a space X through an action $G \times X \to X$, and if $x \in X$, then the *orbit* $\mathcal{O}_x$ of the point x is the set $\{gx : g \in G\}$. In a natural sense the orbit is the image of the group G. Indeed, $\mathcal{O}_x$ is naturally identified with the quotient G/I_x, where $I_x = \{g \in G \colon gx = x\}$. We shall be particularly interested in boundary points that are accumulation points of some orbit for the action of the automorphism group $\operatorname{Aut}(\Omega)$ on Ω. If the orbit $\mathcal{O}_x \subseteq \Omega$, considered as a point set, has a boundary point $p \in \partial\Omega$ as an accumulation point then we call p a *boundary orbit accumulation point*. These will be discussed in detail in Section 1.5.

Corollary 1.3.3 immediately yields the following observation. Fix $p_0 \in \Omega$. Then each $f \in \operatorname{Aut}(\Omega)$ is uniquely determined by $f(p_0)$ and $df|_{p_0}$. Now the possibilities for $f(p_0)$ range at most over Ω and for $df|_{p_0}$ over $\mathbb{C}^{n^2}$ (identifying $df|_{p_0}$ with its complex $n \times n$ matrix). So in a general sense $\operatorname{Aut}(\Omega)$ is parameterized by a subset of $\mathbb{C}^n \times \mathbb{C}^{n^2}$. Thus one might expect $\operatorname{Aut}(\Omega)$ to be a finite-dimensional group, and hence a Lie group. This expectation turns out to be justified. But of course this depends on adding the topology into the picture of $\operatorname{Aut}(\Omega)$: as it stands, this "parameterization" is only set-theoretic. We have already discussed the appropriate topology for $\operatorname{Aut}(\Omega)$, first the compact-open topology. Clearly the association $f \mapsto (f(p_0), df|_{p_0}) \in \mathbb{C}^n \times \mathbb{C}^{n^2}$ is continuous (for the second factor, by Cauchy estimates). To pursue this matter further, we shall need some results from normal families, to which we shall turn next.

Among results also associated to normal families and the closure properties of the group $\operatorname{Aut}(\Omega)$, when Ω is a bounded domain in $\mathbb{C}^n$, the following principle will in particular play an important role in our later considerations. While in a sense this is just an application of standard normal families ideas, the details are surprisingly subtle in this general, multi-variable situation.

Theorem 1.3.4 (Normal Families of Automorphisms). *Let Ω be a bounded domain in $\mathbb{C}^n$. If $\{f_j\}$ is a sequence in $\operatorname{Aut}(\Omega)$ which converges* *uniformly on compact subsets of Ω and if, for some $p_0 \in \Omega$, the limit $\lim_{j\to\infty} f_j(p_0)$ is a point of Ω, then the limit holomorphic mapping $f_0 \equiv \lim f_j : \Omega \to cl(\Omega)$ has image equal precisely to Ω and $f_0 \in \mathrm{Aut}\,(\Omega)$.*

Without the hypothesis about the point p_0, the conclusion can fail. For example, if $\Omega = D = \{z \in \mathbb{C} : |z| < 1\}$ and

$$f_j(z) = \frac{z-(1-1/j)}{1-(1-1/j)z},$$

then $f_j \in \mathrm{Aut}\,(\Omega)$, but

$$\lim f_j = \text{the constant function } -1.$$

In one complex variable, such "degenerate limits," where $\lim f_j(p_0) \in \mathrm{cl}(\Omega)\setminus\Omega$ for some p_0 and hence (by the theorem) all $p_0 \in \Omega$, are necessarily constant functions. This is an easy consequence of Hurwitz's theorem on the limits of sequences of zero-free holomorphic functions. For, suppose to the contrary that $\lim f_j(p_0) = q \in \mathrm{cl}(\Omega) \setminus \Omega$. Then the limit of the zero-free functions $f_j(z) - q$ for $z \in \Omega$ has a zero at p_0 and is hence $\equiv 0$ on Ω.

This argument indeed shows that, under the hypotheses of the theorem, $\lim f_j$ is "interior," i.e., $(\lim f_j)(\Omega) \subset \Omega$, in the one-variable case. But the argument needed in general (i.e., higher dimensions) is much more intricate—even though Hurwitz's theorem on limits of sequences of zero-free holomorphic functions continues to play a role.

Proof of Theorem 1.3.4. Let $\mathcal{J}_{f_j}$ be the holomorphic Jacobian determinant of f_j as discussed earlier. Then $\mathcal{J}_{f_j}$ is zero-free on Ω. Write f_0 for the limit of the f_j. By Hurwitz's theorem, $\mathcal{J}_{f_0}$ is either identically 0 or is zero-free. To rule out the first possibility, we show that $\mathcal{J}_{f_0}(p_0) \neq 0$. For this, note that

$$\mathcal{J}_{f_0}(p_0) = \lim_{j\to\infty} \mathcal{J}_{f_j}(p_0) = \lim_{j\to\infty} \frac{1}{\mathcal{J}_{g_j}(f_j(p_0))},$$

where $g_j \equiv f_j^{-1}$.

Since $\lim f_j(p_0)$ exists by hypothesis and belongs to Ω, it follows that the set $\{f_j(p_0)\}$ belongs to a compact subset of Ω. Indeed it belongs to $\{\lim_j f_j(p_0)\} \cup \{f_j(p_0)\}$, which is surely compact. By Cauchy estimates, $\mathcal{J}_{g_j}$ is bounded on this compact set. Thus $\lim_j 1/\mathcal{J}_{g_j}(f_j(p_0)) \neq 0$, and that is what we wanted.

It would be pleasant if the fact that we just established, first that $\mathcal{J}_{f_0}$ is zero-free on Ω, implied immediately that $f_0(\Omega) \subset \Omega$. In the special case that Ω has a "nice boundary" (e.g., a regularly embedded C^2 hypersurface in $\mathbb{C}^n$), the result would actually follow. For in that case $\mathcal{J}_{f_0}$ being nowhere zero implies that $f_0(\Omega)$ is open in $\mathbb{C}^n$ and for a domain Ω with smooth boundary, every subset of the closure $\mathrm{cl}(\Omega)$ of Ω that is open in $\mathbb{C}^n$ is contained in Ω. But of course in a more general setting, wherein the boundary of Ω is not smooth,

$\mathrm{cl}(\Omega)$ can in fact contain points of $\mathrm{cl}(\Omega) \setminus \Omega$ in its interior (e.g., consider the case of Ω a punctured open ball). Thus a more refined argument is needed.

Fix a point $p \in \Omega$. Then $\mathcal{J}_{f_0}(p) \neq 0$ and of course the entire holomorphic Jacobian matrix of first derivatives of f_j at p converges to the matrix for f_0, which is nonsingular. Moreover, the second derivatives of the f_j on any fixed, closed ball $\mathrm{cl}(B^n(p,\epsilon)) \subset \Omega$, $\epsilon > 0$, are bounded uniformly in j by Cauchy estimates. Now it follows from the inverse function theorem (see, e.g., [Krantz/Parks 2002]) that there is a $\delta > 0$ such that $f_j(\Omega)$ contains an open ball of radius δ around $f_j(p)$. Here δ can be taken to be independent of j. In particular, since $f_j(\Omega) = \Omega$, the distance of $f_j(p)$ to $\mathbb{C}^n \setminus \Omega$ is at least δ for all j. It follows that $\lim_j f_j(p) = f_0(p)$ is in Ω, *not* in $\mathrm{cl}(\Omega) \setminus \Omega$. Thus, $f_0(\Omega) \subset \Omega$.

Now that we know that f_0 is "interior," i.e., *it maps the interior points to the interior points and hence no interior points are mapped to a boundary point*, we want to show that $f_0 \in \mathrm{Aut}\,(\Omega)$, i.e., that $f_0 : \Omega \to \Omega$ is one-to-one and onto. Passing to a subsequence if necessary, we can suppose that $\{g_j\} = \{f_j^{-1}\}$ converges uniformly on compact subsets to a limit $g_0 : \Omega \to \mathrm{cl}(\Omega)$. Our next goal is to show that g_0 is interior. By the argument used to show that f_0 was interior, it suffices to show that $g_0(f_0(p_0))$ belongs to Ω, not to $\mathrm{cl}(\Omega) \setminus \Omega$.

For this, choose $\lambda > 0$ such that the closed ball $\mathrm{cl}\big(B^n(f_0(p_0), 2\lambda)\big) \subset \Omega$. Notice that $f_j(p_0) \in \mathrm{cl}\big(B^n(f_0(p_0), \lambda)\big)$ whenever j is sufficiently large. Hence, by Cauchy estimates, there is a constant $M > 0$, independent of j, such that

$$\|g_j(f_j(p_0)) - g_j(f_0(p_0))\| \leq M \|f_j(p_0) - f_0(p_0)\|$$

for all j sufficiently large. But $g_j(f_j(p_0)) = p_0$. Hence

$$\|p_0 - g_j(f_0(p_0))\| \leq M \|f_j(p_0) - f_0(p_0)\|.$$

Since the righthand side goes to 0 as $j \to +\infty$, so does the lefthand side and hence

$$g_0(f_0(p_0)) = \lim_{j \to \infty} g_j(f_0(p_0)) = p_0.$$

We conclude that $g_0(f_0(p_0)) \in \Omega$ and therefore g_0 is interior.

We now must show that $f_0 \circ g_0 : \Omega \to \Omega$ and $g_0 \circ f_0 : \Omega \to \Omega$ are both identity maps of Ω to Ω. This of course will establish that $f_0 \in \mathrm{Aut}\,(\Omega)$. This final result is a consequence of the next lemma.

Lemma 1.3.5. *If $\{f_j : \Omega \to \Omega\}$ and $\{g_j : \Omega \to \Omega\}$ are sequences of holomorphic mappings which converge uniformly on compact subsets of Ω to interior limits $f_0 : \Omega \to \Omega$ and $g_0 : \Omega \to \Omega$, then the sequence $\{g_j \circ f_j : \Omega \to \Omega\}$ converges uniformly on compact subsets of Ω to $g_0 \circ f_0 : \Omega \to \Omega$.*

Assuming this lemma for the moment, we may apply it to f_j and g_j as before. Since $g_j \circ f_j$ is the identity map of Ω to Ω, for all j, it follows that $g_0 \circ f_0$ is also the identity map. Applying the lemma again with the roles of f and g interchanged gives that $f_0 \circ g_0$ is the identity. This completes the proof of the theorem. Thus, it remains to prove the lemma.

Proof of Lemma 1.3.5. Suppose that $K \subset \Omega$ is a compact subset. Then choose $\epsilon > 0$ such that

$$L_\epsilon \equiv \{z \in \Omega : \|z - w\| \leq \epsilon \text{ for some } w \in f_0(K)\}$$

is a compact subset of Ω. This choice is possible since $f_0(K)$ is a compact subset of Ω. For all j sufficiently large, $f_j(K) \subset L_\epsilon$. Furthermore, the members of $\{g_j\}$ are uniformly Lipschitz continuous on L_ϵ by Cauchy estimates. Thus, for $z \in K$ and j large, there is a j-independent constant M such that

$$\begin{aligned}\|g_j(f_j(z)) - g_0(f_0(z))\| &\leq \|g_j(f_j(z)) - g_j(f_0(z))\| + \|g_j(f_0(z)) - g_0(f_0(z))\| \\ &\leq M\|f_j(z) - f_0(z)\| + \|g_j(f_0(z)) - g_0(f_0(z))\|.\end{aligned}$$

Now $\|f_j(z) - f_0(z)\| \to 0$ uniformly for $z \in K$. Also, since $\{f_0(z) : z \in K\}$ is compact, $\|g_j(f_0(z)) - g_0(f_0(z))\| \to 0$ uniformly for $z \in K$. Thus $\lim_j g_j(f_j(z)) = g_0(f_0(z))$ uniformly for $z \in K$ as required. □

The proof of Theorem 1.3.4 is now complete. □

Corollary 1.3.6. *For each $p \in \Omega$, the orbit $\mathcal{O}_p := \{f(p) : f \in \mathrm{Aut}\,(\Omega)\}$ is closed in Ω.*

Proof. We need to show that, if $\{f_j(p)\}$ converges to $q \in \Omega$, then $q \in \mathcal{O}_p$, i.e., that $q = f(p)$ for some $f \in \mathrm{Aut}\,(\Omega)$. Choose a subsequence of $\{f_j\}$ which converges uniformly on compact subsets of Ω to $f : \Omega \to \mathrm{cl}(\Omega)$.[5] By Theorem 1.3.4, $f \in \mathrm{Aut}\,(\Omega)$ and clearly $f(p) = \lim_j f_j(p) = q$. □

Corollary 1.3.7. *The injective homomorphism $f \mapsto df|_p$ of I_p (the isotropy group $\{f \in \mathrm{Aut}\,(\Omega)\colon f(p) = p\}$) onto dI_p is a homeomorphism of I_p (in the compact-open topology) onto a compact subgroup of $GL(n, \mathbb{C})$.*

Proof. That $f \mapsto df|_p$ is an injective homomorphism of I_p onto dI_p has already been established (Corollary 1.3.3). The continuity is an immediate consequence of the Cauchy estimates for first derivatives. For the compactness, note that a sequence $\{df_j|_p : f_j \in I_p\}$ has a subsequence $\{df_{j_k}|_p : f_{j_k} \in I_p\}$ for which $\{f_{j_k}\}$ converges uniformly on compact subsets of Ω and, by Theorem 1.3.4, to an element $f_0 \in \mathrm{Aut}\,(\Omega)$ that fixes p. Again by the Cauchy estimates, $df_{j_k}|_p$ converges in $GL(n, \mathbb{C})$ to $df_0|_p \in dI_p$. □

The compactness part of Corollary 1.3.7 is a special case of a more general result which has essentially the same proof.

Corollary 1.3.8. *If K is a compact subset of Ω and $p \in \Omega$, then $\{f \in \mathrm{Aut}\,(\Omega) : f(p) \in K\}$ is a compact subset of* $\mathrm{Aut}\,(\Omega)$.

[5] We shall use the notation $\mathrm{cl}(\Omega)$ for the closure of Ω, instead of the more familiar $\overline{\Omega}$, to avoid confusion with the complex conjugate.

Proof. Let $\{f_j\}$ be a sequence in $\mathrm{Aut}\,(\Omega)$ with $f_j(p) \in K$ for all j. Since K is compact, we see by passing to a subsequence (still called f_j) that $\lim_j f_j(p)$ exists and lies in K. By normal families considerations, a further passage to a subsequence yields a sequence that converges uniformly on compact sets. By Theorem 1.3.4, this sequential limit is itself an automorphism. Obviously this limit takes p to some point in K. $\square$

Corollary 1.3.9. *If, for some* $p \in \Omega$, $\{f(p) : f \in \mathrm{Aut}\,(\Omega)\}$ *is compact, then* $\mathrm{Aut}\,(\Omega)$ *is compact.*

Proof. In the corollary before this one, we simply take $K = \{f(p) : f \in \mathrm{Aut}\,(\Omega)\}$. $\square$

For all $p \in \Omega$, $\{f(p) : f \in \mathrm{Aut}\,(\Omega)\}$ is compact if $\mathrm{Aut}\,(\Omega)$ is compact, just because for a given p the mapping

$$
\begin{aligned}
F : \mathrm{Aut}\,(\Omega) &\to \Omega \\
f &\mapsto f(p)
\end{aligned}
$$

is continuous. Thus we have proved the following result.

Proposition 1.3.10. *If one orbit of* $\mathrm{Aut}\,(\Omega)$ *is compact, then* $\mathrm{Aut}\,(\Omega)$ *is compact and all of its orbits are compact.*

We know from Corollary 1.3.6 that any orbit of $\mathrm{Aut}\,(\Omega)$ is closed in Ω. Thus the only way that an orbit of $\mathrm{Aut}\,(\Omega)$ can be noncompact is to "run out to the boundary" of Ω, i.e., the closure must contain an element of $\mathrm{cl}(\Omega) \setminus \Omega$. One of the main points of the present book is to study what happens when $\mathrm{Aut}\,(\Omega)$ is noncompact. And one of the main approaches will be to study $\mathrm{cl}(\Omega) \setminus \Omega$ in a neighborhood of such a "boundary orbit accumulation point," that is, an element of $\mathrm{cl}(\Omega) \setminus \Omega$ that lies in the closure of some orbit of the automorphism group action.

We now see that the automorphism group of a bounded domain is a (finite-dimensional) Lie group. For this we shall use the following general theorem.

Theorem 1.3.11 ([Bochner/Montgomery 1946]). *Let G be a subgroup of the diffeomorphism group of a smooth manifold. If it is locally compact, then G is a Lie group.*

When the action of the automorphism group is proper, the group is necessarily locally compact. first, as before, we define the action map $A\colon \mathrm{Aut}\,(\Omega) \times \Omega \to \Omega \times \Omega$ by $A(\varphi, z) = (\varphi(z), z)$. Then A^{-1} of a compact-closure neighborhood of (z, z) for any $z \in \Omega$ has compact closure in $\mathrm{Aut}\,(\Omega) \times \Omega$, when A is a proper map. This gives a compact-closure neighborhood of the identity in $\mathrm{Aut}\,(\Omega)$, by projection to the first factor of $\mathrm{Aut}\,(\Omega) \times \Omega$. Thus to show that $\mathrm{Aut}\,(\Omega)$ is a Lie group when Ω is a bounded domain in $\mathbb{C}^n$, it suffices, in the presence of the Bochner–Montgomery theorem (Theorem 1.3.11), to show:

Theorem 1.3.12. *If Ω is a bounded domain in $\mathbb{C}^n$, then the action of* $\mathrm{Aut}\,(\Omega)$ *on Ω is proper, i.e., the map $(\varphi, z) \mapsto (\varphi(z), z) : \mathrm{Aut}\,(\Omega) \times \Omega \to \Omega \times \Omega$ is proper.*

Proof. Properness means explicitly that, if $C \subset \Omega \times \Omega$ is a compact set, then $\{(\varphi, z) : (\varphi(z), z) \in C\}$ is a compact set in $\mathrm{Aut}\,(\Omega) \times \Omega$. To check this property for $\mathrm{Aut}\,(\Omega)$, suppose that $\{(\varphi_j, z_j) : j = 1, 2, \ldots\}$ is a sequence with $(\varphi_j(z_j), z_j) \in C$ for all j. Passing to a subsequence if necessary, one can assume that $\{z_j\}$ converges to a point $z_0 \in \Omega$ and that the sequence $\{\varphi_j(z_j)\}$ converges to $w_0 \in \Omega$.

Since Ω is bounded, Cauchy estimates imply that $\varphi_j(z_0)$ converges to w_0: in more detail, this follows by noting from the Cauchy estimates that, for some $\epsilon > 0$, $B(z_0, 2\epsilon) \subset \Omega$, so that there is a constant $M > 0$ independent of j such that the norm of the (real) differential of φ_j is less than M at each point of $B(z_0, \epsilon)$. Thus the distance from $\varphi_j(z_j)$ to $\varphi_j(z_0)$ is bounded by $M\|z_j - z_0\|$, and hence goes to 0.

Since $\varphi_j(z_0)$ converges now to $w_0 \in \Omega$, it follows from Corollary 1.3.8 that $\{\varphi_j\}$ has a subsequence that converges to some $\varphi_0 \in \mathrm{Aut}\,(\Omega)$. The compactness of $\{(\varphi, z) : (\varphi(z), z) \in C\}$ has thus been established. □

Corollary 1.3.13. *If Ω is a bounded domain in $\mathbb{C}^n$, then* $\mathrm{Aut}\,(\Omega)$ *is a Lie group.*

Proof. Combine Theorem 1.3.12 with the Bochner–Montgomery theorem (Theorem 1.3.11). □

As already noted at the end of Section 1.1, this result implies, from the result of Palais [Palais 1961], the existence of a smooth Riemannian metric on Ω invariant under $\mathrm{Aut}\,(\Omega)$. Averaging this with respect to the almost complex structure produces a Hermitian metric on Ω invariant under $\mathrm{Aut}\,(\Omega)$. In Chapter 3, an explicit construction of such a metric will be presented, but it is worth noting that the existence of such an invariant metric is guaranteed by the general principles we have discussed.

The general situation just described gives at least a philosophical idea of why $\mathrm{Aut}\,(\Omega)$ is a Lie group when Ω is a bounded domain. The precise version of this idea is Theorem 1.3.11 by Bochner and Montgomery. The main point is to describe the elements of $G := \mathrm{Aut}\,(\Omega)$ locally, in a neighborhood of the identity element, by a finite number of parameters so as to make the group itself a manifold (of finite dimension). A way to think of this is to look for a point of minimal isotropy dimension. This idea makes sense because all the isotropy groups are closed subgroups of $GL(n, \mathbb{C})$ (actually $U(n)$), so the idea of dimension is just submanifold dimension. If p is such a point, and its orbit $\mathcal{O}_p := \{\gamma(p) : \gamma \in G\}$, then elements γ near the identity can be determined by specifying $\gamma(p)$, which is near p, and $d\gamma\big|_p$, which is near the "identity map," where the "identity map" is just the map from the tangent space at p to the tangent space at $\gamma(p)$ arising from the coordinates in $\mathbb{C}^n$. The set of such $d\gamma$ in Euclidean coordinates is a submanifold of $GL(n,\mathbb{C})$, although it is not in general a subgroup (if $\gamma(p) \neq p$). Using submanifold coordinates from that observation and submanifold-of-$\mathbb{C}^n$ coordinates of $\mathcal{O}_p$ near p gives a local parameterization of $G = \mathrm{Aut}\,(\Omega)$ near the identity.

This picture will be clearer if one thinks of the case of Ω the unit disc and $p = 0$. Let γ be an element of $\mathrm{Aut}\,(\Omega)$. Near the identity, we can parameterize $\mathrm{Aut}\,(\Omega)$ by the image $\gamma(0)$ together with $d\gamma\big|_0$. The set of such $d\gamma\big|_0$ (when $\gamma(0)$ is near 0) is a submanifold of $GL(1,\mathbb{C}) = \mathbb{C} \setminus \{0\}$. It generally is not a subgroup:

$$\{d\gamma\big|_0 : \gamma(0) = a\} = \{\omega T_{-a}\big|_0 : |\omega| = 1\},$$

where $T_{-a} \in \mathrm{Aut}\,(\Omega)$ is defined by $T_{-a}(z) = (z + a)/(1 + \overline{a}z)$. But we still get a legitimate smooth parameterization of $\mathrm{Aut}\,(\Omega)$ near the identity.

The reader is invited to consider the corresponding local parameterization of $\mathrm{Aut}\,(\Omega)$ when Ω is the unit ball in $\mathbb{C}^2$—after this group is discussed in some detail in the next section.

Note that one obtains here a view of the general fact that, for $G = \mathrm{Aut}\,(\Omega)$,

$$\dim \mathcal{O}_p + \dim\,(I_p) = \dim\, G,$$

when

$$\mathcal{O}_p = \text{orbit of } p = \{\gamma(p) : \gamma \in G\}.$$

[This holds in general: the restriction to minimal isotropy, maximal orbit dimensions we made was just for convenience of visualization purposes.]

A closed subgroup of $GL(n,\mathbb{C})$ which acts on $\mathbb{C}^n$ isometrically is necessarily a closed subgroup of $U(n)$ and is hence compact. Conversely, if a subgroup of $GL(n,\mathbb{C})$ is compact, then there is a Hermitian metric on $GL(n,\mathbb{C})$ for which the subgroup acts isometrically and hence belongs to the $U(n)$ associated to the Hermitian metric. This follows from a standard argument using averaging of the standard metric with respect to the group action of the given subgroup of $GL(n,\mathbb{C})$.

The fact that every compact subgroup of $GL(n,\mathbb{C})$ acts isometrically relative to some Hermitian metric combined with Corollary 1.3.7 implies that, at each point $p \in \Omega$, there is a Hermitian metric for which I_p acts isometrically on the tangent space at p. This strongly suggests that one ought to seek a Hermitian metric on Ω which is $\mathrm{Aut}\,(\Omega)$-invariant. In other words, one ought to look for a C^∞ family h_p, $p \in \Omega$, of Hermitian metrics such that, for all $\gamma \in \mathrm{Aut}\,(\Omega)$ and $p \in \Omega$, the map $d\gamma\big|_p$ from the tangent space at p with metric h_p is an isometry onto the tangent space at $\gamma(p)$ with metric $h_{\gamma(p)}$. Indeed, it even suggests a way to do this: for some selection of distinguished points p, one in each orbit, choose h_p more or less arbitrarily except that in some sense it varies nicely with the choices of orbit. Then, for q in the orbit of such a point p, determine h_q by the requirement that $d\gamma\big|_p$ must be isometric for a γ_q with $\gamma_q(p) = q$. This is well defined by Corollary 1.3.7, independently of which γ_q is chosen. Thus the only question is whether this can be done so that the resulting metric on all of Ω is C^∞. This involves finding smooth "slices" for orbits. This is the point addressed in [Palais 1961]. But since we shall construct such Aut (Ω)-invariant metrics directly later on, we leave Palais's general construction as a philosophical observation.

1.4 The Basic Examples

In this section we shall collect a number of examples for which the automorphism groups are obtained explicitly. Some of these are well known and elementary, and the derivations of their automorphism groups need be outlined only briefly. But it will be convenient to have them all in one place; and looking at them all at once will suggest various paths of exploration that we follow later.

(1) $\mathrm{Aut}\,(\mathbb{C}) = \{z \mapsto az + b : a, b \in \mathbb{C}, a \neq 0\}$.

If $f : \mathbb{C} \to \mathbb{C}$ is injective, then the only possible singularity of f at ∞ is a simple pole. If instead ∞ were a removable singularity, then f would be constant by Liouville's theorem. If ∞ were an essential singularity, then f would not be injective in any neighborhood of ∞. Similarly, a pole at ∞ of higher order than 1 would preclude injectivity in a neighborhood of ∞. Thus the nonconstant injective function f is a polynomial of degree one. That any polynomial of degree one is an automorphism is clear. $\square$

(2) $\mathrm{Aut}\,(D) = \{z \mapsto \omega \cdot (z - a)/(1 - \overline{a}z) : a, \omega \in \mathbb{C}, |\omega| = 1, |a| < 1\}$.

That

$$T_a : z \longmapsto \frac{z - a}{1 - \overline{a}z}$$

is defined and injective from D to D is easy algebra. Also $T_a(T_{-a}(z)) = z$; hence T_a is surjective.

Conversely, suppose that $f \in \mathrm{Aut}\,(D)$. Let $a = f^{-1}(0)$. Then $g := f/T_a$ is holomorphic and zero-free on D and

$$\lim_{|\zeta|\to 1} |g(\zeta)| = \lim_{|\zeta|\to 1} \left|\frac{f(\zeta)}{T_a(\zeta)}\right| = 1.$$

By the maximum principle applied to both g and $1/g$, we see that $|T_a/f| \equiv 1$ on D, hence $f = \omega T_a$ for some constant ω with $|\omega| = 1$.[6] $\square$

[6] An alternative argument is to note that $T_a \circ f$ maps the disc to the disc and fixes 0. Then Schwarz's lemma implies that $|(T_a \circ f)(z)| \leq |z|$. Applying the same reasoning to the inverse of this mapping gives $|(T_a \circ f)(z)| \geq |z|$. Hence $|T_a \circ f(z)| \equiv |z|$ on D, and $T_a \circ f$ equals $w \cdot \mathrm{id}$ on D for some ω with $|\omega| = 1$.

(3) $\mathrm{Aut}\,(\mathbb{C} \setminus \{0\}) = \{z \mapsto az^{\epsilon} : \epsilon = \pm 1, a \in \mathbb{C}, a \neq 0\}$.

If $f \in \mathrm{Aut}\,(\mathbb{C}\setminus\{0\})$, then a connectivity argument shows that $\lim_{z\to 0} f(z) = 0$ or $\lim_{z\to 0} |f(z)| = +\infty$. Composing with an inversion, we may assume that the first alternative holds. But then f, considered as a holomorphic function, has a removable singularity at the origin. Thus the extension $f(0) = 0$ makes f an entire function that is an automorphism of the entire plane. From part (1), $f(z) = az$, for some $a \neq 0$. In case $\lim_{z\to 0} f(z) = \infty$, the same reasoning applied to $1/f$ gives $1/f(z) = az$. $\square$

(4) $\mathrm{Aut}\,(\{z \in \mathbb{C} : 0 < r_1 < |z| < r_2\}) = \{z \mapsto \omega z : \omega \in \mathbb{C}, |\omega| = 1\} \cup \{z \mapsto \omega r_1 r_2 / z : \omega \in \mathbb{C}, |\omega| = 1\}$.

Denote the annulus by A. By a connectivity argument, for each $f \in \mathrm{Aut}\,(A)$, either
(a) $\lim_{|z|\to r_2} |f(z)| = r_2$ and $\lim_{|z|\to r_1} |f(z)| = r_1$;

or

(b) $\lim_{|z|\to r_2} |f(z)| = r_1$ and $\lim_{|z|\to r_1} |f(z)| = r_2$.

In either case, repeated application of Schwarz reflection to the boundary circles extends f to an automorphism $\widehat{f} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ of $\mathbb{C} \setminus \{0\}$. Thus, by Example (3), $f(z) = az$ or $f(z) = a/z$ for some nonzero $a \in \mathbb{C}$. The condition $f(A) = A$ tells us then that $a = \omega$ in the first instance and that $a = \omega r_1 r_2$ in the second instance. $\square$

(5) $\mathrm{Aut}\,(\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\})$.

The set

$$B^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$$

is of course the unit ball in $\mathbb{C}^2$. First notice that $I_{(0,0)} = U(2) \subset GL(2, \mathbb{C})$. Obviously $U(2) \subset I_{(0,0)}$. If $f \in I_{(0,0)}$, then f is $\mathbb{C}$-linear according to Corollary 1.3.2. Since f has to preserve the unit sphere (the boundary of B^2), it is immediate that $f \in U(2)$.

Now a direct calculation, analogous to that for the disc, shows that the mapping

$$T_{(a,0)}(z_1, z_2) \equiv \left(\frac{z_1 - a}{1 - \overline{a} z_1}, \frac{\sqrt{1 - |a|^2}\, z_2}{1 - \overline{a} z_1} \right)$$

sends the ball B^2 into itself. Furthermore, the inverse mapping to $T_{(a,0)}$ is $T_{(-a,0)}$. Thus $T_{(a,0)}$ is an automorphism.

If (z_1, z_2) is any point of B^2, then there is an element $\lambda \in U(2)$ that takes (z_1, z_2) to a point of the form $(a, 0)$. Also $T_{(a,0)}(a, 0) = (0, 0)$. These two pieces of information combined tell us that $\mathrm{Aut}\,(B^2)$ acts transitively on B^2: this means that any point of B^2 may be moved to any other by some element of the automorphism group. first, B^2 is *homogeneous*.

Let G denote the subgroup of $\mathrm{Aut}\,(B^2)$ generated by $U(2)$ together with $\{T_{(a,0)} : a \in \mathbb{C}, |a| < 1\}$. Then the isotropy subgroup of G at the origin obviously contains $U(2)$. Thus it equals $U(2)$. It follows that G is the full automorphism group, by Theorem 1.3.1.[7] For future reference, note that if $\varphi \in \mathrm{Aut}\,(B^2)$, then one can always express φ in the form $\mu_1 \circ T_{(b,0)} \circ \mu_2$, where μ_1, μ_2 are unitary rotations. first, let $\lambda_2 =$ a unitary rotation taking $\varphi^{-1}((0,0))$ to a point of the form $(b,0)$. Then $T_{(b,0)} \circ \lambda_2$ takes $\varphi^{-1}((0,0))$ to $(0,0)$. Hence $T_{(b,0)} \circ \lambda_2 \circ \varphi$ takes $(0,0)$ to $(0,0)$. Thus, from our earlier observations, $T_{(b,0)} \circ \lambda_2 \circ \varphi$ is a unitary rotation, say λ_1. Hence $\varphi = \lambda_2^{-1} \circ T_{(b,0)} \circ \lambda_1$, which has the desired form. □

(6) $\mathrm{Aut}\,(\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^4 + |z_2|^4 < 1\})$.

By Corollary 1.3.2, the elements of I_0 (the isotropy group at $0 = (0,0)$) are $\mathbb{C}$-linear. Such a map must take a point of the boundary of the form $(\alpha, 0)$ or $(0, \alpha)$ to another point with one coordinate 0. This is so because boundary points with one coordinate 0 are exactly those boundary points where $\partial\Omega$ makes higher than first-order contact with its complex tangent plane, a condition preserved by invertible complex linear maps. Thus

$$\begin{aligned} I_0 = &\big\{(z_1, z_2) \mapsto (\omega_1 z_1, \omega_2 z_2) : \omega_1, \omega_2 \in \mathbb{C}, |\omega_1| = |\omega_2| = 1\big\} \\ &\cup \big\{(z_1, z_2) \mapsto (\omega_1 z_2, \omega_2 z_1) : \omega_1, \omega_2 \in \mathbb{C}, |\omega_1| = |\omega_2| = 1\big\}. \end{aligned}$$

Next, we claim that any element of $\mathrm{Aut}\,(\Omega)$ must in fact fix the origin. Let ϕ be an automorphism. By standard results in several complex variables, ϕ and ϕ^{-1} are C^∞ up to the boundary of Ω (see [Bell 1981]). Weakly pseudoconvex boundary points must consequently be mapped only to weakly pseudoconvex boundary points. So ϕ must take the union of the two circles to itself. Thus ϕ must (after composition with the map permuting the coordinates if necessary) preserve the circle $\{(\alpha, 0) \in \partial\Omega\}$, and it must also preserve the circle $\{(0, \alpha) \in \partial\Omega\}$. By the Cauchy integral formula and continuity of ϕ at the boundary, it follows that ϕ preserves the entire discs $\{(\alpha, 0) : |\alpha| \le 1\}$ and $\{(0, \alpha) : |\alpha| \le 1\}$. We conclude that $\phi(0) = 0$. Hence ϕ is linear and in fact $\phi \in I_0$. So we have completely identified all elements of $\mathrm{Aut}\,(\Omega)$, and this verifies that $\mathrm{Aut}\,(\Omega) = I_0$. □

(7) $\mathrm{Aut}\,(\Omega)$ for $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : 0 < \alpha < |z_1|^2 + |z_2|^2 < 1\}$.

By the Hartogs extension phenomenon, each element $f \in \mathrm{Aut}\,(\Omega)$ extends uniquely to a holomorphic mapping $\widehat{f} : B^2 \to B^2$, where B^2 is the unit ball in $\mathbb{C}^2$ as usual. These extensions must all be invertible since

[7] Determining the automorphism group of B^2 as a recognizable Lie group requires additional work. It turns out that it is $PSL(2, \mathbb{C})$. See [Helgason 1962] for more on this matter.

clearly $\widehat{f \circ g} = \widehat{f} \circ \widehat{g}$ for all $f, g \in \mathrm{Aut}\,(\Omega)$ (and of course the extension of the identity map is the identity map). Each such $\widehat{f}$, $f \in \mathrm{Aut}\,(\Omega)$, is a unitary rotation. To see this, note that, by the remark at the end of Example (5), $\widehat{f} = \mu_1 \circ T_{(a,0)} \circ \mu_2$ for some unitary rotations μ_1, μ_2 with $T_{(a,0)}$, as in the discussion there. Both μ_1 and μ_2 preserve Ω, but $T_{(a,0)}$ definitely does not preserve Ω if $a \neq 0$. This point is simple to check algebraically by looking at points of the form $ta/|a|$ with $-1 < t < 1$. Thus $\widehat{f}$ can preserve Ω only if $a = 0$ and, hence, $\widehat{f}$ is a unitary rotation. Consequently, $\mathrm{Aut}\,(\Omega)$ consists of the restrictions to Ω of the set of unitary rotations around the origin $(0, 0)$. $\square$

(8) $\mathrm{Aut}\,(\Omega)$ for

$$\begin{aligned}\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : 1/100 < |z_1|^2 + |z_2|^2 < 1\} \\ \setminus \Big[\{(z_1, z_2) \in \mathbb{C}^2 : |z_1 - 3/4|^2 + |z_2|^2 \leq r_1\} \\ \cup \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2 - 7/8|^2 \leq r_2\}\Big],\end{aligned}$$

with some small positive numbers r_1 and r_2.

Notice first that each element of $\mathrm{Aut}\,(\Omega)$ again extends uniquely to an element of $\mathrm{Aut}\,(B^2)$, by the Hartogs extension theorem. Then each automorphism of Ω must either preserve the sphere $\Sigma = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1/100\}$ or map this sphere to one of the other deleted spheres, by topological considerations. Algebraic considerations show that the image of a Euclidean sphere around the origin under an automorphism of B^2 is a Euclidean sphere only if the automorphism fixes the origin and hence is a rotation.

The algebraic determination that the image of a sphere with a center at the origin is again a sphere only if the origin is fixed can be done conveniently as follows. Consider $T_{(a,0)}$, for $-1 < a < 1$, acting on $S(r) =$ the sphere of radius $0 < r < 1$ around the origin $(0, 0)$. Then $T_{(a,0)}(r, 0)$ and $T_{(a,0)}(-r, 0)$ are diametrically opposite on the image sphere. Again, if the image is a sphere, it then follows that the vector from $T_{(a,0)}(0, r)$ to $T_{(a,0)}(-r, 0)$ is perpendicular to the vector from $T_{(a,0)}(0, r)$ to $T_{(a,0)}(r, 0)$. But direct calculation shows that the inner product of these two vectors is 0 if and only if $a = 0$.

As in the arguments for Example (7) above, f is now an automorphism of B^2 preserving the origin, that is the center of Σ. Consequently, any automorphisms of this Ω must be elements of $U(2)$. Since the elements of $U(2)$ are Euclidean isometries, and since the removed balls around $(3/4, 0)$ and $(0, 7/8)$ have centers that are at different distances from the origin, each of these balls must be mapped to itself. It follows that the automorphism which is an element of $U(2)$ must in fact be the identity mapping. Thus $\mathrm{Aut}\,(\Omega) = \{\mathrm{id}\}$: the automorphism group has just the single element, which is the identity. In this circumstance, we say that the domain Ω is *rigid*. $\square$

(9) Aut (Ω) for

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2k} < 1\} , \quad k > 1 .$$

First we note that I_0 is linear from Corollary 1.3.3. Also this isotropy group clearly contains all linear maps of the form

$$(z_1, z_2) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2) , \quad \theta_1, \theta_2 \in \mathbb{R}.$$

By the same logic as in Example (6), the set $\{(\alpha, 0) \in \Omega\}$ must be mapped to itself by any element of this isotropy group. This and the compactness of I_0 imply that

$$I_0 = \{(z_1, z_2) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2) : \theta_1, \theta_2 \in \mathbb{R}\},$$

as follows. The invariance of the disc $\{(\alpha, 0) \in \Omega\}$ implies that the matrices in I_0 have the form

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix}$$

with $\alpha_{11} \neq 0$ and $\alpha_{22} \neq 0$. If also α_{12} were not zero, then the powers of this matrix (which arise under multiple compositions of the mapping) would not be contained in a compact set in $GL(n, \mathbb{C})$. Thus in fact $\alpha_{12} = 0$.

For $a \in \mathbb{C}$, $|a| < 1$, consider the mapping

$$S_a : (z_1, z_2) \longmapsto \left(\frac{z_1 - a}{1 - \bar{a} z_1}, \frac{(1 - |a|^2)^{1/2k}}{(1 - \bar{a} z_1)^{1/2k}} \, z_2 \right) .$$

We see that S_a belongs to Aut (Ω). This assertion can be easily checked by direct calculation. Also S_{-a} is the inverse mapping of S_a. The orbit of $\mathbf{0}$ under Aut (Ω) consequently contains $\{(\alpha, 0) \in \Omega\}$. Again, by the logic of Example (6) using [Bell 1981] etc., it follows that the set $\{(\alpha, 0) \in \Omega\}$ is preserved by elements of Aut (Ω). Hence the Aut (Ω)-orbit of $\mathbf{0}$ is equal to $\{(\alpha, 0) \in \Omega\}$. This information then completely determines the automorphism group. □

(10) Aut (D^2), where $D^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$.

We write $\tau_a(z) = (z - a)/(1 - \bar{a}z)$ for $z \in D \subseteq \mathbb{C}$. The maps of the form $(z_1, z_2) \mapsto (\tau_{a_1}(z_1), \tau_{a_2}(z_2))$ act transitively on D^2. Also the isotropy subgroup I_0 at the origin $(0, 0)$ consists of linear maps only by Corollary 1.3.2. These linear maps must have the form $(z_1, z_2) \mapsto (\omega_1 z_1, \omega_2 z_2)$ or $(z_1, z_2) \mapsto (\omega_2 z_2, \omega_1 z_1)$ with $|\omega_1| = |\omega_2| = 1$, since they must preserve the *distinguished boundary* $\{(z_1, z_2)\colon |z_1| = 1, |z_2| = 1\}$: this set is exactly the points where ∂D^2 is not smooth, and the property of being not smooth is preserved by linear maps. It follows that Aut (D^2) is exactly the group generated by the maps $(z_1, z_2) \mapsto (\tau_{a_1}(z_1), \tau_{a_2}(z_2))$, $(z_1, z_2) \mapsto (\omega_1 z_1, \omega_2 z_2)$ with $|\omega_1| = |\omega_2| = 1$, and $(z_1, z_2) \mapsto (z_2, z_1)$.

Examples (5) and (10) yield the following historical theorem of Poincaré, which, as already discussed, shows that the Riemann mapping theorem does not hold in complex dimension higher than 1. The proof of this theorem by Poincaré (see below) demonstrated that automorphism groups could play an important role—especially in complex dimensions greater than 1. Of course we have already shown in the remarks after Corollary 1.3.2 that the ball and the polydisc are not biholomorphic, but Poincaré's proof is of historical interest.

Theorem 1.4.1 (Poincaré). *In complex dimension 2, the ball and the bidisc are not biholomorphic to each other.*

Proof. Suppose that there exists a biholomorphic map $f : B^2 \to D^2 = D \times D$. Composing with an automorphism of D^2, we may assume without loss of generality that f maps the origin to itself. Then the map $f_* : \mathrm{Aut}\,(B^2) \to \mathrm{Aut}\,(D^2)$ defined by $f_*(\gamma) \equiv f^{-1} \circ \gamma \circ f$ is a continuous group isomorphism. So, this map generates a group isomorphism between the identity components of the isotropy subgroups at the origin. Note that the identity component of the isotropy subgroup of $\mathrm{Aut}\,(B^2)$ at the origin contains $U(2)$, the group of 2×2 unitary matrices (and indeed $= U(2)$). On the other hand, the identity component of the isotropy subgroup of $\mathrm{Aut}\,(D^2)$ at the origin is the torus group consisting of rotations in each variable separately. But the torus group is commutative, while $U(2)$ is noncommutative. This is a contradiction. Therefore the desired conclusion follows immediately. □

1.5 Orbit Accumulation Boundary Points Are Pseudoconvex

In the preceding section, we have rather few examples in higher dimensions (i.e., $\mathbb{C}^n$, $n \geq 2$) of domains Ω with $\mathrm{Aut}\,(\Omega)$ noncompact. But the examples that we do have—numbers (5), (9), (10) in the last section—all have the notable property that they are convex and hence pseudoconvex. It turns out that if Ω is a bounded domain and p is a point of the boundary with the boundary smooth near p, then accumulation of an $\mathrm{Aut}\,(\Omega)$-orbit at p implies pseudoconvexity at p. More precisely:

Theorem 1.5.1 (Greene/Krantz [Greene/Krantz 1991]). *If $p_0 \in \partial\Omega$ is a boundary point of a bounded domain Ω in $\mathbb{C}^n$ whose boundary is C^2 smooth in a neighborhood of p_0, and if there exists a sequence $\varphi_j \in \mathrm{Aut}\,(\Omega)$ such that $\lim_{j\to\infty} \varphi_j(x_0) = p_0$ for some $x_0 \in \Omega$, then $\partial\Omega$ is Levi pseudoconvex at p_0.*

Proof. Assume the contrary, that $\partial\Omega$ is not pseudoconvex at p_0. Then there exists a compact set K contained in Ω such that the holomorphic hull $\widehat{K}$ of K contains a set of the form $\Omega \cap U$ where U is an open set in $\mathbb{C}^n$

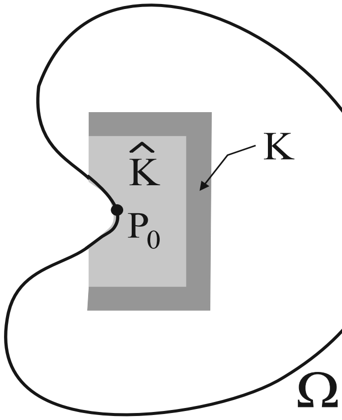

Fig. 1.1. The Hartogs figure and its holomorphic hull.

containing p_0.[8] [Recall that the holomorphic hull $\widehat{K}$ of a compact set K is by definition the set $\{p \in \Omega : |f(p)| \le \max_K |f|, \forall f : \Omega \to \mathbb{C} \text{ holomorphic}\}$.] $\square$

Now choose an $\epsilon > 0$ such that $B^n(x_0, 3\epsilon) \subset \Omega$. Let A_M be the set of $\varphi \in \mathrm{Aut}\,(\Omega)$ such that $\|d\varphi^{-1}|_{\varphi(x_0)}\| \le M$, where $\|\cdot\|$ here represents the usual operator norm. Then we show:

Lemma 1.5.2. *There exists $\delta > 0$ such that $\varphi(B^n(x_0, \epsilon))$ contains $B^n(\varphi(x_0), \delta)$ for every $\varphi \in A_M$.*

Proof of the lemma. Since $d\varphi^{-1}|_{\varphi(x_0)} = (d\varphi|_{x_0})^{-1}$, we see that $\|(d\varphi|_{x_0})^{-1}\| \le M$ whenever $\varphi \in A_M$. Consider the map

$$T(z) := (d\varphi|_{x_0})^{-1} \circ \varphi(z), \quad z \in B^n(x_0, \epsilon).$$

The differential at x_0 of this map is equal to the identity. And its second derivatives on $B^n(x_0, \epsilon)$ are bounded (Cauchy estimates on φ) by a constant depending only on M and the bound on $\|(d\varphi|_{x_0})^{-1}\|$ (and Ω and ϵ) but not on $\varphi \in A_M$. Hence, by standard information about the inverse function theorem, $T(B^n(x_0, \epsilon))$ contains a ball of radius $\alpha > 0$ centered at x_0, where α is independent of which φ is chosen from A_M: here α depends only on M (and ϵ and Ω). Thus the image of the map $\varphi = d\varphi|_{x_0} \circ T$ contains a ball of radius $\delta > 0$ centered at $\varphi(x_0)$, with δ independent of the choice of φ. [The radius δ depends only on M, ϵ, and Ω for the following reason: since $d\varphi|_{x_0}$ is a linear transformation with its inverse bounded above in operator norm, no such φ

[8] The usual construction of a compact set in Ω with holomorphic hull running out to a nonpseudoconvex boundary is casually called a "Hartogs tin can" in several complex variables (Figure 1.1). See [Grauert/Fritzsche 1976] for example. In case one "Hartogs tin can" does not provide a U of the sort we are after, one can perturb it and take the set K as the union of the perturbations to get the desired situation.

can take a given radius ball to a set not containing a definite radius ball. In fact, it cannot contract anything by more than a factor of $1/M$.] Thus the assertion of the lemma follows.

Altogether, one obtains that, if $\varphi_j(x_0) \to p_0 \in \partial\Omega$ as $j \to \infty$, then $\|d\varphi_j^{-1}|_{\varphi_j(x_0)}\| \to \infty$. Let $\psi_j = (\psi_j^1, \ldots, \psi_j^n)$ be the component representation of φ_j^{-1} for a moment. Passing to a subsequence, we may assume that

$$\left| \frac{\partial \psi_j^\ell}{\partial z_m} \Big|_{\varphi_j(z_0)} \right| \to \infty$$

for some $\ell, m \in \{1, \ldots, m\}$. [Otherwise these φ_js would belong to A_M for some $M > 0$, and hence the image of φ_j contains a ball of radius δ, independent of j. A contradiction.] However, this is impossible, because $|\partial\psi_j^\ell/\partial z_m|$ is bounded near p_0 by its absolute value on the compact Hartogs figure K, and that is bounded by a constant independent of j, by Cauchy estimates. This completes the proof. □

We shall return to related considerations later in Chapter 7 (Proposition 7.6.2), using somewhat different, albeit related, methods.

1.6 Holomorphic Vector Fields and Their Flows

From the viewpoint of the Lie theory of transformation groups, it is natural to ask which (real) vector fields have the property that their flows consist of holomorphic mappings. We shall have explicit use for these ideas later (e.g., in Chapter 6), in addition to their general interest. To explore the matter in some detail, we recall first the general viewpoint.

Suppose that $\mathbf{V} : U \to \mathbb{R}^N$ is a "vector field" (at this state, it is just a vector-valued function) on an open set $U \subset \mathbb{R}^N$. If $\mathbf{V}$ has suitable regularity—even Lipschitz continuity will suffice—then, for each $p \in U$, there are an $\epsilon > 0$ and a neighborhood W of p, $p \in W \subset U$, such that, for each $q \in W$, there is a differentiable function $\gamma_q : (-\epsilon, \epsilon) \to U$ with

$$\frac{d\gamma_q}{dt}\Big|_t = \mathbf{V}(\gamma_q(t))$$

for each $t \in (-\epsilon, \epsilon)$. Such a γ_q is called an *integral curve* of $\mathbf{V}$ with initial point q. Integral curves are unique up to the domain of definition in t if their initial point is given.

Such a vector field $\mathbf{V} : U \to \mathbb{R}^N$ thus defines a (local) flow $q \mapsto \gamma_q(t)$. We call this function φ_t so that $\varphi_t : W \to U$ is defined for all $t \in (-\epsilon, \epsilon)$. Also, $\varphi_0 =$ the identity map. Uniqueness of integral curves shows that

$$\varphi_{t_1} \circ \varphi_{t_2} = \varphi_{t_1+t_2}$$

for all t_1, t_2 with both $|t_1|$ and $|t_2|$ small enough that the φ-maps are defined.

This all makes sense for vector fields defined on an open subset of a manifold M of dimension n. In this case, the vector field $\mathbf{V}$ is a function from M into the tangent bundle $TM := \bigcup_{p\in M} T_pM$, where T_pM is the tangent space of M at p, and it is required that $\mathbf{V}(p) \in T_pM$ for every $p \in M$. The definitions of properties are the same as for the Euclidean space case, *mutatis mutandis*.

Now we are interested specifically in the question, either for $\mathbb{C}^n = \mathbb{R}^{2n}$, or on a complex manifold (locally the same as $\mathbb{C}^n$), of which vector fields $\mathbf{V}$ have the property that the associated local flows φ_t are holomorphic functions. Such a flow is called *holomorphic*, that is, a flow of a vector field is called holomorphic, if for each t, φ_t is holomorphic (where it is defined).

The answer to this question is straightforward, but it will be most easily explainable if we introduce some notation.

First we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by setting $z_j = x_j + iy_j$, $j = 1, \ldots, n$, and then identifying $(z_1, \ldots, z_n) \in \mathbb{C}^n$ with $(x_1, y_1, \ldots, x_n, y_n)$. We set $\frac{\partial}{\partial x_j}$ = the $\mathbb{R}^{2n}$ vector with the $(2j-1)$-th component 1 and all other components 0, and then $\frac{\partial}{\partial y_j}$ = the $\mathbb{R}^{2n}$ vector with $(2j)$-th component 1 and all other components 0, for $j = 1, 2, \ldots, n$. [This notation makes sense because the directional derivative of a function along one such vector just considered is equal to the corresponding partial derivative, e.g., $\frac{\partial}{\partial x_1}$ of a function is its directional derivative along the vector $(1, 0, \ldots, 0) \in \mathbb{R}^{2n}$.] As usual, we set $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$ as a differential operator.

If $\mathbf{V}$ is a real vector field on $U \subset \mathbb{C}^n = \mathbb{R}^{2n}$, then $\mathbf{V}$ has the form

$$\sum_{j=1}^n a_j \frac{\partial}{\partial x_j} + \sum_{j=1}^n b_j \frac{\partial}{\partial y_j}$$

for some real-valued functions a_j and b_j and these are uniquely determined. We define

$$J\mathbf{V} = \sum_{j=1}^n a_j \frac{\partial}{\partial y_j} - \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}.$$

One can easily verify that

$$\mathbf{V} - iJ\mathbf{V} = 2\left(\sum_{j=1}^n (a_j + ib_j)\frac{\partial}{\partial z_j}\right).$$

We define the real vector field $\mathbf{V}$ to be *holomorphic* if, for each j, the function $a_j + ib_j$ is holomorphic. Thus a real vector field $\mathbf{V}$ is holomorphic if and only if $\mathbf{V}$ is the real part of a complex vector field of the form $\sum_{j=1}^n f_j \frac{\partial}{\partial z_j}$ where the f_j are holomorphic functions. In these terms, we can answer the question about which real vector fields have (local) flows that are holomorphic.

Theorem 1.6.1 (Lie Theory Lemma). *A C^1 real vector field $\mathbf{V}$ has holomorphic local flows φ_t if and only if $\mathbf{V}$ is a holomorphic vector field in the sense just defined.*

If one is willing to use the standard methods of "Lie derivatives," then this assertion is easy to check. We shall present that proof first. Then we shall recast it in more concrete form in which the concept of Lie derivative is not used explicitly.

Proof of the lemma using Lie derivatives. The local flow φ_t for a fixed t value is holomorphic if and only if $d\varphi_t$ commutes with the J-mapping already defined. (This latter is just a restatement of the Cauchy–Riemann equations.) Here $d\varphi_t$ denotes the real differential of φ_t. Since $\varphi_0 =$ the identity map, to check that $d\varphi_t \circ J = J \circ d\varphi_t$ for all t, we need only check that $L_{\mathbf{V}} J = 0$ where $L_{\mathbf{V}} J$ denotes the Lie derivative of the tensor J with respect to $\mathbf{V}$. Thus we need only check that, for each $j = 1, \dots, n$,

$$(L_{\mathbf{V}} J)\frac{\partial}{\partial x_j} = 0 \qquad \text{and} \qquad (L_{\mathbf{V}} J)\frac{\partial}{\partial y_j} = 0.$$

Now

$$(L_{\mathbf{V}} J)\frac{\partial}{\partial x_j} = L_{\mathbf{V}}\left(J\frac{\partial}{\partial x_j}\right) - J\left(L_{\mathbf{V}}\left(\frac{\partial}{\partial x_j}\right)\right)$$

by the Leibniz rule for Lie derivatives. But

$$\begin{aligned} L_{\mathbf{V}}\left(J\frac{\partial}{\partial x_j}\right) &= L_{\mathbf{V}}\left(\frac{\partial}{\partial y_j}\right) \\ &= -\sum_{\ell=1}^{n} \frac{\partial a_\ell}{\partial y_j}\frac{\partial}{\partial x_\ell} - \sum_{\ell=1}^{n} \frac{\partial b_\ell}{\partial y_j}\frac{\partial}{\partial y_\ell} \end{aligned}$$

while

$$\begin{aligned} J\left(L_{\mathbf{V}}\left(\frac{\partial}{\partial x_j}\right)\right) &= -J\left(\sum_{\ell=1}^{n} \frac{\partial a_\ell}{\partial x_j}\frac{\partial}{\partial x_\ell} + \sum_{\ell=1}^{n} \frac{\partial b_\ell}{\partial x_j}\frac{\partial}{\partial y_\ell}\right) \\ &= \sum_{\ell=1}^{n} \frac{\partial b_\ell}{\partial x_j}\frac{\partial}{\partial x_\ell} - \frac{\partial a_\ell}{\partial x_j}\frac{\partial}{\partial y_\ell}. \end{aligned}$$

Thus, $L_{\mathbf{V}}\big(J\frac{\partial}{\partial x_j}\big) = J\big(L_{\mathbf{V}}\big(\frac{\partial}{\partial x_j}\big)\big)$ if and only if

$$\frac{\partial a_\ell}{\partial y_j} = -\frac{\partial b_\ell}{\partial x_j} \qquad \text{and} \qquad \frac{\partial a_\ell}{\partial x_j} = \frac{\partial b_\ell}{\partial y_j}$$

for $\ell = 1, \dots, n$, in both cases. But these are precisely the Cauchy–Riemann equations for $a_\ell + ib_\ell$ to be holomorphic in the z_j variable. It is clear that if these hold, then $(L_{\mathbf{V}} J)\big(\frac{\partial}{\partial y_j}\big)$ is also 0 since

$$\begin{aligned} J\left(L_{\mathbf{V}}\left(J\frac{\partial}{\partial x_j}\right) - JL_{\mathbf{V}}\left(\frac{\partial}{\partial x_j}\right)\right) &= L_{\mathbf{V}}\left(\frac{\partial}{\partial x_j}\right) + J\left(L_{\mathbf{V}}\left(\frac{\partial}{\partial y_j}\right)\right) \\ &= -L_{\mathbf{V}}\left(J\left(\frac{\partial}{\partial y_j}\right)\right) + J\left(L_{\mathbf{V}}\left(\frac{\partial}{\partial y_j}\right)\right). \end{aligned}$$

For the converse direction, just trace the steps backwards. The conclusion follows.

To carry out essentially the same proof without introducing the Lie derivatives explicitly, we compute first, for each $j = 1, \ldots, n$,

$$\frac{\partial}{\partial t}\left(Jd\varphi_t|_{p,0}\left(\frac{\partial}{\partial x_j}\right) - d\varphi_t|_{p,0}\left(\frac{\partial}{\partial y_j}\right)\right).$$

For this, note that $p = \varphi_0(p)$ and write, for all q near p,

$$\varphi_t(q) = (x_{1,t}(q),\, y_{1,t}(q),\, \ldots,\, x_{n,t}(q),\, y_{n,t}(q)).$$

Then

$$d\varphi_t\left(\frac{\partial}{\partial x_j}\right) = \left(\frac{\partial x_{1,t}}{\partial x_j}, \frac{\partial y_{1,t}}{\partial x_j}, \ldots, \frac{\partial x_{n,t}}{\partial x_j}, \frac{\partial y_{n,t}}{\partial x_j}\right)$$

and

$$d\varphi_t\left(\frac{\partial}{\partial y_j}\right) = \left(\frac{\partial x_{1,t}}{\partial y_j}, \frac{\partial y_{1,t}}{\partial y_j}, \ldots, \frac{\partial x_{n,t}}{\partial y_j}, \frac{\partial y_{n,t}}{\partial y_j}\right),$$

while

$$J\left(d\varphi_t\left(\frac{\partial}{\partial x_j}\right)\right) = \left(\frac{\partial y_{1,t}}{\partial x_j},\ -\frac{\partial x_{1,t}}{\partial x_j}, \ldots\right).$$

So

$$\frac{\partial}{\partial t}J\left(d\varphi_t\left(\frac{\partial}{\partial x_j}\right)\right) = \left(\frac{\partial^2 y_{1,t}}{\partial t \partial x_j}, -\frac{\partial^2 x_{1,t}}{\partial t \partial x_j}, \ldots\right) = \left(\frac{\partial}{\partial x_j}\left(\frac{\partial y_{1,t}}{\partial t}\right), \ldots\right)$$

and

$$\frac{\partial}{\partial t}d\varphi_t\left(\frac{\partial}{\partial y_j}\right) = \left(\frac{\partial^2 x_{1,t}}{\partial t \partial y_j}, \ldots\right) = \left(\frac{\partial}{\partial y_j}\left(\frac{\partial x_{1,t}}{\partial t}\right), \ldots\right).$$

Note that

$$\left.\frac{\partial x_{\ell,t}}{\partial t}\right|_{t=0,p} = a_\ell(p) \qquad \text{and} \qquad \left.\frac{\partial y_{\ell,t}}{\partial t}\right|_{t=0,p} = b_\ell(p).$$

Translating the Cauchy–Riemann equations for the functions $a_\ell + ib_\ell$ back into the x, y notation gives

$$\frac{\partial}{\partial t}\left\{J\left(d\varphi_t\left(\frac{\partial}{\partial x_j}\right)\right) - d\varphi_t\left(J\left(\frac{\partial}{\partial x_j}\right)\right)\right\} = 0$$

when $t = 0$.

Working through the details of this calculation gives that this implication goes in both directions.

Now note that $d\varphi_{t+h}(\cdot) = d\varphi_t(d\varphi_h(\cdot))$ for small h and so $d\varphi_{t+h} - d\varphi_t = d\varphi_t(d\varphi_h - d\varphi_0)$. Hence $\lim_{h\to 0} \frac{1}{h}(d\varphi_{t+h} - d\varphi_t) = 0$, if $\lim_{h\to 0} \frac{1}{h}(d\varphi_h -$ identity$) = 0$. Thus, if $\mathbf{V}$ is holomorphic then $Jd\varphi_t(\frac{\partial}{\partial y_j}) - d\varphi_t(J(\frac{\partial}{\partial y_j})) = 0$. first, φ_t is holomorphic. These calculations also work in the opposite direction. □

This proof is essentially the same as the Lie derivative one: the Lie derivative concept has been replaced by equality of mixed partials, in effect.

Corollary 1.6.2. *If the local flow functions φ_t of a real vector field $\mathbf{V}$ are holomorphic, then so are the local flow functions of $J\mathbf{V}$.*

Proof. If $\mathbf{V} - iJ\mathbf{V}$ is a holomorphic linear combination of $\frac{\partial}{\partial z}$ vector fields, then so is $i(\mathbf{V} - iJ\mathbf{V})$. But Re $(i(\mathbf{V} - iJ\mathbf{V})) = J\mathbf{V}$. □

If $\mathbf{V}$ is a (real) vector field defined on an open set $U \subset \mathbb{R}^N$ (or on a manifold M), and if $q \in U$ (or, $q \in M$, respectively), then it may not be the case that the integral curve $\gamma_q(t)$ of $\mathbf{V}$ with $\gamma_q(0) = q$ is defined for all $t \in \mathbb{R}$. So the local flow functions φ_t of $\mathbf{V}$ may not be defined on all U for all t.

Note, however, that if there is an $\epsilon > 0$ such that $\varphi_t(q)$ is defined for all $t \in (-\epsilon, \epsilon)$ and all $q \in U$ (or $q \in M$), then φ_t is defined for all $t \in \mathbb{R}$: this result follows by "patching together" via uniqueness of integral curves the local flows for $|t| < \epsilon/2$. That is, one notes that φ_t should equal $\varphi_{t/k} \circ \cdots \circ \varphi_{t/k}$ (k-times) for any positive integer k and that, if k is large enough, then $|t/k| \leq \epsilon/2$. Then one uses $\varphi_{t/k} \circ \cdots \circ \varphi_{t/k}$ as the definition of φ_t and verifies easily that this indeed has the defining property that $\frac{d}{dt}\varphi_t(q) = \mathbf{V}(\varphi_t(q))$.

Consequently, if M is a compact manifold and $\mathbf{V}$ a vector field on it, then the φ_t flows associated to $\mathbf{V}$ are defined for all $t \in \mathbb{R}$ since the existence of an ϵ uniform over M follows from the basic local existence result for ordinary differential equations and the compactness of M.

In noncompact complex instances, it can happen that a holomorphic vector field $\mathbf{V}$ has integral curves and flow functions φ_t defined for all $t \in \mathbb{R}$ but $J\mathbf{V}$, also a holomorphic vector field, does not. Consider, for instance, the vector field $\mathbf{V}(x, y) = (y, -x)$ on $U := \{z \in \mathbb{C} \mid |z| < 1\}$. The vector field $\mathbf{V}$ is the"infinitesimal generator" of rotations around the origin, and its flow φ_t, defined for all $t \in \mathbb{R}$, is the rotation clockwise around the origin through angle t. As guaranteed by the fact that V is holomorphic ($\mathbf{V} = \text{Re }(-2iz\,\frac{\partial}{\partial z})$), these φ_t are indeed holomorphic. The vector field $J\mathbf{V}$ is (x, y). This too is holomorphic: $J\mathbf{V} = \text{Re } 2z\,\frac{\partial}{\partial z}$. Its local flow functions φ_t are given by $\varphi_t(x, y) = (e^t x, e^t y)$, as is easily verified. But of course these are not defined for all t: when t is large positive, $(e^t x, e^t y)$ no longer lies in U, unless $(x, y) = (0, 0)$, the origin $(0, 0)$ being a fixed point of the flow since $J\mathbf{V}(0, 0) = (0, 0)$.

But, when one passes to the compact case, things change. The vector field $\mathbf{V}$ extends to be a vector field on $\mathbb{C} \cup \{\infty\}$, the "Riemann sphere": it is again the infinitesimal generator of the one-parameter group of rotations around the origin (in the clockwise direction). Since this is a group of holomorphic mappings, it must be that $\mathbf{V}$ extended is holomorphic on $\mathbb{C} \cup \{\infty\}$. (One can of course check directly that V is holomorphic at ∞, using $w = 1/z$ as a local coordinate around ∞.) But now the flow of $J\mathbf{V}$ is defined for all t: the point $(e^t x, e^t y)$ is in $\mathbb{C}$ for all $(x, y) \in \mathbb{R}^2$ with $(x, y) \neq (0, 0)$, and the flow has $(0, 0)$ and ∞ as fixed points, with t going to $-\infty$ corresponding to motion towards 0. Thus one sees in action the important difference between the compact and noncompact cases. These themes will reappear in Chapter 6.

2

Riemann Surfaces and Covering Spaces

In this chapter, we shall discuss the automorphisms of Riemann surfaces as a preview of the higher-dimensional results to come later. In no sense are we going to try to survey completely the enormous collection of results on the subject obtained in the nineteenth century (cf. [Farkas/Kra 1992], and historically [Fricke/Klein 1897]) nor the continuing investigation of the subject up to our own time. Even less shall we explore the interaction of the theory of Riemann surface automorphisms with number theory, dynamical systems, and so on. Rather, we are going to focus concretely on the circle of ideas involving invariant metrics, since that subject will be one of our major themes in higher dimensions.

The Riemann surface situation has the attractive property that explicit determination of automorphism-invariant metrics is possible, when they exist, as they do in almost all instances. In particular, when a Riemann surface is a quotient of the unit disc, then the "push-down" of the Poincaré metric will be invariant, as we shall see.

It is worth noting, however, that this particular method of constructing automorphism-invariant metrics on (almost all) Riemann surfaces does not extend to higher dimensions. In particular, the celebrated uniformization theorem (Theorem 2.5.1) of Poincaré and Koebe does not really have a higher-dimensional analogue, as already noted in Chapter 1. Thus the ideas in the chapter are suggestive of the power and attractiveness of invariant metrics, but the extension to higher dimensions requires an alternative construction. It is interesting in this regard that when the higher-dimensional construction is applied to the case of bounded domains in $\mathbb{C}$, it gives different results, except in the case of the unit disc and its biholomorphic images: first, the Bergman metric of a bounded domain in $\mathbb{C}$ is complete and has constant Gauss curvature if and only if the domain is biholomorphic to the disc. (This will be proved for all dimensions in Chapter 4.) But the invariant metrics we construct on bounded domains in $\mathbb{C}$ in this present Chapter 2 are all complete and of constant Gauss curvature. This is of course possible because in general the automorphism group of a bounded domain is not so large that the

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invariance of a metric under automorphisms implies anything like the essential uniqueness of the metric (up to a constant factor). Invariance does imply the uniqueness of the metric up to a constant multiple for the unit disc (and its biholomorphic images): every automorphism-invariant metric is a constant multiple of the Poincaré metric. But, in other cases, the automorphism group is too small to force such uniqueness.

2.1 Coverings of Riemann Surfaces

Let M be a Riemann surface.[1] As a topological surface, M has a "universal covering space." That is, there is a simply connected surface $\widetilde{M}$ and a "projection" $\pi : \widetilde{M} \to M$ that is a covering space in the topological sense of the word. This means by definition that every point $p \in M$ has a neighborhood U_p with $\pi^{-1}(U_p)$ equaling a disjoint union of open sets $\{V_j\}$ such that $\pi|_{V_j} : V_j \to M$ is a homeomorphism onto U_p. One says here that U_p is *evenly covered.*

The topological surface $\widetilde{M}$ can be given a Riemann surface structure by "pullback," that is, by declaring $\pi|_{V_j}$ to be holomorphic. Thus $\widetilde{M}$ becomes a simply connected Riemann surface and $\pi : \widetilde{M} \to M$ is a holomorphic mapping. Koebe's celebrated uniformization theorem (Theorem 2.5.1) asserts that $\widetilde{M}$ can only be one of three things (up to biholomorphic equivalence): the plane, the unit disc, or the Riemann sphere. And, as we shall see, the plane and the sphere arise only in a few special cases. So the uniformization theorem says that all but a finite number (topologically) of Riemann surfaces have simply connected covering space biholomorphic to the disc.

It is a standard and easily-checked fact from topological covering space theory (for which see, e.g., [Gamelin/Greene 1999], [Greenberg/Harper 1981], [Spanier 1966]) that the projection $\pi : \widetilde{M} \to M$ can be interpreted as a quotient by a group action. There is a group Γ, the group of covering transformations, acting as homeomorphisms on $\widetilde{M}$, such that M is the orbit space of the action of Γ. In the Riemann surface case, Γ consists of homeomorphisms which are holomorphic, i.e., elements of $\mathrm{Aut}\,(\widetilde{M})$.

The possibilities $\widetilde{M} = \mathbb{C}$ and $\widetilde{M} = \mathbb{C} \cup \{\infty\}$ yield few Riemann surfaces. It is again part of general covering space theory that the (nonidentity) elements of Γ act without fixed points, that is, if $\sigma \in \Gamma$ and $\sigma(p) = p$ for some $p \in \widetilde{M}$, then σ is the identity map of $\widetilde{M}$ to $\widetilde{M}$. Now direct calculation shows that the linear fractional transformations that are automorphisms of $\mathbb{C} \cup \{\infty\}$ all have fixed points—this is just elementary algebra. Thus $\mathbb{C} \cup \{\infty\}$ has no covering-space quotients except itself. In other words, in the language of the uniformization theorem, the sphere can cover only the sphere.

When $\widetilde{M} = \mathbb{C}$, quotients are possible. The automorphisms of $\mathbb{C}$, $\{z \mapsto az + b : a \neq 0, a, b \in \mathbb{C}\}$, do include maps without fixed points. Specifically,

[1] A *Riemann surface*, by definition, is a one-dimensional complex manifold.

any map of the form $z \mapsto z + b$ has no fixed point; these are in fact the only ones, since, if $a \neq 1$, then $z = b/(1 - a)$ is a fixed point. A familiar analysis shows that the possible Riemann surface quotients that can thus arise are $\mathbb{C} \setminus \{0\}$, topologically a cylinder, and surfaces of genus 1, topologically tori. And these possibilities do indeed occur. But these are the only topological possibilities.

All other Riemann surfaces must be quotients of the disc D. Clearly the possibilities for groups acting on D to yield Riemann surface covering quotients must be many and varied. But, as we shall see, much can be said about this situation in spite of its generality.

2.2 Covering Spaces and Invariant Metrics, I: Quotients of $\mathbb{C}$

The examples in Section 1.4 suggest at least two themes that will be prominent in what follows. First, topological complexity tends to make the automorphism group small. The "holes" in Examples (4), (7), and (8) made the group compact; in Example (8), in fact, the group was the identity alone. Second, Example (6) illustrates the fact that, in complex dimension 2 or greater, the structure of the boundary can restrict the group even when the topology is as simple as possible. In that case, the isotropy subgroup I_0 at 0 had to be a relatively small subgroup of $U(n)$ because the boundary of the domain had some special boundary points that had to be taken only to other boundary points with the same geometry. Later, we shall see that domains in $\mathbb{C}^n$, $n \geq 2$, can be homeomorphic to the ball but have no automorphisms other than the identity.

The examples of Section 1.4, however, did not really show anything like the full potential of the idea of invariant metrics. Of course, in accord with general principles described in Section 1.3 (see the discussion that follows Theorem 1.3.12), invariant metrics exist for bounded domains. These will be constructed explicitly in generality later, without use of [Palais 1961]. But, in the examples considered, the invariant metrics, and the automorphism groups themselves, appeared in rather *ad hoc* and case-specific ways.

In the present section we shall describe a much more general construction for invariant metrics in the Riemann surface cases for which such metrics exist. This will be achieved by the systematic use of the uniformization theorem (Theorem 2.5.1) and covering-space techniques. Since uniformization as such has no analogue in higher dimensions, we shall need other constructions there, as noted in the introduction to this chapter. But the Riemann surface cases are even so of interest.

As already noted earlier, almost all Riemann surfaces are quotients of the unit disc D rather than of $\mathbb{C}$ or $\mathbb{C} \cup \{\infty\}$. But, as it happens, it will be best to dispose of the latter two possibilities first; this illustrates the general idea in a couple of specific situations in which the calculations are as simple as possible.

First it should be noted that the cases of $\mathbb{C}$ and $\mathbb{C} \cup \{\infty\}$ do not fit into the invariant metric picture.[2] This is clear without any generalities since the maps $\{z \mapsto az : a \in \mathbb{C}\}$ belong to $\mathrm{Aut}\,(\mathbb{C})$ and, interpreted as taking ∞ to ∞, they also lie in $\mathrm{Aut}\,(\mathbb{C} \cup \{\infty\})$. These maps fix 0, but obviously preserve no metric at 0 (since they dilate without bound in the infinitesimal sense).

Now let us return to Example (3) of Section 1.4—the punctured plane $\mathbb{C} \setminus \{0\}$. This time around we shall analyze it from the viewpoint of uniformization and covering spaces.

The mapping $\exp : z \mapsto e^z$ takes $\mathbb{C}$ to $\mathbb{C} \setminus \{0\}$. It is easy to see that this is a covering map in the topological sense: For if $z_0 \in \mathbb{C} \setminus \{0\}$, then $\exp^{-1}(\{z : |z - z_0| < |z_0|\})$ consists of disjoint connected open sets, on each of which exp is a homeomorphism onto the disc $\{z : |z - z_0| < |z_0|\}$. This is simply a geometric expression of the familiar fact that the logarithm function has well-defined holomorphic "branches" on discs in $\mathbb{C} \setminus \{0\}$.

Now suppose that $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is an automorphism of $\mathbb{C} \setminus \{0\}$. We shall prove that there exists a holomorphic function $f^* : \mathbb{C} \to \mathbb{C}$ such that

$$f \circ \exp = \exp \circ f^* . \tag{2.1}$$

The meaning of (2.1) is clearest if we think of it in terms of a commutative diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f^*} & \mathbb{C} \\ \exp \downarrow & \searrow & \downarrow \exp \\ \mathbb{C} \setminus \{0\} & \xrightarrow{f} & \mathbb{C} \setminus \{0\} \end{array}$$

In topological language, f^* is a "lift" of f.

Proof of (2.1). Set $f^*(0) = L(f(1))$, where L is a "holomorphic branch of the logarithm" defined on a neighborhood of $f(1)$, i.e., $\exp(L(z)) \equiv z$ on that neighborhood. The function $L \circ f \circ \exp$ is defined and holomorphic on a neighborhood of 0. And it is easily seen to admit unrestricted analytic continuation (along every curve with initial point 0) in $\mathbb{C}$. By the monodromy theorem, there is a global holomorphic function $f^* : \mathbb{C} \to \mathbb{C}$ such that, in a neighborhood of 0, $f^* \equiv L \circ f \circ \exp$. In that neighborhood,

$$\exp \circ f^* = (\exp \circ L) \circ f \circ \exp = f \circ \exp .$$

Hence $\exp \circ f^* = f \circ \exp$ on all of $\mathbb{C}$. $\square$

[2]By an *invariant metric*, we mean a metric that is invariant under the action of the automorphism group.

Remark. We gave the proof of (2.1) in terms of function theory and the monodromy theorem out of respect for tradition and familiarity. But actually the best way to think of it is in terms of pure topology. The mapping

$$f \circ \exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$$

lifts because $\mathbb{C}$ is simply connected, and every map of a simply connected space lifts to any covering.

Next we want to show that the lift f^* is an automorphism of $\mathbb{C}$. To prove this statement, we first obtain from (2.1) that

$$\begin{array}{ccccccccc} \mathbb{C} & \xrightarrow{f^*} & \mathbb{C} & & & \mathbb{C} & \xrightarrow{f^{-1*}} & \mathbb{C} \\ \exp\downarrow & \searrow & \downarrow\exp & \quad\text{and}\quad & & \exp\downarrow & \searrow & \downarrow\exp \\ \mathbb{C}\setminus\{0\} & \xrightarrow{f} & \mathbb{C}\setminus\{0\} & & & \mathbb{C}\setminus\{0\} & \xrightarrow{f^{-1}} & \mathbb{C}\setminus\{0\} \end{array}$$

Clearly $f^* \circ (f^{-1})^*$ is a lift of the identity map, i.e.,

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f^*\circ(f^{-1})^*} & \mathbb{C} \\ \exp\downarrow & \searrow & \downarrow\exp \\ \mathbb{C}\setminus\{0\} & \xrightarrow{\text{id}} & \mathbb{C}\setminus\{0\} \end{array}$$

This means that, for each z, we have

$$\exp(f^* \circ (f^{-1})^*(z)) = \exp(z).$$

It follows that, for each z, $f^* \circ (f^{-1})^*(z) - z \in 2\pi i\mathbb{Z}$. By continuity, $f^* \circ (f^{-1})^*(z) = z + 2\pi ik$ for some fixed integer k and all $z \in \mathbb{C}$. Similarly, $(f^{-1})^* \circ f^*$ is also a translation. We then have that f^* is one-to-one and onto. Hence $f^* \in \text{Aut}\,(\mathbb{C})$. $\square$

Note that the lift $f^* : \mathbb{C} \to \mathbb{C}$ of a given $f \in \text{Aut}\,(\mathbb{C} \setminus \{0\})$ is far from unique: we were free to choose the "branch" L of the logarithm in the proof of (2.1) arbitrarily. The set of all lifts of the identity map of $\mathbb{C}\setminus\{0\}$ to $\mathbb{C}\setminus\{0\}$ forms a subgroup of $\text{Aut}\,(\mathbb{C})$. This subgroup is called the *group of covering transformations* of the covering space $\mathbb{C} \overset{\exp}{\to} \mathbb{C}\setminus\{0\}$. Note that a covering transformation is uniquely determined by the image of 0 (or of any pre-chosen point). Moreover, every element of $\exp^{-1}(1)$ is obtainable as an image under a covering transformation of the point 0. These general statements from covering space theory are easy here: The covering transformations are exactly the translation mappings $z \mapsto z + 2\pi ik$, $k \in \mathbb{Z}$, and obviously there is one and only one of these taking 0 to a given element of $\exp^{-1}(1) = 2\pi i\mathbb{Z}$.

Next we want to see which elements of $\mathrm{Aut}(\mathbb{C})$ actually arise as lifts of elements of $\mathrm{Aut}(\mathbb{C}\setminus\{0\})$. One condition for this to happen is clear: an $f^* \in \mathrm{Aut}(\mathbb{C})$ must have the property that

$$\exp(f^*(z+2\pi ik)) = \exp(f^*(z)),$$

for all $z \in \mathbb{C}$, in order for f^* to arise as the "lift" of some holomorphic map of $\mathbb{C}\setminus\{0\}$ to itself. This necessary condition is also almost sufficient: an element $F \in \mathrm{Aut}(\mathbb{C})$ with $\exp(F(z+2\pi ik)) = \exp(F(z))$ does give rise to some mapping $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$ by setting $f(w) = \exp(F(z))$ for some $z \in \exp^{-1}(w)$. The condition on F makes this f well defined, i.e., independent of the choice of z in $\exp^{-1}(w)$. And $f^* = F$. But the resulting f may not be an automorphism of $\mathbb{C}\setminus\{0\}$. For instance, if F is the function $z \mapsto 2z$, then the associated f is $z \mapsto z^2$. The answer to this small conundrum is as follows.

Lemma 2.2.1. *An element $F \in \mathrm{Aut}(\mathbb{C})$ is of the form f^* for some $f \in \mathrm{Aut}(\mathbb{C}\setminus\{0\})$ if and only if F maps each set of the form $\{z+2\pi ik : k \in \mathbb{Z}\}$ in a one-to-one, onto fashion onto another set of the same form.*

Proof. Under the condition given in the lemma, $F = f^*$ for some (not necessarily biholomorphic) $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$. And $F^{-1} = g^*$ for some g. Then $(F \circ F^{-1}) = (f \circ g)^*$ and $(F^{-1} \circ F) = (g \circ f)^*$. From this, $f \circ g$ and $g \circ f$ are both the identity on $\mathbb{C}\setminus\{0\}$. □

Now the automorphisms $z \mapsto az+b$, $a, b \in \mathbb{C}$, $a \neq 0$, that preserve sets of the form $\{z+2\pi ik : k \in \mathbb{Z}\}$, as required by Lemma 2.2.1, are easily checked to be exactly those of the forms

$$z \mapsto z+b, \quad b \in \mathbb{C}$$

or

$$z \mapsto -z+b, \quad b \in \mathbb{C}.$$

The corresponding automorphisms of $\mathbb{C}\setminus\{0\}$ are $z \mapsto e^b z$ and $z \mapsto e^b/z$. These are all the automorphisms of $\mathbb{C}\setminus\{0\}$. Note that this agrees with the result already obtained in Section 1.4, Example (3).

Since the mappings $z \mapsto z+b$ and $z \mapsto -z+b$ preserve a metric on $\mathbb{C}$ (first, the standard Euclidean metric), it follows that we can obtain an invariant metric on $\mathbb{C}\setminus\{0\}$ by declaring exp to be a local isometry. [This gives a well-defined metric on $\mathbb{C}\setminus\{0\}$ since, in particular, the covering transformations act as isometries.] Since it is Hermitian, this metric on $\mathbb{C}\setminus\{0\}$ is a multiple of the Euclidean metric, first, at $z \in \mathbb{C}\setminus\{0\}$,

$$\|\mathbf{v}\| = \left(\frac{1}{|z|}\right) \cdot \|\mathbf{v}\|_{\text{euclid}}.$$

(This can be checked by direct calculation.) If we express this in terms of $(\text{length})^2$, then the new metric on $\mathbb{C}\setminus\{0\}$ becomes, in polar coordinates (r,θ),

$$\frac{1}{r^2}(dr^2 + r^2 d\theta^2) = \frac{1}{r^2}dr^2 + d\theta^2.$$

If we reparameterize $\mathbb{C}\setminus\{0\}$ by $R = \ln r$ and θ, then $dR = (1/r)dr$, so the metric becomes $dR^2 + d\theta^2$ for $R \in (-\infty, +\infty)$, $\theta \in [0, 2\pi)$. Thus $\mathbb{C}\setminus\{0\}$ in the new metric is isometric to a right circular cylinder of radius 1.

The automorphism $z \mapsto (r_0 e^{i\theta_0})z$ corresponds to translation of R by $\ln r_0$ and rotation in θ by θ_0, i.e., the map $(R,\theta) \mapsto (R + \ln r_0, \theta + \theta_0)$. The automorphism $z \mapsto (r_0 e^{i\theta_0})/z$ corresponds to $(R,\theta) \mapsto (-R + \ln r_0, -\theta + \theta_0)$ (the two minus signs make it an orientation-preserving map as required). Thus the automorphisms of $\mathbb{C}\setminus\{0\}$ are visualized in an explicit geometric form as isometries of the cylinder. Moreover, the covering map $\mathbb{C} \to \mathbb{C}\setminus\{0\}$ is visualized as just wrapping the plane around the cylinder.

The other Riemann surfaces covered by $\mathbb{C}$ are the torus family, the Riemann surfaces obtained as quotients by translations in two different directions. Specifically, one defines $z \sim w$, $z, w \in \mathbb{C}$, if $z - w \in \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$, where ω_1, ω_2 are (fixed) complex numbers linearly independent over $\mathbb{R}$. Then the torus T is obtained as $\mathbb{C}$ modulo this equivalence relation $\sim$. The situation here is much as before: the covering transformations are the maps $z \mapsto z + m\omega_1 + n\omega_2$, $m, n \in \mathbb{Z}$ fixed. Automorphisms of T lift to automorphisms of $\mathbb{C}$ which preserve the covering-transformation orbits, i.e., preserve the equivalence classes. These automorphisms of $\mathbb{C}$ are exactly the translations $z \mapsto z + b$ and the negation-translations $z \mapsto -z + b$, together with, possibly, some additional elements that arise when there is a $\mathbb{C}$-linear mapping other than $z \mapsto -z$ which takes the lattice $\{n\omega_1 + m\omega_2 : m, n \in \mathbb{Z}\}$ in a one-to-one fashion onto itself. Such maps (if any) must be rotations since, if $|c| \neq 1$, then the powers of $z \mapsto cz$ or its inverse would move lattice points $\neq 0 + 0i$ arbitrarily close to $0 + 0i$, a contradiction. The set of lifts of T-automorphisms to $\mathbb{C}$ thus preserves the standard, Euclidean metric on $\mathbb{C}$ so that this metric, pushed down to T, is $\mathrm{Aut}\,(T)$-invariant. As before, the push-down is well defined because the covering transformations act as isometries.

2.3 Covering Spaces and Invariant Metrics, II: Quotients of *D*

The construction in the previous section of automorphism-invariant metrics on $\mathbb{C}\setminus\{0\}$ and on 2-tori involved a certain delicate point: while $\mathbb{C}$ has no automorphism-invariant metrics itself, it does possess a metric which is invariant under all the fixed-point-free automorphisms that can potentially be covering transformations for nontrivial covering-space quotients. For the unit disc D, this subtlety does not arise: There is a Riemannian (actually Hermitian) metric on D which is invariant under every element of $\mathrm{Aut}\,(D)$, fixed-point-free or not. This is of course the well-known Poincaré metric[3] (cf. [Kobayashi 1970])

$$\rho = \frac{4}{(1-|z|^2)^2}(dx^2 + dy^2).$$

In the context of the ideas developed in Section 1.3, the existence of an automorphism-invariant metric on D is anything but a surprise. Indeed, in that context the Poincaré metric arises all but automatically, as follows. Since $I_0 = \{z \mapsto \omega : |\omega| = 1, \omega \in \mathbb{C}\}$, there is, up to a constant factor, only one I_0-invariant Riemannian metric at 0—first the Euclidean metric $dx^2 + dy^2$. This metric is of course Hermitian. Now, if there is to be an Aut (D)-invariant metric on all of D, then it must be that

$$\begin{aligned} T_a : D &\to D \\ z &\mapsto \frac{z-a}{1-\bar{a}z} \end{aligned}$$

is an isometry for all $a \in D$.[4] In particular, the length of the real vector $(1,0)$ at a must be equal to the length of $(dT_a)|_a(1,0)$ at 0. Now the vector $(dT_a)(1,0)$ has length equal to $|T_a'(a)|$ since the real differential of a holomorphic function f on an open subset of $\mathbb{C}$ acts as a rotation composed with a dilation by the factor $|f'|$. Note that

$$T_a'(a) = \frac{1}{1-|a|^2}.$$

Hence the invariant-metric length of $(1,0)$ at $a \in D$ must be

$$\left(\frac{1}{1-|a|^2}\right) \times (\text{the length of a Euclidean unit vector at } 0).$$

In other words, the invariant metric in $(\text{length})^2$ form must be, at $a \in D$, given by

$$\frac{c}{(1-|a|^2)^2}(dx^2 + dy^2).$$

With $c = 4$ we obtain the usual Poincaré metric.

The argument just given shows at first only what the invariant metric must be if it exists. But a similar and more detailed analysis shows that in fact the metric obtained *must* be invariant. Think of it this way: if $F \in$ Aut (D) and

[3]The constant factor 4 is chosen for geometric convenience: it gives the metric's Gaussian curvature the constant value of -1. For our present purposes, the factor 4 can be regarded as simply historically motivated.

[4]These mappings are often, in the context of function theory, called *Möbius transformations*.

$F(z) = w$, then F can be written as $T_w^{-1} \circ r \circ T_z$, where r is some element of I_0. This is just group-theoretic formalism since $T_w^{-1} \circ F \circ T_z(0) = 0$. Since T_z, r, and T_w are by construction isometries at z, 0, and w, then so is F.

This is of course related to the general proper-action concepts introduced in Chapter 1.

Meanwhile, the following result follows easily from the existence of the $\mathrm{Aut}\,(D)$-invariant metric and the covering-space ideas introduced in the previous section.

Proposition 2.3.1 (Invariant Metrics on Quotients of D). *Let M be a Riemann surface. If $\pi : D \to M$ is a covering map with π holomorphic, then there is an Hermitian metric H on M such that π is a local isometry (here D is equipped with the Poincaré metric). Moreover, the metric H, which is uniquely determined by the condition that π be a local isometry, has the property that* $\mathrm{Aut}\,(M)$ *acts on M as orientation-preserving isometries of H. And every orientation-preserving isometry of M with metric H is an element of* $\mathrm{Aut}\,(M)$.

Outline of the proof. Since M is the quotient of D by a subgroup Γ of $\mathrm{Aut}\,(D)$ and since $\mathrm{Aut}\,(D)$ acts as isometries of the Poincaré metric, the Poincaré metric pushes down to M, as in the earlier discussion. Here we use also the fact that π is a local holomorphic diffeomorphism, and in particular $d\pi$ is nonsingular. An automorphism f of M can be lifted to an automorphism f^* of D, also as before. And since f^* is isometric for the Poincaré metric on D, the original element f must be isometric for H. The final statement follows from the fact that H is Hermitian. □

We turn now to some specific examples.

Example 1 $M = D \setminus \{0\}$. An explicit universal covering space of M is given by

$$\begin{aligned} \pi : \{z \in \mathbb{C} : \mathrm{Re}\, z < 0\} &\to M \\ z &\mapsto \exp z. \end{aligned}$$

Calculation via the standard biholomorphic map from the half-plane $\{z \in \mathbb{C} : \mathrm{Re}\, z < 0\}$ to D shows that the Poincaré metric on D transferred to $\{z \in \mathbb{C} : \mathrm{Re}\, z < 0\}$ is $(1/|\mathrm{Re}\, z|)$ (the Euclidean metric). first, the usual biholomorphic map of $\{z \in \mathbb{C} : \mathrm{Re}\, z < 0\}$ to D is the composition of $z \mapsto -iz$ (left half-plane to upper half-plane) and $z \mapsto (z-i)/(z+i)$ (upper half-plane to D). So $F : z \mapsto (z+1)/(z-1)$. The derivative of this map is $-2/(z-1)^2$. For F to be an isometry, the Poincaré-induced length of $(1,0)$ at z must be

$$\begin{aligned} \frac{2|F'(z)|}{1 - |F(z)|^2} &= \frac{2}{-z - \overline{z}} \\ &= \frac{1}{-\mathrm{Re}\, z} \\ &= \frac{1}{|\mathrm{Re}\, z|}. \end{aligned}$$

For a similar direct calculation of the induced metric on $D \setminus \{0\}$, note that the length at $\exp z$ of the $d\pi$-image of the real Euclidean vector $(1,0)$ is $(\exp z)(1,0)$. Since the Poincaré length of $(1,0)$ at $z \in \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta < 0\}$ is $1/|\operatorname{Re} z|$, the induced-metric H-length of $(1,0)$ at $\exp z \in M$ must be $(1/|\exp z|) \cdot (1/|\operatorname{Re} z|)$. Expressed in terms of z itself,

$$\begin{aligned} H\text{-length} &= \frac{1}{|z| \cdot \big|\ln|z|\big|} \cdot (\text{Euclidean length}) \\ &= \frac{-1}{|z| \cdot \ln|z|} \cdot (\text{Euclidean length}). \end{aligned}$$

In polar coordinates $z = re^{i\theta}$:

$$H\text{-length at } z = \left[-\frac{1}{r \ln r}\right] \cdot (\text{Euclidean length}).$$

Readers may check directly for themselves that the H-metric has Gauss curvature $\equiv -1$ using the standard formula[5]

$$\text{Gauss curvature of } e^{2\sigma}(dx^2 + dy^2) = -\frac{\Delta\sigma}{e^{2\sigma}}.$$

Here $e^{2\sigma} = (1/r^2 \ln^2 r)$ so that $\sigma = -\ln r - \ln(|\ln r|)$. Of course, since Gaussian curvature is preserved by local isometry, the fact that the H-metric has Gauss curvature $\equiv -1$ follows from the fact that the Poincaré metric on Δ has Gauss curvature -1. This latter fact can be verified with the same formula but with $\sigma = \ln(1 - |z|^2) + \ln 2$ on D. This corresponds to the Poincaré metric $[4/(1 - |z|^2)^2](dx^2 + dy^2)$. $\square$

Example 2 Let $M = \{z \in \mathbb{C} : e^{-\alpha} < |z| < e^{\alpha}\}$, some $\alpha > 0$. Here the covering map requires a bit more effort to construct. The exponential $z \mapsto e^z$ maps $U_1 = \{z \in \mathbb{C} : -\alpha < \operatorname{Re} z < \alpha\}$ onto M, and this is a covering. On the other hand, $z \mapsto (\pi i/2\alpha)z$ maps U_1 biholomorphically onto $U_2 = \{z \in \mathbb{C} : -\pi i/2 < \operatorname{Im} z < \pi i/2\}$, and $z \mapsto \exp z$ takes U_2 biholomorphically to $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. The Poincaré-induced metric for this last region is

$$\text{Poincaré length} = \left(\frac{1}{\operatorname{Re} z}\right) \cdot (\text{Euclidean length}).$$

Tracing through the derivatives of these maps as in Example 1 gives the H-metric on M:

$$H\text{-length} = \left(\frac{\pi/[2\alpha]}{r \cos([\pi \ln r]/[2\alpha])}\right) \cdot \text{Euclidean length}$$

[5]See [Kobayashi/Nomizu 1963], Volume II, p. 184.

or, in length^2 notation,

$$H = \left(\frac{(\pi/[2\alpha])^2}{r^2 \cos^2([\pi \ln r]/[2\alpha])} \right) \cdot (dx^2 + dy^2).$$

It is instructive to write this metric in polar coordinates, where $dx^2+dy^2 = dr^2 + r^2 d\theta^2$. Thus

$$H = \left(\frac{(\pi/[2\alpha])^2}{r^2 \cos^2((\pi \ln r)/(2\alpha))} \right) dr^2 + \left(\frac{(\pi/[2\alpha])^2}{\cos^2((\pi \ln r)/(2\alpha))} \right) d\theta^2.$$

The homology generator of M which goes once around the point 0 counterclockwise has as representatives closed curves along which the total change of θ is $+2\pi$. For such a curve that is, say, piecewise C^1 and is written in the form $(r(t), \theta(t))$, $t \in [0,1]$, one has

$$\int_0^1 \frac{d\theta}{dt} dt = 2\pi.$$

Now the H-length of the curve is, with $\beta = \pi/(2\alpha)$,

$$\begin{aligned}
&\int_0^1 \sqrt{\left(\frac{\beta^2}{r^2 \cos^2(\beta \ln r)} \right) \left(\frac{dr}{dt} \right)^2 + \left(\frac{\beta^2}{\cos^2(\beta \ln r)} \right) \left(\frac{d\theta}{dt} \right)^2}\, dt \\
&\quad \geq \int_0^1 \frac{\beta}{\cos(\beta \ln r)} \cdot \left| \frac{d\theta}{dt} \right| dt \\
&\quad \geq \int_0^1 \beta \cdot \frac{d\theta}{dt}\, dt \\
&\quad = 2\pi\beta.
\end{aligned}$$

When $(r(t), \theta(t)) \equiv (1, 2\pi t)$, the inequalities become equalities and that is, up to parameterization, the only case when they do. Thus the "central circle" $r = 1$ is the unique minimum-length curve in its homology class. As such, by standard geometry, it must be a geodesic for the H-metric. Refer to Figure 2.1.

In particular, this central circle must be taken to itself by an automorphism, except of course its direction may be reversed, since the circle traversed clockwise is the other homology generator.

Now we are able to interpret the automorphisms of $\{z \in \mathbb{C} : e^{-\alpha} < |z| < e^{+\alpha}\}$, $\alpha > 0$, as determined in Example (4) of Section 1.4. The automorphisms, being isometries, come in two classes. The first is that in which the central circle and its orientation are preserved. These are rotations $z \mapsto \omega z$, $|\omega| = 1$, and these act as isometries since the expression for the H-metric has no θ-dependence of its coefficients.

The second class of isometries that preserve orientation and are hence automorphisms is that in which the central circle is mapped to itself with

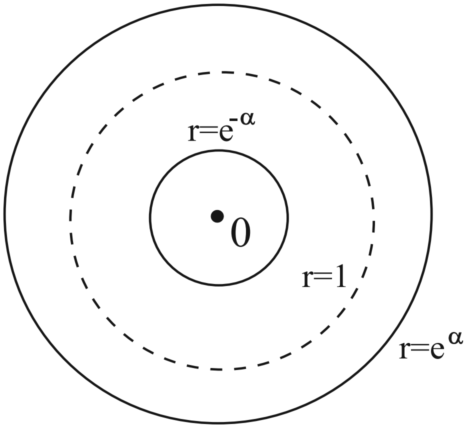

Fig. 2.1. The central geodesic.

opposite direction. For orientation to be preserved, the outward directions perpendicular to the circle must be mapped to inward directions. The maps have the form $(r, \theta) \mapsto (1/r, -\theta + \theta_0)$ or, equivalently, $z \mapsto \omega/z$, $|\omega| = 1$. That these are isometries arises from the θ-independence of the H-metric coefficients together with the observations that

$$\frac{\beta}{(1/r)\cos^2\left(\frac{\pi}{2\alpha} \cdot \ln \frac{1}{r}\right)} = r^2 \cdot \frac{\beta}{r\cos^2\left(\frac{\pi}{2\alpha} \cdot \ln r\right)}$$

while $|(1/z)'| = 1/|z|^2$. The reader is invited to trace through the logic to see that consequently $z \mapsto 1/z$ is length-preserving for the H-metric.

One first observes (with $\beta = \pi/[2\alpha]$ as before) that

$$\int_1^{e^\alpha} \frac{\beta}{r\cos(\beta \ln r)}\, dr = +\infty.$$

This follows from elementary estimates. Similarly,

$$\int_{e^{-\alpha}}^1 \frac{\beta}{r\cos(\beta \ln r)}\, dr = +\infty.$$

From this information, and from the H-metric in polar form, it follows that any curve $\gamma : [0, 1) \to \Omega = \{z \in \mathbb{C} : e^{-\alpha} < |z| < e^{+\alpha}\}$ with $|\gamma(0)| = 1$ and $\lim_{t \uparrow 1} |\gamma(t)| = e^\alpha$ has infinite length. Similarly, if $\lim_{t\uparrow 1} |\gamma(t)| = e^{-\alpha}$, then γ has infinite length. In an obvious sense, it is "infinite H-distance" to the boundary of Ω from one (and hence any) interior point of Ω. This property is (equivalent to) the domain being complete in the H-metric in the sense of Riemann-geometric completeness. This completeness property holds for all the metrics that arise in the situation of Proposition 2.3.1. This matter will be discussed in more detail and in greater generality later in the book. Meanwhile, one can check that the H-metric on $\{z \in \mathbb{C} : 0 < |z| < 1\}$ also gives infinite distance to the boundary from each interior point.

It is reassuring to be able to visualize the covering maps of Examples 1 and 2 geometrically, analogous to visualizing the covering $\mathbb{C} \to \mathbb{C} \setminus \{0\}$ as mapping a plane onto a cylinder. This visualization is in fact almost exactly analogous to the cylinder case.

In detail, the curve in polar coordinates with $\theta = 0$ and r varying from $e^{-\alpha}$ to $e^{+\alpha}$ is, up to parameterization, a geodesic. This is clear since it is a shortest curve joining any two of its points (cf. the argument for why it is infinite distance to the boundary). Alternatively, one can see that this curve is a component of the fixed-point set of the (orientation-reversing, nonholomorphic) isometry $(r, \theta) \mapsto (r, -\theta)$ and hence must be a geodesic. As in basic Riemann surface theory for function elements, we can use this geodesic as a "cut" and obtain the simply connected covering space of $\{z \in \mathbb{C} : e^{-\alpha} < |z| < e^{+\alpha}\}$ by connected "sheets" along the cut. Translated into the unit disc D, this process "develops" $\{z \in \mathbb{C} : e^{-\alpha} < |z| < e^{+\alpha}\}$ on D isometrically. A fundamental domain, analogous to the strip with parallel sides glued together to form a cylinder, is given by the region between two Poincaré-metric geodesics in D, each perpendicular to a third geodesic, with the points of intersection separated by Poincaré distance $2\pi\beta = \pi/\alpha$. This is of course the length of the "central circle" geodesic $\{z \in \mathbb{C} : |z| = 1\}$ of $\{z \in \mathbb{C} : e^{-\alpha} < |z| < e^{+\alpha}\}$ in the H-metric.

Without loss of generality, we can take the third geodesic in D to be the x-axis and the points of intersection to be $-a + i0$ and $a + i0$, where $a > 0$. Here a must satisfy

$$\int_{-a}^{a} \frac{2}{1 - t^2}\, dt = \frac{\pi^2}{\alpha}.$$

Therefore $a = (e^{\pi^2/[2\alpha]} - 1)/(e^{\pi^2/[2\alpha]} + 1)$. In this situation, the group Γ of covering transformations is generated by $T_{-a} \circ T_{-a}$, which takes $-a + 0i$ to $a + 0i$. [Note also that the map $T_{-a} \circ T_{-a}$ takes z to $(z+b)/(1+bz)$, where $b = 2a/(1+a^2)$.] Then D/Γ becomes the fundamental domain (the region between the two geodesics perpendicular to the x-axis) with its edges *glued together* via the identification map $T_{-a} \circ T_{-a}$. As noted, this is truly the analogue of the flat cylinder situation $\mathbb{C} \to \mathbb{C} \setminus \{0\}$. In this setup, the *generators* of the cylinder, the parallel lines that form the cylinder, are replaced by H-geodesics from the inner edge of the annulus to the outer edge, first those of constant θ value (lying in rays from the origin). And these H-geodesics are images of the Poincaré metric geodesics perpendicular to the x-axis. The boundary of our fundamental domain, consisting of two Poincaré metrics, has each of these Poincaré metric geodesic mapping onto the H-metric geodesic $re^{-i\pi}$ in $\{z \in \mathbb{C} : e^{-\alpha} < |z| < e^{+\alpha}\}$, where r varies from $e^{-\alpha}$ to $e^{+\alpha}$. Refer to Figure 2.2.

The situation of $D \to D \setminus \{0\}$ in Example 1 is conceptually similar but different in that there is no geodesic "central circle." As before, we can use the

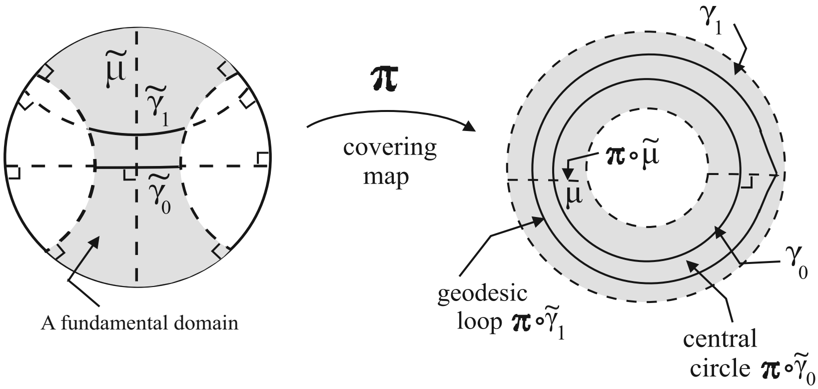

Fig. 2.2. The mapping $D \to A \equiv \{z \in \mathbb{C} : e^{-\alpha} < |z| < e^{\alpha}\}$.

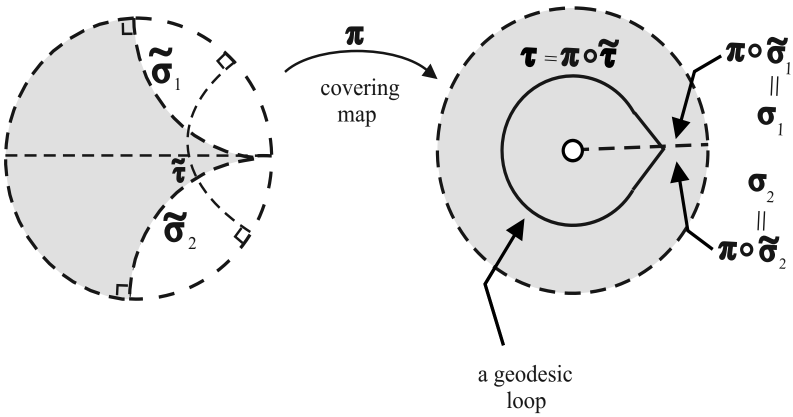

Fig. 2.3. Fundamental domain for $D \to D \setminus \{0\}$.

$\theta = 0$, $r \in (0, 1)$ curve (which is a geodesic for the same reasons as before) as a "branch cut." We can begin our development map unwrapping the universal cover of $D \setminus \{0\}$ onto D by sending the $\theta = \pi$ geodesic to the y-axis, with $r \to 1$ corresponding to $y \to 1$: to normalize the precise correspondence, we could, for instance, send $r = 1/2$, $\theta = -\pi$, to $0 + i0 \in D$. The fundamental domain becomes the region between the two geodesics σ_1, σ_2 as shown, where the distance between the two goes to 0 at the boundary point. See Figure 2.3.

The two geodesics σ_1, σ_2 in D are each a lift of the $\theta = 0$ geodesic in $D \setminus \{0\}$ with r varying. The fact that they approach each other in the limit corresponding to $r \to 0^+$ is a consequence of the fact that the H-length of a (Euclidean) circle of radius r around $0 + i0$ goes to 0 as $r \to 0^+$. The H-length of the circle is $2\pi/|\ln r|$.

The group of covering transformations $\mathit{\Gamma}$ on D needs to have as generator the mapping that moves σ_1 to σ_2 while fixing the boundary point $1 \in \partial D$ at

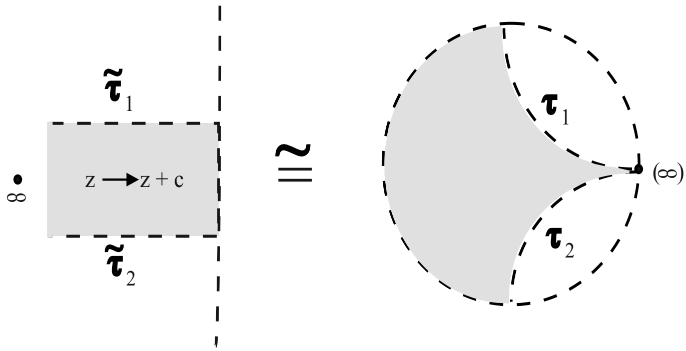

Fig. 2.4. Half-plane model vs. disc model for the covering of $D \setminus \{0\}$.

which σ_1 and σ_2 "meet." Thus Γ should be a cyclic subgroup of the classical horocycle transformations at 1.

The form of the transformations, which are supposed to be automorphisms of D, is determined by requiring that $z \mapsto \omega \cdot (z-a)/(1-\overline{a}z)$ should fix 1, i.e.,

$$1 = \omega \cdot \frac{-i-a}{1+\overline{a}i} \qquad \text{or} \qquad \omega = \frac{1+\overline{a}i}{-i-a}.$$

Since $|\omega| = 1$, it must follow that a is real. Furthermore, for each real a with $|a| < 1$, there is one and only one possible ω value. It is easy to check that there is exactly one such transformation taking σ_1 to σ_2 with 1 fixed, and this transformation "glues" the edges of the fundamental domain (between σ_1 and σ_2) to yield $D \setminus \{0\}$ with its H-metric. This whole situation is even easier to visualize if one looks at the upper half-plane version of the Poincaré metric: the geodesics σ_1, σ_2 become vertical half-lines emanating from points on the x-axis and meeting "at infinity," and the covering transformation group is generated by a translation $z \mapsto z + c$, c real. This picture really does look cylindrical. The reader is invited to explore this matter in more detail. Refer to Figure 2.4.

It is also intriguing to move the situation of Example 2 over to the upper half-plane model. Take the unwrapping of the central circle (which is the x-axis in the D-model) to the (geodesic) x-axis. Then the two perpendicular geodesics which bound the fundamental domain in our D-model become half-circles around the origin. And the group of covering transformations, again of course a cyclic group, is now generated by $z \mapsto cz$, $c > 0$, where c is the ratio of the radii of the two half-circles. Again the reader is invited to explore the details.

2.4 Covering Spaces and Automorphisms in General

While much of the material of the previous two sections was very specific, underlying the considerations there and illustrated by them were some general principles. In this section, we shall make these principles explicit. This is well worthwhile in spite of the tedium of the notation involved because these principles have wide applications.

Proposition 2.4.1. *Suppose that $\widetilde{M}$ and M are complex manifolds with $\widetilde{M}$ simply connected and further suppose that $\pi : \widetilde{M} \to M$ is a holomorphic mapping which is also a covering in the sense of topology so that, in particular, π is a local holomorphic diffeomorphism. Let Γ be the subgroup of* $\mathrm{Aut}\,(\widetilde{M})$ *consisting of the covering transformations of $\widetilde{M}$ over M. In this case these transformations are necessarily holomorphic. Then:*

(a) *For each $f \in \mathrm{Aut}\,(M)$, there is an element $\widehat{f}$ of $\mathrm{Aut}\,(\widetilde{M})$ such that $f \circ \pi \equiv \pi \circ \widehat{f}$. Any such $\widehat{f}$ maps each orbit of Γ bijectively onto another orbit of Γ. Moreover, given $p \in M$, $\widehat{p} \in \widetilde{M}$ with $\pi(\widehat{p}) = p$, and $\widehat{q} \in \widetilde{M}$ with $\pi(\widehat{q}) = f(p)$, there is exactly one such $\widehat{f}$ with $\widehat{f}(\widehat{p}) = \widehat{q}$.*

(b) *Conversely, if $F \in \mathrm{Aut}\,(\widetilde{M})$ and if F maps each orbit of Γ bijectively onto another orbit of Γ, then there is exactly one $f \in \mathrm{Aut}\,(M)$ such that $\pi \circ F \equiv f \circ \pi$.*

Proof. If $f : M \to M$ is a holomorphic mapping, whether an automorphism or not, then there is exactly one continuous mapping $\widehat{f} : \widetilde{M} \to \widetilde{M}$ with $f \circ \pi = \pi \circ \widehat{f}$ and $\widehat{f}(\widehat{p}) = \widehat{q}$. This is just the standard lifting argument for covering spaces $\widetilde{M} \to M$ with $\widetilde{M}$ simply connected. Since π is a holomorphic local diffeomorphism, it follows that $\widehat{f}$ must be holomorphic. To see that such an $\widehat{f}$ is an automorphism of $\widetilde{M}$, let $\widehat{g}$ be the lift to $\widetilde{M}$ of $f^{-1} : M \to M$ with $\widehat{g}(\widehat{q}) = \widehat{p}$, i.e., $\pi \circ \widehat{g} = f^{-1} \circ \pi$. Now, since π is a local diffeomorphism and since $g \circ f$ is the identity, it follows that $\widehat{g} \circ \widehat{f}$ is the identity in a neighborhood of $\widehat{p}$. By uniqueness of analytic continuation, $\widehat{g} \circ \widehat{f}$ is the identity on all of $\widetilde{M}$. Similarly, $(\widehat{f} \circ \widehat{g})(\widehat{q}) = \widehat{f}(\widehat{p}) = \widehat{q}$ and $\widehat{f} \circ \widehat{g}$ is also the identity on $\widetilde{M}$. So $\widehat{f} \in \mathrm{Aut}\,(\widetilde{M})$.

Now, for $x \in \widetilde{M}$ and $\gamma \in \Gamma$,

$$\pi \widehat{f}(\gamma(x)) = f\pi(\gamma(x)) = f(\pi(x)).$$

So $\widehat{f}(\gamma(x))$ is in the Γ-orbit of $\widehat{f}(x)$ since the orbits of Γ are exactly the sets of the form $\pi^{-1}(\{\text{points of } M\})$. Now, with $\widehat{g}$ as given, the same holds for $\widehat{g}$ since $\widehat{g}$ is a lift of f^{-1}: the map $\widehat{g}$ sends Γ-orbits to Γ-orbits. Since $\widehat{f}$ and $\widehat{g}$ are inverses, the maps by $\widehat{f}$ of Γ-orbits to Γ-orbits must each be bijective.

For the second statement, note that under the hypothesis that $F : \widetilde{M} \to \widetilde{M}$ maps Γ-orbits (that is, sets of the form $\pi^{-1}(\{p\})$ to other sets of the same form), set-theoretic considerations show that F induces a map on M. first, set $f(x) = \pi \circ F(\widehat{x})$ for any $\widehat{x} \in \pi^{-1}(\{x\})$; this f is well defined by hypothesis. The full hypothesis on F, including the bijectivity on inverse image sets, means that F^{-1} also pushes down to a well-defined map $g : M \to M$. It is easy to check that $f \circ g$ and $g \circ f$ are the identity on M, so that $f \in \mathrm{Aut}\,(M)$. □

This "general nonsense" argument becomes clearer if one thinks of it in terms of the orbits of the covering transformation group Γ acting on $\widetilde{M}$. The hypothesis of the second statement says then that F maps each Γ-orbit bijectively onto another Γ-orbit, and so does F^{-1}. The latter condition is needed, since the orbits are in general infinite sets. As an instance, the map $z \mapsto 2z$ maps orbits of Γ for $\exp : \mathbb{C} \to \mathbb{C}\setminus\{0\}$ into themselves: if $z_1 - z_2 = 2\pi i k$ then $2z_1 - 2z_2 \in 2\pi i\mathbb{Z}$. But the induced map on $\mathbb{C} \setminus \{0\}$ is $z \mapsto z^2$, which has no inverse (see Section 2.2).

Now the set of all $F \in \mathrm{Aut}\,(\widetilde{M})$ such that both F and F^{-1} map orbits to orbits is a subgroup of $\mathrm{Aut}\,(\widetilde{M})$. Moreover, this subgroup is closed. One can see this from the definition of covering space and the group Γ. Suppose that the sequence $\{F_j\}$ converges uniformly on compact sets to some F_0. For fixed $\gamma_0 \in \Gamma$ and $p_0 \in \widetilde{M}$, the images $F_j(\gamma_0(p_0))$ by hypothesis belong to the same orbit of $F_j(p_0)$ so that $F_j(\gamma_0(p_0)) = \gamma_j(F_j(p_0))$ for some unique $\gamma_j \in \Gamma$. Now choose a neighborhood of $F_0(\gamma_0(p_0))$, say U, (sufficiently small) so that the conditions $\sigma_1(U) \cap \sigma_2(U) \neq \emptyset$ and $\sigma_1, \sigma_2 \in \Gamma$ imply $\sigma_1 = \sigma_2$. This is possible by the definition of covering space. Then, for j_0 so large that $j \geq j_0$ implies that $F_j(\gamma_0(p_0)) \in U$, it must be that γ_j is independent of j. For clearly $F_0(p_0) \in \gamma_{j_1}^{-1}(U) \cap \gamma_{j_2}^{-1}(U)$ for any $j_1, j_2 > j_0$. Thus $F_0(\gamma_0(p_0)) = \gamma_\infty(F_0(p_0))$, where γ_∞ is the eventually constant value of γ_j. In particular, F_0 maps orbits to orbits. (A similar argument applies to F^{-1} to show that F^{-1} also maps orbits to orbits.)

Proposition 2.4.2. *With $\pi : \widetilde{M} \to M$ as in Proposition 2.4.1, let G be the subset of* $\mathrm{Aut}\,(\widetilde{M})$ *consisting of those $F \in$* $\mathrm{Aut}\,(\widetilde{M})$ *such that F maps each orbit of Γ bijectively to another orbit of Γ. The covering transformation group Γ of $\widetilde{M}$ over M is a closed, normal subgroup of G; and* $\mathrm{Aut}\,(M)$ *is homeomorphically isomorphic to G/Γ.*

Proof. Almost everything follows from arguments already presented. We check only the normality of Γ in G. Suppose that $F \in G$ and $\gamma \in \Gamma$. We need to see that $F \circ \gamma \circ F^{-1}$ is a covering transformation. Obviously $F \circ \gamma \circ F^{-1} \in \mathrm{Aut}\,(\widetilde{M})$, so we need only check that $\pi \circ (F \circ \gamma \circ F^{-1}) = \pi$. Now, for $p \in \widetilde{M}$, we see that $\gamma \circ F^{-1}(p)$ is an element of the set of Γ-orbits of $F^{-1}(p)$, and hence $F(\gamma(F^{-1}(p))$ is an element of the Γ-orbit of $F(F^{-1}(p))$. This is so because F applied to a point of an orbit of $q \in \widetilde{M}$ is equal to the orbit of the point $F(q)$ by hypothesis. Note that

$$\pi(\Gamma\text{-orbit of } F(F^{-1}(p))) = \pi(\Gamma\text{-orbit of } p) = \{\pi(p)\}.$$

The remainder of the proof of the proposition, which follows the patterns already established, is left to the reader. $\square$

Corollary 2.4.3. *If* $\mathrm{Aut}\,(\widetilde{M})$ *is a Lie group, then* $\mathrm{Aut}\,(M)$ *is a Lie group.*

Proof. Let G be the same as in Proposition 2.4.2. Since a closed subgroup of a Lie group is a Lie group, G is a Lie group. A quotient of a Lie group by a closed, normal subgroup is again a Lie group. Hence G/Γ is a Lie group. $\square$

Proposition 2.4.4. *If* $\mathrm{Aut}\,(\widetilde{M})$ *has compact isotropy subgroups, then so does* $\mathrm{Aut}\,(M)$.

Proof. Given $p \in M$, choose $\widehat{p} \in \widetilde{M}$ such that $\pi(\widehat{p}) = p$. By Proposition 2.4.1(a), we see that, for each $f \in I_p$, there is exactly one $\widehat{f} \in I_{\widehat{p}}$ with $\pi \circ \widehat{f} = f \circ \pi$. And $\widehat{f}$ maps each orbit of Γ bijectively onto another orbit. Also, by 2.4.1(b), the set of all $F \in I_{\widehat{p}}$ that map each Γ-orbit bijectively to a Γ-orbit is exactly the set of such $\widehat{f}$. Moreover, this set of F is a closed subgroup, say $G_{\widehat{p}}$ of $I_{\widehat{p}}$, by the proof of Proposition 2.4.2. $G_{\widehat{p}}$ is compact since $I_{\widehat{p}}$ is, by hypothesis. And with G as in Proposition 2.4.2, $G_{\widehat{p}} \subset G$ maps, under the continuous map $G \to G/\Gamma$, onto a subgroup corresponding to I_p under the isomorphism $\mathrm{Aut}\,(M) \cong G/\Gamma$. Indeed $G_{\widehat{p}}$ maps isomorphically onto the subgroup of G/Γ corresponding to I_p. In particular, since $G \to G/\Gamma$ is continuous and $G_{\widehat{p}}$ is compact, it follows that I_p is compact. $\square$

In the case when M has an $\mathrm{Aut}\,(M)$-invariant metric, all the previous considerations become considerably simplified.

Theorem 2.4.5. *Let M be a compact, complex manifold. Suppose that $d : M\times M \to \mathbb{R}$ is a metric space structure on M, with the property that the metric space topology for d is the same as the manifold topology. If d is* $\mathrm{Aut}\,(M)$*-invariant in the sense that $d(f(p), f(q)) = d(p, q)$ for all $f \in$* $\mathrm{Aut}\,(M)$ *and all $p, q \in M$, then* $\mathrm{Aut}\,(M)$ *is a compact Lie group.*

Proof. If $\{f_j\}$ is a sequence in $\mathrm{Aut}\,(M)$ then, by the invariance of d, the sequence is d-equicontinuous and indeed d-Lipschitz continuous with constant 1. By the Arzela–Ascoli theorem and the compactness of M, there is a subsequence $\{f_{j_k}\}$ that converges uniformly on M to a continuous d-isometric map $f_0 : M \to M$. Passing again to a subsequence if necessary, we can suppose that the sequence $\{f_{j_k}^{-1}\}$, which is also d-equicontinuous, converges uniformly on M to a limit $g_0 : M \to M$, also d-isometric. It is now elementary to see that $f_0 \circ g_0$ and $g_0 \circ f_0$ are both the identity map of M to M. Also, d-uniform convergence implies uniform-on-compact-subsets convergence in local coordinates, since the d-topology is equivalent to the manifold topology. So f_0 is holomorphic and invertible, and $f_0 \in \mathrm{Aut}\,(M)$.

Since $\mathrm{Aut}\,(M)$ is compact, it is certainly locally compact. By the Bochner–Montgomery theorem (Theorem 1.3.11), $\mathrm{Aut}\,(M)$ is a Lie group. $\square$

We turn now to some general circumstances under which a complex manifold M has an $\mathrm{Aut}\,(M)$-invariant metric.

Proposition 2.4.6. *If $\pi : \widetilde{M} \to M$ is a holomorphic covering space with $\widetilde{M}$ simply connected and if $\widetilde{H}$ is an Hermitian metric on $\widetilde{M}$ that is invariant* *under the subgroup Γ of* $\mathrm{Aut}\,(\widetilde{M})$ *consisting of the covering transformations of* $\pi : \widetilde{M} \to M$*, then there is a Hermitian metric H on M such that π is a local isometry, and H is uniquely determined by this condition. If, in addition, $\widetilde{H}$ is* $\mathrm{Aut}\,(\widetilde{M})$*-invariant, then H is* $\mathrm{Aut}\,(M)$*-invariant.*

Proof. The proof of the existence and uniqueness of H follows a by-now-familiar pattern. To determine H at $p \in M$, choose $\widehat{p}$ with $\pi(\widehat{p}) = p$. Then define H at p by requiring $d\pi|_{\widehat{p}}$ to be isometric from $\widetilde{H}$ at $\widehat{p}$ to H at p. Since $d\pi|_{\widehat{p}}$ is a vector space isomorphism, this gives one and only one H at p. The metric H is well defined since any other element of $\pi^{-1}(p)$ has the form $\gamma(\widehat{p})$ for some $\gamma \in \Gamma$ and γ is an isometry of $\widetilde{H}$.

For the second statement, it suffices to apply Proposition 2.4.1. To wit, if $f \in \mathrm{Aut}\,(M)$, then there is an $\widehat{f} \in \mathrm{Aut}\,(\widetilde{M})$ with $f \circ \pi = \widehat{f} \circ \pi$. Since $\widehat{f}$ is $\widetilde{H}$-isometric and since π is locally $\widetilde{H}$-to-H isometric, we see that f must be H-isometric. □

Proposition 2.4.6 was more or less obvious after our experience with the examples of Sections 2.1 and 2.2. What is somewhat less obvious is that the completeness of $\widetilde{H}$ and of H happen together. An even more general result holds:

Proposition 2.4.7. *Let $\widetilde{M}$ and M be Riemannian manifolds equipped with metrics $\widetilde{H}$ and H, respectively. Suppose that $\pi : \widetilde{M} \to M$ is a covering, and that π is locally isometric. Then the $\widetilde{M}$ metric $\widetilde{H}$ is complete if and only if the M metric is complete.*

Remark. This is a standard result in Riemannian geometry.

Sketch of the proof. Use the fact that completeness in the Cauchy sense is equivalent for a Riemannian manifold to infinite extendibility of geodesics (see [Kobayashi/Nomizu 1963]). Geodesics in M can be lifted to geodesics in $\widetilde{M}$ because continuous curves can always be lifted. It is also the case that the image under π of a geodesic in $\widetilde{M}$ is a geodesic in M. Thus infinite extendibility in $\widetilde{M}$ happens precisely when it does so in M. □

The most obvious application of Proposition 2.4.7 is to quotients of D with its Poincaré metric.

Corollary 2.4.8. *If $\pi : D \to M$ is a holomorphic covering space of a Riemann surface M, then the H-metric (in our previous terminology) on M is complete.*

The Poincaré metric on the disc D is of course complete. To check the corollary by hand, from the geodesic extendibility viewpoint, it is enough to deal with geodesics through 0. These are, up to parameter, straight lines: the form of the metric in polar coordinates shows immediately that these are minimal-length connections, hence geodesics. With Poincaré metric arc length parameter, they have the form in polar coordinates given[6] by

$$\sigma_{\theta_0}(t) = \left(\frac{e^t - 1}{e^t + 1}, \theta_0\right), \quad \theta_0 \in [0, 2\pi).$$

In particular, the geodesics σ_{θ_0} are defined on $(-\infty, +\infty)$. So D is complete. This completeness is also clear from the compactness of closed metric balls (the equivalence of all these different notions of completeness is the famous Hopf–Rinow theorem, treated in all Riemannian geometry texts, e.g., [Petersen 2006], [Kobayashi/Nomizu 1963], [Helgason 1962]). Explicitly

$$\{z \in \mathbb{C} : \mathrm{dis}_{\widetilde{H}}(0, z) \leq r\} = \{z \in \mathbb{C} : |z| \leq (e^r - 1)/(e^r + 1)\},$$

which is compact in D. So the (quotient) H-metrics on Riemann surfaces covered by D are necessarily complete.

2.5 Compact Quotients of *D* and Their Automorphisms

In this section, we shall investigate the automorphism groups of compact Riemann surfaces. We have already discussed the situation for the Riemann sphere and for the torus quotients of $\mathbb{C}$. In actuality, all other compact Riemann surfaces are quotients of D or, in other words, their universal covering space is biholomorphic to D. We almost have this result already, but not quite. The "not quite" arises from the fact that we need to show that a torus can only arise as a quotient of $\mathbb{C}$; it cannot have D as a (holomorphic) universal cover. Before proving this statement, we quickly remind the reader of some basic results about the topology of compact surfaces (see, e.g., [Massey 1967]).

According to nineteenth century work of Jordan and Möbius, every compact orientable surface M (i.e., topological 2-manifold) is homeomorphic either to the 2-sphere S^2 or or to the 2-sphere with g handles attached (here g is a finite, positive integer). The number g is uniquely determined; g is called the *genus* of the surface.[7] The homology groups of the surface M are:

- $H_0(M, \mathbb{Z}) \cong \mathbb{Z}$;
- $H_1(M, \mathbb{Z}) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ ($2g$ times);

---

[6] To check for arc length parameterization, calculate

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{e^t - 1}{e^t + 1}\right) = \frac{2e^t}{(1 + e^t)^2};$$

hence, at the point $\big((e^t - 1)/(e^t + 1), \theta_0\big)$, the length of the vector $\sigma'(t)$ is

$$\left(\frac{2e^t}{(1 + e^t)^2}\right) \Big/ \left(\frac{2}{1 - \left(\frac{e^t - 1}{e^t + 1}\right)^2}\right) = 1.$$

[7] By convention, the sphere S^2, having no handles, is considered to have genus 0.

- $H_2(M, \mathbb{Z}) \cong \mathbb{Z}$;
- $H_k(M, \mathbb{Z}) = 0$ if $k > 2$.

In particular, the Euler characteristic of M is $2 - 2g$.[8] No two surfaces of different genus are homeomorphic or even of the same homotopy type. This assertion follows immediately from the homology results, e.g., because the first homology group is the abelianization of the first homotopy group (i.e., the fundamental group) and that group is clearly a homotopy type invariant. Or, one can use directly the homotopy type invariance of homology. If $g = 0$ then the fundamental group of a surface M of genus g is the one-element group $\{0\}$. If $g \geq 1$, then the fundamental group has $2g$ generators $a_1, b_1, \ldots, a_g, b_g$ subject to the one relation

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1.$$

The group $\pi_1(M)$ is noncommutative if $g \geq 2$.

We now turn to the result which shows the relationship between the topology of the surface and the complex structure of the universal cover.

Theorem 2.5.1 (Uniformization of Compact Riemann Surfaces: Poincaré and Koebe). *Let M be a compact Riemann surface of genus g and $\pi : \widetilde{M} \to M$ its (holomorphic) universal cover. Then:*

(1) *If $g = 0$, then $\widetilde{M} = M$ and M is conformally equivalent to the Riemann sphere $\mathbb{C} \cup \{\infty\}$;*
(2) *If $g = 1$, then $\widetilde{M}$ is conformally equivalent to $\mathbb{C}$ and M is obtained from $\mathbb{C}$ by quotienting out by a group of the form $\Gamma = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$, with ω_1 and ω_2 being $\mathbb{R}$-linearly independent complex numbers.*
(3) *If $g \geq 2$, then $\widetilde{M}$ is conformally equivalent to D and M is D/Γ. Here Γ is a subgroup of* $\mathrm{Aut}\,(D)$.

Proof. We have already covered all the issues involved with this argument save one: We need to show that a surface of genus 1 cannot arise as a quotient of D. This can be established by working with the geometry of D and its holomorphic quotients in terms of the group actions of covering transformations, but it is much quicker to use a result of classical differential geometry.

The torus, the topological surface of genus 1, has Euler characteristic equal to 0. The Gauss–Bonnet theorem thus yields that $\int K\, dA = 0$ for any Riemannian metric on the torus, where K is the Gauss curvature and dA denotes integration with respect to the Riemannian metric area element. Now the Gauss curvature of the Poincaré metric is $\equiv -1$, and hence so is the Gauss curvature of the H-metric on any covering space quotient of D. Thus no topological torus admits such an H-metric. $\square$

[8] Note that when $g = 0$, or M is the sphere, then $H_1(M, \mathbb{Z}) = 0$, so the formulae still apply.

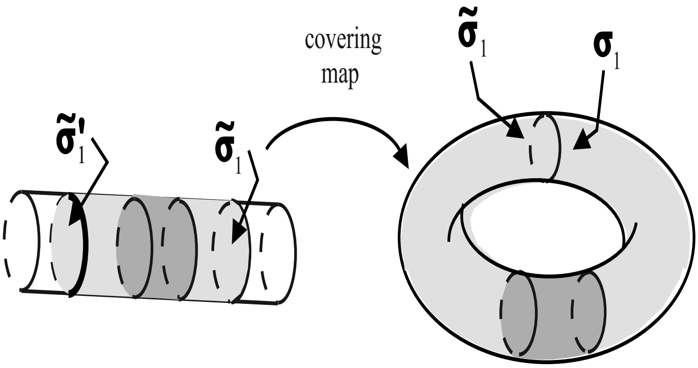

Fig. 2.5. The covering of the torus.

The sudden incursion of a result from pure Riemannian geometry may at first be startling. So we outline how the impossibility of covering a torus holomorphically by the unit disc D can be proved using only direct ideas about the quotienting possibilities that could arise. Refer to Figure 2.5.

Specifically, if M is a Riemann surface that is topologically $S^1 \times S^1$, then associated to this situation we could find a surface $\widetilde{M}$ covering $S^1 \times S^1$ where $\widetilde{M}$ is topologically $\mathbb{R} \times S^1$. This is done simply by "unwinding" the first S^1 factor and leaving the second factor alone. Now if M were holomorphically covered by D, then $\widetilde{M}$ could also be taken to be holomorphically covered by D, and the covering $\widetilde{M} \to M$ to be holomorphic as well. Both coverings would be isometric for the "H-metrics" as usual. Now choose a sequence $\{p_j\}$ diverging to infinity in one end of the topological cylinder $\widetilde{M}$ and $\{q_j\}$ diverging to infinity in the other end. Choose a closed curve C in $\widetilde{M}$ going around the cylinder in the obvious sense of those words so that $\widetilde{M} \setminus C$ has two components, both unbounded. Let, for each j, γ_j be an H-metric geodesic from p_j to q_j with minimal length. Then γ_j intersects C at least once. Let ξ_j be one such point of intersection. Now parameterize γ_j by arc length with $\gamma_j(0) = \xi_j$. By compactness of C, there is an arc-length parameter geodesic $\gamma_0 : (-\infty, +\infty) \to \widetilde{M}$ with $\gamma_0(0)$ a point of C which is minimizing between any two of its points. It is in a sense a minimal connection between the two ends of $\widetilde{M}$. [Note: To get γ_0, pass to a subsequence γ_{j_k} of the γ_js with ξ_j converging and with $\gamma_j'(0)$ converging. Then let γ_0 be the geodesic with $\gamma_0(0) = \lim \gamma_j(0)$ and $\gamma_0'(0) = \lim \gamma_j'(0)$.] Then $\widetilde{M} \setminus \{\gamma_0(t) : t \in (-\infty, +\infty)\}$ is isometric to an open subset of D (in the Poincaré metric) bounded by two nonintersecting geodesics, say σ and τ: $\widetilde{M}$ is obtained by gluing the two geodesics together in the same way that a circular cylinder is obtained by gluing together two parallel lines that bound a strip in the plane. We can parameterize σ and τ so that $\sigma(t)$ is glued to $\tau(t)$ to give $\gamma_0(t)$. See Figure 2.6.

Two such nonintersecting (Poincaré metric) geodesics in D can be related in only one of two ways: the geodesics diverge from each other at both ends,

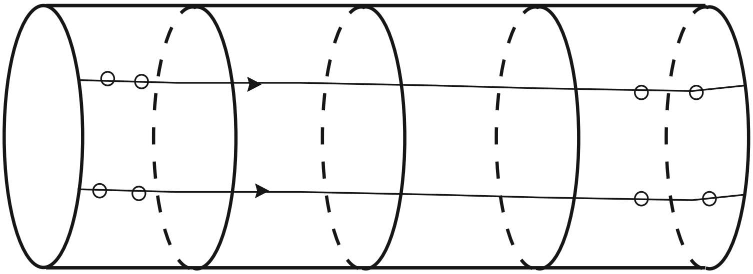

Fig. 2.6. Behavior of geodesics on $S^1 \times \mathbb{R}$.

or they converge at one end but diverge at the other. These cases correspond to the Euclidean circles perpendicular to the boundary, which are the Poincaré metric geodesics on D having distinct boundary endpoints or having one boundary endpoint in common. [Two boundary endpoints in common is not possible: the geodesics then coincide.] In the convergence-at-one-end case, one gets an immediate contradiction of our situation: in this case, one obtains arbitrarily short curves from one point far out on the geodesic (near the common boundary point) to the corresponding point on the "other side" of $\widetilde{M}$ cut open. This corresponds on M to arbitrarily short curves representing the free homotopy class of "once around" the first factor S^1. Such an eventuality is impossible: such a curve cannot be shorter than the injectivity radius of M.

The case of divergence of the pair of geodesics at both ends is also impossible, but the reason is slightly more subtle. Note that, by a simple compactness argument, no covering spaces needed, there is a number L such that, for each $p \in M$, there is a curve starting and ending at p with length $\leq L$ which lies in the free homotopy class of "once around the first factor" of $S^1 \times S^1$. Now choose p very far out on the geodesic γ_0 in the direction of divergence of the two copies σ, τ of γ_0 bounding the region in D that glues together at the edge to give $\widetilde{M}$. Choose a closed curve of length $\leq L$ at the projection of p into M in the free-homotopy class of "once around" the first S^1 factor. The lift of this curve to $\widetilde{M}$ that starts at a point $\sigma(t)$ projecting to p must end at $\tau(t)$ (or vice versa depending on direction). But this lift has length $\leq L$. This is impossible if p is sufficiently far out on γ since σ and τ diverge from each other. [Note: One could use this argument twice to show convergence of σ and τ at both ends and detour around the argument we have used for the impossibility of the convergence-at-one-end case.] See Figure 2.7.

These arguments are intricate to express in words. But if the reader will consider suitable pictorial representations, then the ideas will become clear.

The geometric analysis of possible groups Γ of holomorphic covering transformations acting on D can be pushed much further. It can be shown that, for such a Γ, every abelian subgroup is cyclic. This recovers the part of Theorem 2.5.1 ruling out the covering by D of a topological torus. For if M were a torus and $\pi : D \to M$ a holomorphic covering, then the fundamental group

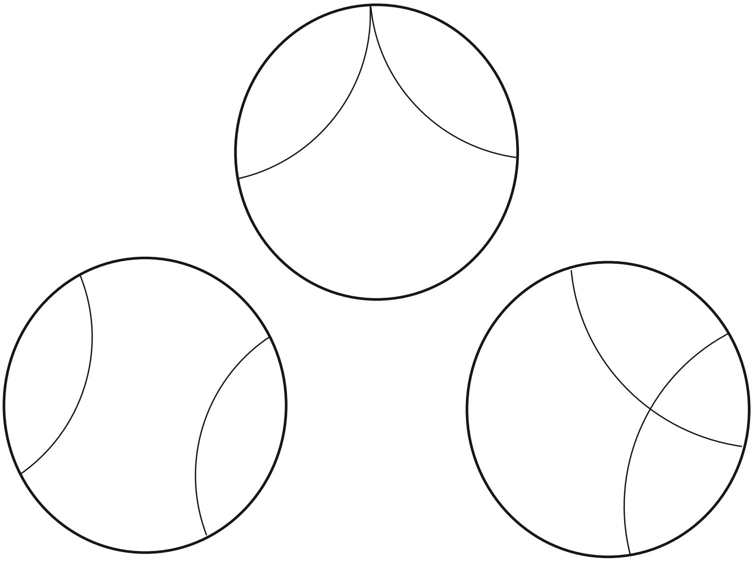

Fig. 2.7. The behavior of geodesics in D.

$\pi_1(M)$ would be isomorphic to the group Γ of covering transformations. But π_1 of a torus is $\mathbb{Z} \oplus \mathbb{Z}$, abelian but not cyclic. The cyclic nature of abelian subgroups actually holds in much more general situations: it applies to subgroups of the fundamental group of a compact Riemannian manifold of any dimension having negative Riemannian sectional curvature (see Preissmann's theorem [Preissmann 1943]), or, for a recent treatment, see [Petersen 2006].

It is not difficult to see by direct construction using hyperbolic geometry that, for each $g = 2, 3, 4, \ldots$, there is a (compact) covering-space quotient of D with genus g. This amounts just to realizing in hyperbolic geometry the representation of the topological surface of genus g via a polygon of $4g$ sides with suitable identifications. However, this is far from showing that *every* Riemann surface of genus $g \geq 2$ is holomorphically a covering space quotient of D. There are infinitely many complex structures on a compact surface of genus $g \geq 2$ (or indeed $g \geq 1$). The direct geometric construction of some of them does not *a priori* guarantee that all are obtained by that means. To see that all are uniformized by D requires the uniformization theorem (Theorem 2.5.1), or at least some method beyond pure hyperbolic geometry.[9]

We turn now to the subject of the automorphism groups $\mathrm{Aut}\,(M)$ for compact Riemann surfaces M of genus ≥ 2.

Proposition 2.5.2. *If M is a compact Riemann surface of genus $g \geq 2$, then* $\mathrm{Aut}\,(M)$ *is a finite group.*

[9]Proofs that use differential equations without appeal to uniformization are also known—see [Berger 1971], [Kazdan/Warner 1974].

Remark. A compact Riemann surface can certainly have an infinite automorphism group; even a compact surface with an automorphism-invariant metric can have such a group. For instance, the quotient T of $\mathbb{C}$ by the integer lattice $\{m+ni : m, n \in \mathbb{Z}\}$, which is topologically a torus, has a positive-dimensional automorphism group induced by the translations $z \mapsto z + \alpha$, $\alpha \in \mathbb{C}$. Thus the condition that the genus be at least 2 in the proposition is essential.

Proof of the proposition. By Theorem 2.4.5 and the existence of the invariant H-metric on M (Proposition 2.4.6 with $\widetilde{M} = D$), $\mathrm{Aut}\,(M)$ is a compact Lie group. Thus $\mathrm{Aut}\,(M)$ is a finite group if and only if the identity component of $\mathrm{Aut}\,(M)$ consists of the identity element alone (i.e., the group must be topologically discrete). By standard Lie group theory, this happens if and only if M contains no nontrivial one-parameter subgroup. For a proof by contradiction, suppose that $\{\varphi_t\}$ is a nontrivial one-parameter subgroup of $\mathrm{Aut}\,(M)$. Then $X(p) = d\varphi_t(p)/dt|_{t=0}$ for all $p \in M$ is a C^∞ vector field on M. [It is a part of the Bochner–Montgomery theorem (Theorem 1.3.11) that the action of the Lie group $\mathrm{Aut}\,(M)$ on M is smooth, so that X is indeed defined and C^∞.] The vector field X on M lifts to a C^∞ vector field $\widehat{X}$ on D by setting $\widehat{X}(w)$ equal to the unique vector $\mathbf{v}$ at w such that $d\pi|_w(\mathbf{v}) = X(\pi(w))$, where $\pi : D \to M$ is the universal covering of M. The vector field $\widehat{X}$ generates a one-parameter group $\{\psi_t\}$ of diffeomorphisms of D; indeed, one obtains (complete) integral curves for $\widehat{X}$ by lifting the integral curves of X. Clearly the elements of the group $\{\psi_t\}$ are isometries of D. In particular, the vector field $\widehat{X}$ can have at most one zero. For if $\widehat{X}(w_0) = 0$, then $\{\psi_t\}$ consists of a one-parameter group of Poincaré rotations around the fixed point w_0, and such a one-parameter group has generator that vanishes only at w_0—as one sees by writing it out explicitly (setting $w_0 = 0$ without loss of generality). On the other hand, if $\widehat{X}(w_0) = 0$, then $\widehat{X}(\gamma(w_0)) = 0$ for all $\gamma \in \Gamma$. Since Γ is infinite by the compactness of M, it must be that $\widehat{X}$, having at most one zero, has in fact no zeros at all. Thus X itself has no zeros. But a compact manifold with a nowhere vanishing vector field has Euler characteristic 0 (by the Poincaré–Hopf theorem, (cf. [Munkres 1966])), while M has Euler characteristic $2 - 2g < 0$. That is a contradiction. □

In the next section, we shall obtain an explicit estimate for the number of elements in the finite group $\mathrm{Aut}\,(M)$. But some work is required.

2.6 The Automorphism Group of a Riemann Surface of Genus at Least 2

The proof in Section 2.5 that the automorphism group $\mathrm{Aut}\,(M)$ of a compact Riemann surface M of genus $g \geq 2$ is necessarily a finite group involved very general considerations: the group $\mathrm{Aut}\,(M)$ was compact because the universal cover of M was D and hence M inherited an $\mathrm{Aut}\,(M)$-invariant metric from the Aut (D)-invariant Poincaré metric of D. The identity component of Aut (M) was the identity map alone because otherwise M would have had a zero-free vector field, contradicting the fact that its Euler characteristic $2-2g$ was nonzero. The specific structure of M played no role in these considerations, and the result was correspondingly general: finiteness of Aut (M) with no explicit estimate of the number of elements in Aut (M). It is quite surprising in this context that an explicit estimate is possible in the same generality.

Theorem 2.6.1 (Hurwitz). *If M is a compact Riemann surface of genus $g \geq 2$, then the order of the finite group* Aut (M) *does not exceed* $84(g-1)$.

This theorem seems mysterious at first sight. And the apparent mystery is deepened by the fact that the estimate is sharp. In fact there are Riemann surfaces with genus $g > 2$ for which the order of Aut (M) is precisely equal to $84(g-1)$ (see, for instance, [Farkas/Kra 1992], p. 325, VII.3.10). The exact origin of the specific number 84 will become apparent from the proof that follows.

The proof of the Hurwitz theorem that we shall present is closely related to Hurwitz's original argument. The subject has tended meanwhile to become burdened with abstract Riemann surface theory. In our treatment we shall, like Hurwitz himself, stick to the basics. However, we shall use our metric viewpoint rather than Hurwitz's analogous pure function theory. The theorem will then turn out to be completely geometric in nature.

Before beginning the proof itself, we need to develop some general ideas about quotients by group actions. Specifically, we want to consider the orbit space of Aut (M) on M. An orbit of Aut (M) is as usual, by definition, for a given point p, the set $\{\gamma(p) : \gamma \in \text{Aut}\,(M)\}$. The manifold M is a disjoint union of orbits, because Aut (M) is a group, and "being in the same orbit" is an equivalence relation, with each orbit an equivalence class.

Let $\mathcal{M}$ be the collection of orbits. Then $\mathcal{M}$ has a natural topology: the quotient topology. In other words, a set of orbits is said to be open if and only if the union of the orbits as sets is an open subset of M. For an arbitrary group action on a manifold, the topological orbit space can be a mess. But in our case, where Aut (M) is finite, $\mathcal{M}$ is a decent topological space; in particular, it is necessarily a Hausdorff space, as is easily checked. This is a special property of orbit spaces for finite groups acting on manifolds (or even just on Hausdorff topological spaces).[10]

Lemma 2.6.2. *If M is a compact, orientable Riemannian 2-manifold and if Γ is a finite group of orientation-preserving isometries of M, then the orbit space M/Γ is an orientable 2-manifold.*

[10] However, for a general finite group action on a manifold, the quotient need not be a manifold. For example, the orbit space of the action $\{x \mapsto x, x \mapsto -x\}$ on $\mathbb{R}$ is homeomorphic to $\{x \in \mathbb{R} : x \geq 0\}$. This is *not* a manifold. This difficulty does not arise in our situation.

Proof. If two points $p, q \in M$ lie in the same Γ-orbit, then $I_p = \{\gamma \in \Gamma : \gamma(p) = p\}$ and $I_q = \{\gamma \in \Gamma : \gamma(q) = q\}$ are conjugate subgroups of Γ. first, if $\sigma(p) = q$, $\sigma \in \Gamma$, then $\sigma \circ \gamma \circ \sigma^{-1} \in I_q$ if and only if $\gamma \in I_p$. With this idea in mind, we divide the orbits of Γ (that is to say, the points of M/Γ) into two classes:

(a) nonsingular orbits $\mathcal{O}$: For these, $I_p = \{\mathrm{id}_M\}$ if $p \in \mathcal{O}$;
(b) singular orbits $\mathcal{O}$: For these, $I_p \neq \{\mathrm{id}_M\}$ if $p \in \mathcal{O}$.

Similarly, we call points of M itself singular or nonsingular according to whether they belong to singular or nonsingular orbits. These ideas are all well defined by our conjugacy observation made at first.

The next step in the proof of Lemma 2.6.2 is the following subsidiary result.

Sublemma 2.6.3. *The singular orbits of M/Γ and the set of singular points of M are both finite sets.*

Proof of Sublemma 2.6.3. By standard results from Riemannian geometry, there is an $\epsilon > 0$ such that, for any two points $p, q \in M$ with $\mathrm{dist}_M(p, q) < \epsilon$, there is a unique geodesic from p to q of length $\mathrm{dist}_M(p, q)$. Now suppose that $p_1, p_2, \ldots$ were an infinite sequence of distinct singular points of M. Since Γ is finite, we can suppose, by passing to a subsequence, that there is a fixed nonidentity element $\gamma_0 \in \Gamma$ such that $\gamma_0 \in I_{p_j}$ for all j. By the compactness of M, there are two points p_{j_1} and p_{j_2} with $\mathrm{dist}_M(p_{j_1}, p_{j_2}) < \epsilon$. Clearly γ fixes the (short) geodesic $c(t)$ from p_{j_1} to p_{j_2}. But the only nonidentity isometry that fixes every point of a geodesic segment in dimension 2 is locally given by reflection in that geodesic. Such an isometry is orientation-reversing. Thus by contradiction the set of singular points of Γ in M is finite, and hence so is the set of singular orbits. $\square$

Returning now to the proof of Lemma 2.6.2, we need to exhibit a Euclidean neighborhood of each point of $\mathcal{M}$, that is, a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^2$. For a nonsingular orbit $\mathcal{O}$, this is easy: Choose a point $p \in \mathcal{O} \subset M$ and choose a Euclidean neighborhood U of p in M with every point of U nonsingular (this is possible by Sublemma 2.6.3) and with the natural map $U \to \mathcal{M}$ injective. The latter choice is possible by the finiteness of Γ and the fact that p is nonsingular. For if $\{(p_j, q_j)\}$ were Γ-equivalent pairs of distinct points with $\lim_j p_j = \lim_j q_j = p$, then there would be an element $\gamma \neq \mathrm{id}_M$ in I_p.[11] Thus U maps homeomorphically onto a neighborhood of $\mathcal{O}$, giving a Euclidean neighborhood as required.

The situation is more complicated when we want to find a Euclidean neighborhood of a singular orbit $\mathcal{O}$; but it is only a little more complicated. Let $p \in \mathcal{O}$. The group I_p, being a group of orientation-preserving isometries that

[11] More precisely, if $q_j = \gamma_j(p_j)$, $\gamma_j \neq \mathrm{id}_M$, then by the finiteness of Γ, and passing to a subsequence if necessary, γ_j is independent of j and (by continuity) $\gamma_j \in I_p$.

all fix the point p, has a simple structure—both as a group and as a space acting on a neighborhood of p. Choose a small ball B around p, and let (r, θ) be a geodesic polar coordinate system centered at p on this ball. The I_p acts in this coordinate system as a finite group of Euclidean rotations. In particular, I_p is cyclic, say of order m_p.

Recall from Sublemma 2.6.3 that, if the ball B around p is small enough, then every point of the ball except p is nonsingular. Moreover, $B \to M/\Gamma$ is exactly m_p-to-1, except on the orbit $\mathcal{O}$ corresponding to p (which has only the pre-image p). The same limit argument as in the nonsingular case shows that, for B small enough, no quotienting occurs on B except via I_p. But it is then clear by Euclidean geometry that $\mathcal{O}$ has a Euclidean neighborhood, since a disc modulo a finite group of rotations around its center is locally Euclidean—it is a one-nappe circular cone with p as vertex.

The orientability is clear since Γ contains only orientation-preserving transformations, and the singular orbits do not separate the collection of orbits $\mathcal{M}$, even locally. □

Geometrically, $\mathcal{M}$ is a C^∞ manifold with C^∞ metric pushed down from M except for a finite number of cone-point singularities of the structure already described. Topologically, $\mathcal{M}$ remains a compact, orientable manifold as indicated, although it is likely to be a quite different manifold from M itself. Let g^* denote the genus of $\mathcal{M}$.

Now we want to describe a relationship between the genus g of M, the genus g^* on $\mathcal{M}$, and the action of Γ. To express this idea, we need some notation. Write $|\Gamma|$ for the order of Γ, let $\mathcal{O}_1, \ldots, \mathcal{O}_m$ be the singular orbits (finite in number by Sublemma 2.6.3) of Γ, and let $k_1, \ldots, k_m$ be the orders of the corresponding isotropy subgroups $I_{p_1}, \ldots, I_{p_m}$ for $p_1 \in \mathcal{O}_1, p_2 \in \mathcal{O}_2, \ldots, p_m \in \mathcal{O}_m$. [The groups I_{p_j} are determined only up to conjugacy, but their orders are of course uniquely determined.]

Lemma 2.6.4. *In the notation just given,*

$$2 - 2g = |\Gamma| \cdot (2 - 2g^*) - |\Gamma| \cdot \sum_{j=1}^{m} \left(1 - \frac{1}{k_j}\right).$$

Lemma 2.6.4, which will turn out to be the crucial ingredient for the proof of Theorem 2.6.1, is a special case of the well-known Riemann–Hurwitz formula for branched coverings. But an exact description of the Riemann–Hurwitz general setup would be almost as long as the proof in our case. So we give the direct proof instead. In fact, we shall give two proofs. The first one is a bit disingenuous, since we shall assume without proof the existence of a triangulation of $\mathcal{M}$ of a certain type; but it is still quite instructive. The second proof uses the Gauss–Bonnet theorem and is completely self-contained: we push the triangulation question back to the corresponding difficulty in the proof of the Gauss–Bonnet theorem.

Proof I of Lemma 2.6.4. Choose a triangulation of $\mathcal{M}$ such that the singular orbits $\mathcal{O}_1, \ldots, \mathcal{O}_m$ (thought of as points of $\mathcal{M}$) occur as vertices of the triangulation. This triangulation pulls back to a triangulation of M. In this triangulation of M, every edge of the $\mathcal{M}$-triangulation gives rise to $|\Gamma|$ edges and every face from $\mathcal{M}$ to $|\Gamma|$ faces. Let F_M denote the number of faces, E_M the number of edges, and V_M the number of vertices for the M-triangulation. We have similar notation for the $\mathcal{M}$-triangulation. Then

$$E_M = |\Gamma| \cdot E_{\mathcal{M}}$$

and

$$F_M = |\Gamma| \cdot F_{\mathcal{M}}.$$

However, the pre-image $\pi^{-1}(\mathcal{O}_j)$ contains not $|\Gamma|$ points but rather $|\Gamma|/k_j$ points. Of course pre-images of vertices of the $\mathcal{M}$-triangulation that are non-singular still contain $|\Gamma|$ points. Clearly the errors arising from the singular orbits are accounted for correctly by

$$V_M = |\Gamma| \cdot V_{\mathcal{M}} - \sum_{j=1}^{m} \left(|\Gamma| - \frac{|\Gamma|}{k_j} \right).$$

Thus the Euler characteristic $\chi(M) = V_M - E_M + F_M$ is related to $\chi(\mathcal{M}) = V_{\mathcal{M}} - E_{\mathcal{M}} + F_{\mathcal{M}}$ by

$$\chi(M) = |\Gamma| \cdot \chi(\mathcal{M}) - \sum_{j=1}^{m} \left(|\Gamma| - \frac{|\Gamma|}{k_j} \right)$$

or

$$2 - 2g = |\Gamma| \cdot (2 - 2g^*) - |\Gamma| \cdot \sum_{j=1}^{m} \left(1 - \frac{1}{k_j} \right). \quad \square$$

Proof II of Lemma 2.6.4. We cannot apply the Gauss–Bonnet theorem directly to $\mathcal{M}$ because of the “cone point” singularities of the metric at $\mathcal{O}_1, \ldots, \mathcal{O}_m$. We instead set $\mathcal{M}^\epsilon$ equal to the image under the mapping $M \to M/\Gamma$ of $M \setminus \bigcup_{p \text{ singular}} B(p, \epsilon)$. Then $\mathcal{M}^\epsilon$ is, for $\epsilon > 0$ small enough, an orientable, compact, C^∞ manifold with boundary having a C^∞ metric inherited from M (we use here the fact that the elements of Γ are isometries). The manifold with boundary $\mathcal{M}^\epsilon$ is in effect $\mathcal{M}$ with the cone points and ϵ-discs around them removed. The Gauss–Bonnet theorem does apply to $\mathcal{M}^\epsilon$ as a manifold with boundary. Thus

$$2\pi\chi(\mathcal{M}^\epsilon) = \int_{\mathcal{M}^\epsilon} K \, dA - \int_{\partial\mathcal{M}^\epsilon} K_g \, ds,$$

where K_g is the geodesic curvature of the boundary (with $K_g > 0$ in our situation).

Now $\chi(\mathcal{M}^\epsilon) = \chi(\mathcal{M}) - m$ by standard topology. Also write C^j_ϵ for the component of $\partial\mathcal{M}^\epsilon$ at $\mathcal{O}_j$ and $\widehat{C}^j_\epsilon$ for its pre-image in M at a particular point $p_j \in \mathcal{O}_j$; then

$$\lim_{\epsilon\to 0^+} \int_C K_g \, ds = \frac{1}{k_j} \lim_{\epsilon\to 0^+} \int_{\widehat{C}} K_g \, ds = 2\pi \cdot \frac{1}{k_j}.$$

Here, for typographical convenience, $C = C^j_\epsilon$ and $\widehat{C} = \widehat{C}^j_\epsilon$. Combining these results gives

$$\chi(\mathcal{M}) = m - \sum_{j=1}^{m} \frac{1}{k_j} + \lim_{\epsilon\to 0^+} \frac{1}{2\pi} \int_{\mathcal{M}^\epsilon} K \, dA$$

or

$$\chi(\mathcal{M}) = \sum_{j=1}^{m} \left(1 - \frac{1}{k_j}\right) + \lim_{\epsilon\to 0^+} \frac{1}{2\pi} \int_{\mathcal{M}^\epsilon} K \, dA.$$

But

$$\lim_{\epsilon\to 0^+} \frac{1}{2\pi} \int_{\mathcal{M}^\epsilon} K \, dA = \frac{1}{2\pi|\Gamma|} \int_M K \, dA = \frac{1}{|\Gamma|} \cdot \chi(M).$$

Thus

$$\chi(M) = |\Gamma| \cdot \chi(\mathcal{M}) - |\Gamma| \sum_{j=1}^{m} \left(1 - \frac{1}{k_j}\right). \quad \square$$

With Lemma 2.6.4 now in hand, we are ready to prove Hurwitz's theorem.

Proof of Hurwitz's Theorem 2.6.1. Using the notation already introduced,

$$\frac{2-2g}{|\Gamma|} = (2-2g^*) - \sum_{j=1}^{m} \left(1 - \frac{1}{k_j}\right)$$

or

$$\frac{2g-2}{|\Gamma|} = (2g^*-2) + \sum_{j=1}^{m} \left(1 - \frac{1}{k_j}\right).$$

To get an upper bound for $|\Gamma|$, we need a lower bound for the righthand side of this last equation (for g fixed). Now $2g - 2 > 0$ by hypothesis. So the righthand side is required to be positive. If $g^* \geq 2$, then the righthand side is at least 2. If $g^* = 1$, then $2g^* - 2 = 0$ and there must be at least one term of the form $1 - 1/k_j$, with k_j a positive integer greater than 1. Hence, in this case, the righthand side is at least $1/2$. Finally, if $g^* = 0$, so that $2g^* - 2 = -2$, then there must be at least three terms of the form $1 - 1/k_j$ with $k_j \geq 2$.

If there are five or more of these, then the righthand side would be at least $-2+5(1/2)=1/2$. If there are four, then the righthand side—in order to be positive—must be at least $-2+3(1/2)+2/3=1/6$. Finally if there are three terms of the form $1-1/k_j$, then only one of these can have $k_j=1/2$. A little further experimentation shows that $-2+3-(1/k_1+1/k_2+1/k_3)$ has the minimum possible value $1/42$, obtained when $k_1=2$, $k_2=3$, and $k_3=7$.

Thus the minimum possible positive value in all cases for the righthand side is $1/42$, from which

$$|\Gamma| \leq 42(2g-2) = 84(g-1). \quad \square$$

As already noted earlier in this section, the number 84 is sharp: the *Klein quartic* is a Riemann surfaces of genus 3 with $168 = 84(3-1)$ automorphisms. There is a considerable literature on the subject of when the $84(g-1)$ bound is attained (cf. [Lucchini/Tamburini/Wilson 2000] and [Wilson 2001]). Since the automorphism group of a Riemann surface of genus $g>1$ is precisely the group of orientation-preserving automorphisms of the "push-down" metric from its uniformization by the unit disc (Proposition 2.3.1), the examples where the order of the automorphism group has order $84(g-1)$ also serve as examples for the sharpness of the bound for the order of the group of orientation-preserving isometries.

Another application of the idea that geometric quotients give rise to holomorphic ones is to the question of which compact Riemann surfaces cover others in the sense of covering spaces. This corresponds essentially to Lemma 2.6.4 without branch points. Specifically, one can see either by the triangular argument of Proof I of Lemma 2.6.4 or via the Gauss–Bonnet argument used in Proof II that the following holds:

> If $\pi : M \to M'$ is a covering map from one compact Riemann surface to another, then, with g denoting the genus of M and g' denoting the genus of M',
>
> $$2-2g = k\cdot(2-2g').$$
>
> Here k is a positive integer, and indeed k is the "sheeting number" of π, i.e., the number of points in $\pi^{-1}(\{p\})$ for each $p\in M$.

The necessary condition that, when $g' \geq 2$, then $g'-1$ is a divisor of $g-1$, is in fact also sufficient. This is clear geometrically: consider M as a torus of revolution in $\mathbb{R}^3$ with $g-1$ tubes smoothly attached in such a way that rotation by $2\pi/(g-1)$ takes each one to the next. Let Γ be the group of rotations by multiples of $2\pi/[(g-1)/k]$. Then $M \to M/\Gamma$ is a covering onto a surface with $(g-1)/k = g^*-1$ tubes attached. That is a Riemann surface of genus $1+(g^*-1)=g^*$. This geometric construction is then, according to the principle of associating complex structures to metrics, holomorphic in suitable complex structures. first, there is a unique complex structure for which the metric of M as a surface in $\mathbb{R}^3$ is Hermitian. Since

Γ acts as orientation-preserving isometries on M, it acts holomorphically for this complex structure. And thus $M \to M/\Gamma$ is a holomorphic covering space.

2.7 Automorphisms of Multiply Connected Domains

A simply connected, bounded domain in the plane $\mathbb{C}$ is biholomorphic to the unit disc D—this is the Riemann mapping theorem. A bounded domain U in $\mathbb{C}$ such that $\mathbb{C} \setminus U$ has exactly two connected components is biholomorphic to $D \setminus \{z : |z| \leq r\}$ for some $r \in [0, 1)$. This assertion follows by a Dirichlet problem argument using boundary values of the absolute value of the desired biholomorphic mapping f. One notes that $\log |f|$ is harmonic with boundary value 0 on one component of the boundary of U and constant value α to be determined on the other component of the boundary of U. Let h be the harmonic function with these boundary values. Choosing this value α correctly, one can arrange that $* \, dh$ has integral 2π around an oriented homology generator of U, from which it follows that the associated map $z \mapsto \exp(h + i \oint * \, dh)$ is well defined and one-to-one onto $D \setminus \{z : |z| \leq r\}$, where $r = e^{\alpha}$ (α being the boundary value noted above). Strictly speaking, one needs some regularity of the boundary of U to solve the Dirichlet problem here, but this technical difficulty is easily disposed of by approximating U from the inside by domains with smooth boundary and the same connectivity as U (cf. [Greene/Krantz 2002] or [Ahlfors 1978] for further details).

More generally, one can show that any bounded domain U in $\mathbb{C}$ such that $\mathbb{C} \setminus U$ has finitely many connected components is biholomorphic to D with a finite number of closed discs removed (the closed discs can, in principle, be arcs). This result holds even in the case of countably many components of the complement—see [He/Schramm 1993]. As is standard, we say that the domain has connectivity k if its complement has $k+1$ connected components.

Understanding the automorphisms of these discs with holes that serve as models is certainly a reasonable goal in itself, and it is treated in detail in [Remmert 1998]. But these interesting one-dimensional results are not a very good indication of what to expect in higher dimensions. The reason for this is very simple: as discussed in Section 2.3, the unit ball in $\mathbb{C}^n$, for $n \geq 2$, with a finite number of closed balls removed (call this domain Ω), has the property (by the Hartogs extension phenomenon) that every automorphism of Ω arises from an automorphism of the entire ball (restricted to Ω). This property holds by the same argument for any connected open set obtained by removing a finite number of compact subsets from an open ball. So the theory of the automorphisms of such open sets—"balls with holes"—is subsumed by the knowledge of the automorphisms of the ball. What we shall discover instead, as the book develops, is that the boundary geometry is the deciding feature in studying automorphism groups of domains; basic topology tells us little.

In spite of all this, there *are* aspects of the planar situation that offer clues to the higher-dimensional study in general. We now discuss some of these.

Theorem 2.7.1. *If Ω is a bounded planar domain of finite connectivity that is not simply connected, then* $\mathrm{Aut}\,(\Omega)$ *is compact.*

Actually, we have already proved this result for connectivity 1 by explicit determination of the automorphism group in that case, so the interesting consideration will be for higher connectivity.

Proof of Theorem 2.7.1. Suppose that Ω has connectivity $k \geq 1$ and that $\mathrm{Aut}\,(\Omega)$ is noncompact. Choose $q \in \Omega$. Then the orbit $\{f(q) : f \in \mathrm{Aut}\,(\Omega)\}$ is noncompact by Proposition 1.3.10. So there is a sequence $\{f_j\} \subseteq \mathrm{Aut}\,(\Omega)$ such that $f_j(q)$ converges to a point $p \in \partial\Omega$. Passing to a subsequence if necessary, we can assume that $\{f_j\}$ converges uniformly on compact subsets of Ω to some limit function $f_0 : \Omega \to$ (closure of Ω). The functions $z \mapsto f_j(z) - p$ have no zeros in Ω but their limit $f_0 - p$ is 0 at p. By Hurwitz's theorem, $f_0 - p$ is identically 0. Thus $\{f_j\}$ converges uniformly to the constant function with value 0 on each compact subset of Ω.

Now choose a collection of k closed curves, $k =$ connectivity of Ω, $k \geq 1$, which form a generating set for the homology of Ω. For example, we could choose $C_1, C_2, \ldots, C_k$ where each C_ℓ, $\ell = 1, 2, \ldots, k$, winds once around points of the ℓ^{th} bounded component of $\mathbb{C} \setminus \Omega$ and does not wind around points of the other bounded components. To choose these conveniently, we can and do assume that Ω has the "standard form" $D \setminus$ (k closed discs or points). The image, for each j, $f_j(C_\ell)$, $\ell = 1, \ldots, k$, is again a homology basis for Ω. But when j is large enough, these images all lie in an arbitrarily small neighborhood of p. In particular, if p is not in the ℓ^{th} bounded component of $\mathbb{C} \setminus \Omega$, then none of these images wind around any point of the ℓ^{th} bounded component when j is large.

Clearly this situation is impossible if $k > 1$. When $k = 1$, it is possible only if the one bounded component of Ω (with Ω in "standard form") is a point, since to wind around points of a closed disc, a curve cannot lie in an arbitrarily small neighborhood of any given point. But we have already determined that $\mathrm{Aut}\,(\Omega)$, where Ω is the disc D with one point removed, is compact. first, $\mathrm{Aut}\,(\Omega) = \{T_{-a} \circ R_\omega \circ T_a : |\omega| = 1\}$ if a is the removed point, where R_ω is the map from D to D defined by $R_\omega(z) = \omega z$ and T_a is the map $z \mapsto (z - a)/(1 - \overline{a}z)$ already discussed. $\square$

We shall have occasion frequently in the sequel to use this type of argument: an automorphism sequence which pushes one interior point to a given boundary point will tend to push *all* interior points to that same boundary point, and this will yield topological restrictions. Note, however, that in higher dimensions this idea does not apply automatically as it does in one dimension. Consider, for example, a sequence of automorphisms of $D \times D$ in $\mathbb{C}^2$ acting on the first variable only and converging in that first variable to the constant 1. The accumulation set in the boundary is the whole set $\{1\} \times D$.

Just as genus 1 Riemann surfaces and those of genus $g > 1$ differ as to possible automorphism behavior, so do domains of connectivity 1 (which have an infinite group of automorphisms) differ from those of connectivity > 1. But the reason is now different. For compact surfaces, the difference arises from change of the universal cover—from $\mathbb{C}$ to D. But connectivity 1 bounded domains are covered by D just as those of connectivity 2 are. So the difference lies elsewhere.

Theorem 2.7.2. *If Ω is a bounded, planar domain of connectivity > 1, then* $\mathrm{Aut}\,(\Omega)$ *is finite.*

This result was proved, for instance, by Heins [Heins 1946]. The proof here is different, being more geometric and topological, as preparation for later use of the ideas involved.

Proof of the theorem. Suppose to the contrary that $\mathrm{Aut}\,(\Omega)$ is in fact infinite. Since $\mathrm{Aut}\,(\Omega)$ is compact (and a Lie group), as already shown, the identity component must be positive dimensional. In particular, the identity component of $\mathrm{Aut}\,(\Omega)$ must contain a subgroup G that is isomorphic to the circle, that is, the group $\{\omega \in \mathbb{C} : |\omega| = 1\}$ with multiplication as the binary group operation. This is standard Lie group theory: the closure of any nontrivial one-parameter subgroup is a compact abelian subgroup of positive dimension and hence contains a circle subgroup.

The remainder of the proof consists of formalizing the intuition that Ω cannot be a union of the kinds of orbits—circles or points—that would arise in this situation, except in case of connectivity 0 or 1.

We begin by noting that, since the action of G is via orientation-preserving isometries of Ω relative to what we are calling the H-metric of Ω, the fixed points of the action are isolated. That is, the associated vector field that generates the circle action has isolated zeros. [This can also be seen function-theoretically without reference to metrics.]

For each $p \in \Omega$, we define a continuous map $\gamma_p : G = S^1 \to \Omega$ by $\gamma_p(g) = g(p)$. This collection of closed curves in Ω is parameterized continuously by p. In particular, they all belong to the same free homotopy class. Note that any two such curves, say γ_{p_1} and γ_{p_2}, have either disjoint images in Ω or the same image in Ω, and in the latter case, they are the same curve up to a rotation of the circle in itself.

Suppose that, for some $p_0 \in \Omega$, the image $\gamma_{p_0}(G)$ consists of the point p_0 alone. Since G acts by isometries (of the H-metric), and acts nontrivially, it follows that, for p near p_0 but unequal to p_0, the curve γ_p represents a nontrivial element of the free homotopy classes of curves in $\mathbb{C} \setminus \{p_0\}$. Indeed, γ_p is the (possibly multiple) transversal of an H-metric circle around p_0 in a fixed direction, the associated G-action near p_0 being rotation when expressed in H-metric geodesic polar coordinates.

The previous paragraphs imply immediately that there is at most one point p_0 such that the image of γ_{p_0} is p_0 alone. For if there were two such points, say p_0 and q_0, then, for q near q_0 and p near p_0, the curves γ_p and γ_q would be freely homotopic in the G-invariant set $\Omega \setminus \{p_0, q_0\}$ (since this set is connected and γ_x depends continuously on x in this set). Note that the image of γ_x lies in $\Omega \setminus \{p_0, q_0\}$ for each $x \in \Omega \setminus \{p_0, q_0\}$. But clearly this free homotopy is not the case: γ_p winds around p_0 but *not* around q_0 and γ_q winds around q_0 but *not* around p_0.

We now consider the two (exhaustive) cases: **(1)** γ_p does not have a one-point image for any $p \in \Omega$ and **(2)** there is exactly one $p_0 \in \Omega$ such that the image of γ_{p_0} is p_0 alone.

Looking at case **(2)** first, consider a neighborhood of an orbit $\mathcal{O}_p$, that is in fact equal to image γ_p, for some fixed p. Nearby orbits are labelled uniquely by their (unique) point of intersection with a (short) H-metric geodesic γ perpendicular to $\mathcal{O}_p$ at p: here we use the fact that $\mathcal{O}_p$ is a smooth curve. That G acts by isometries shows that the G-action on a neighborhood of $\mathcal{O}_p$ is a tubular-neighborhood product action; diffeomorphically it is exactly the rotation action of S^1 on an annulus around the origin (possibly multiply covered): once around G may rotate the annulus multiple times but the multiplicity is constant—the same for each orbit. This gives the space of orbits the structure of a 1-manifold, hence an open interval here (since it is connected and open). It follows that the domain Ω is homeomorphic to $(0, 1) \times S^1$ and hence that the fundamental group of Ω is the same as that of S^1, first $\mathbb{Z}$. So Ω has connectivity 1. This is a contradiction.

For the first case, one applies the same reasoning to $\Omega \setminus \{p_0\}$, which is G-invariant, to conclude that $\Omega \setminus \{p_0\}$ has connectivity 1. But this is impossible since Ω already has connectivity at least 1 and removing p_0 adds 1 to the connectivity. [Note here that rotation is an S^1 action on D with a single "degenerate" orbit. So case **(1)** occurs—but only when Ω is simply connected.] □

The reader approaching this subject from the strictly analytical viewpoint may find the great use of geometry and topology in these proofs a bit overwhelming. But, for our purposes, it illustrates well the utility of the ideas of metric geometry in this subject. That G operates as isometries gives useful information in determining the possibilities for its action. This way of understanding key ideas will continue to be important in our later development.

3

The Bergman Kernel and Metric

The subject of this chapter is a remarkable construction developed by Stefan Bergman that produces an explicit, smooth, automorphism-invariant Hermitian metric on each bounded domain in $\mathbb{C}^n$. These metrics are in fact biholomorphic invariants in the sense that, if Ω_1 and Ω_2 are bounded domains in $\mathbb{C}^n$ and if $F : \Omega_1 \to \Omega_2$ is a biholomorphic mapping, then F is an isometry relative to the Bergman metrics of Ω_1 and Ω_2.

The construction is based on the observation that the set of holomorphic functions on a domain in $\mathbb{C}^n$ that have finite integral for the squares of their absolute value (casually called *square-integrable functions*) form a Hilbert space with the usual L^2 inner product $\langle f, g\rangle = \int f\bar{g}$. This fact is a straightforward consequence of Cauchy estimates, which also imply that, for each point of the domain, evaluation at that point is a continuous linear functional on this Hilbert space. If the domain is bounded, this Hilbert space is infinite dimensional since it contains all polynomials. For each point w in the domain, there is, by Riesz representation, an L^2 holomorphic function $k_w(z)$, the "Bergman kernel," such that for each L^2 holomorphic function $f(z)$, the value $f(w)$ is the L^2 inner product of $f(z)$ and $k_w(z)$ (conjugated). It turns out that the function $\log k_z(z)$ is, when the domain is bounded, strictly plurisubharmonic, so the Levi form of $\log k_z(z)$ is the complex form of an Hermitian, indeed Kähler, metric. This is the Bergman metric of the domain. The details of this construction and the proof of the biholomorphic invariance of the metrics occupy the beginning sections of this chapter.

While the construction is explicit, it is at first sight not readily computable since it seems to involve knowledge of the entire Hilbert space of L^2 holomorphic functions. But, in actuality, it can be expressed in terms of certain solutions of the $\overline{\partial}$ operator. This process brings in some powerful machinery for analyzing the behavior of the Bergman kernel $k_w(z)$ and hence of the metric. This culminates in the Fefferman asymptotic expansion when the bounded domain is strongly pseudoconvex; this expansion yields in turn information about the geometry of the Bergman metric near the boundary of a bounded strongly pseudoconvex domain. This will be discussed in the latter part of

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the chapter, and applications of the geometric information will be presented. This development represents a profound extension of the geometric methods that were discussed in Chapter 2. In this more general setting, one does not in general have constancy of the curvature in any sense. But one does have "asymptotic constancy" of holomorphic sectional curvature as the boundary of the strongly pseudoconvex domain is approached. From this, much information about automorphism and biholomorphic maps in general can and will be derived later in this chapter and in Chapter 4.

3.1 The Bergman Space and Kernel

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. Let dV denote Euclidean volume measure. We define

$$A^2(\Omega) = \left\{ f \text{ holomorphic on } \Omega : \int_\Omega |f(z)|^2 \, dV(z) < \infty \right\}.$$

It follows from elementary inequalities that $A^2(\Omega)$ is a vector space. Also, the Hermitian inner product

$$\langle f, g \rangle \equiv \int_\Omega f(z)\overline{g(z)} \, dV(z)$$

is well defined; the integral is bounded (by Schwarz's inequality). These definitions make $A^2(\Omega)$ into a pre-Hilbert space. We check next that $A^2(\Omega)$ is complete, and thus is a genuine Hilbert space.

We define the L^2 or square norm on $A^2(\Omega)$ as usual:

$$\|f\|_{A^2} \equiv \left[\int_\Omega |f(\zeta)|^2 \, dV(\zeta) \right]^{1/2}.$$

If $K \subseteq \Omega$ is a compact set, $\delta = \operatorname{dist}(K, \partial\Omega)$, $z \in K$, and $f \in A^2(\Omega)$, then for $\lambda \in (0, \delta)$,

$$\begin{aligned} |f(z)| &= \left| \frac{1}{V(B(z,\lambda))} \int_{B(z,\lambda)} f(\zeta) \, dV(\zeta) \right| \\ &\leq \frac{1}{[V(B(z,\lambda))]^{1/2}} \left(\int_{B(z,\lambda)} |f(\zeta)|^2 \, dV(\zeta) \right)^{1/2} \\ &\leq [V(B(z,\lambda))]^{-1/2} \cdot \|f\|_{A^2}. \end{aligned}$$

In particular,

$$\max_{z \in K} |f(z)| \leq C_K \cdot \|f\|. \tag{3.1}$$

From this inequality it follows that a sequence in $A^2(\Omega)$ that is Cauchy in norm will also converge uniformly on compact sets. It is straightforward to check that this sequence converges in L^2 norm to this same uniform-on-compact-sets limit, which is of course a holomorphic function. So A^2 is indeed complete and is thus a Hilbert space.

The particular case of the estimate (3.1) with K being the singleton set $\{p\}$, with $p \in \Omega$, tells us that the linear functional

$$\phi_p : A^2(\Omega) \ni f \longmapsto f(p)$$

is bounded. By the Riesz representation theorem, there is an element $k_p \in A^2(\Omega)$ such that

$$f(p) = \phi_p(f) = \langle f, k_p \rangle$$

for every $f \in A^2(\Omega)$. This says that

$$f(p) = \int_\Omega f(\zeta)\overline{k_p(\zeta)}\, dV(\zeta).$$

Following tradition, we write $\overline{k_p(\zeta)} = K(p, \zeta)$, so that our formula becomes

$$f(p) = \int_\Omega f(\zeta) K(p, \zeta)\, dV(\zeta). \tag{3.2}$$

This is the Bergman reproducing formula. The function $K(p, \zeta)$ is called the *Bergman kernel.* This kernel function can be used to construct a very useful $\mathrm{Aut}\,(\Omega)$-invariant metric, called the Bergman metric, as follows.

The Bergman space $A^2(\Omega)$ is separable, since $L^2(\Omega)$ is. Observe that if $\{\phi_j\}_{j=1}^\infty$ is any complete orthonormal basis for $A^2(\Omega)$, then

$$K(z, \zeta) = \sum_j \phi_j(z)\, \overline{\phi_j(\zeta)}. \tag{3.3}$$

This representation comes immediately from the reproducing property (3.2) as follows. Since $k_p(\zeta)$ is in $A^2(\Omega)$ as a function of ζ, we see that

$$k_p(\zeta) = \sum_j a_j(p)\phi_j(\zeta)$$

for some coefficients $a_j(p) \in \mathbb{C}$, with $\sum_j |a_j(p)|^2 < +\infty$. Putting $f = \phi_j$ in the reproducing formula (3.2) yields $\overline{a_j(p)} = \phi_j(p)$; hence $a_j(p) = \overline{\phi_j(p)}$. From this we see that $k_p(\zeta) = \sum_j \overline{\phi_j(p)}\phi_j(\zeta)$ so that $K(z, \zeta) = \sum_j \phi_j(z) \cdot \overline{\phi_j(\zeta)}$.

Notice that one can begin the orthonormal basis starting with $\phi_1 = \text{Volume } (\Omega)^{-1/2}$. It follows that $K(z, z) > 0$ for any $z \in \Omega$. Now we define functions $g_{j\overline{k}}(z)$, $k = 1, \ldots, n$, by

$$g_{j\overline{k}}(z) = \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log K(z, z).$$

We would now like to define the Bergman metric on Ω to be the Hermitian form with value at p given by

$$\sum_{j,k=1}^{n} g_{j\overline{k}}(p) dz_j \otimes d\overline{z}_k .$$

This will be geometrically useful only if this form is positive definite or, equivalently, if the Hermitian matrix $g_{j\overline{k}}$ is positive definite for all $p \in \Omega$.

This wish is in fact realized for all bounded domains $\Omega \subseteq \mathbb{C}^n$ and for all $p \in \Omega$. To check this assertion, one makes a clever choice of orthonormal basis $\{\phi_j\}$ for $A^2(\Omega)$.

First, notice that it is enough to check that $g_{1\overline{1}} > 0$ since the whole construction is invariant under unitary rotation of the standard coordinates on $\mathbb{C}^n$. With this point in mind, we choose the elements of the basis $\{\phi_j : j = 0, 1, 2, \ldots\}$ for $A^2(\Omega)$ as follows.

$$\phi_0 = \frac{\psi_0}{\|\psi_0\|},$$

where ψ_0 has minimal norm among all $\psi \in A^2(\Omega)$ with $\psi(p) = 1$. Such a ψ_0 exists because $\{\|\psi\| : \psi \in A^2(\Omega), \psi(p) = 1\}$ is closed and convex. Note that, in this situation, every element of $\{\phi_0\}^\perp = \{\psi_0\}^\perp$ has value 0 at p. To see this, suppose not. Then $\langle \eta, \psi_0 \rangle = 0$ and $\eta(p) \neq 0$ for some $\eta \in A^2(\Omega)$. Without loss of generality, we may suppose that $\eta(p)$ is positive and (after normalization) equal to 1. For small $\epsilon > 0$, let $h = \psi_0 + \epsilon\eta$. Then $h(p) = 1 + \epsilon\eta(p)$. Also

$$\|h\| = \sqrt{\|\psi_0\|^2 + \epsilon^2\|\eta\|^2}.$$

If we set

$$k = \frac{h}{1 + \epsilon\eta(p)},$$

then $k(p) = 1$ and

$$\|k\| = \frac{\sqrt{\|\psi_0\|^2 + \epsilon^2\|\eta\|^2}}{|1 + \epsilon\eta(p)|}.$$

When ϵ is sufficiently small, we see then that $\|k\| < \|\psi_0\|$ and hence the minimality of ψ_0 is contradicted.

Next we choose

$$\phi_1 = \frac{\psi_1}{\|\psi_1\|},$$

where ψ_1 has minimal norm in the closed, convex set

$$\left\{\psi : \psi \in (\psi_0)^\perp \subseteq A^2(\Omega), \frac{\partial}{\partial z_1}\psi\bigg|_p = 1\right\}.$$

Reasoning as before shows that if ψ is orthogonal to ϕ_0 and to ϕ_1, then $\psi(p) = 0$ and $\frac{\partial}{\partial z_1}\psi\big|_p = 0$.

Complete the orthonormal set $\{\phi_0, \phi_1\}$ to an orthonormal basis $\phi_0, \phi_1, \phi_2, \dots$ of $A^2(\Omega)$. Then computing directly gives that

$$\begin{aligned} g_{1\overline{1}}(p) &= \frac{\partial^2}{\partial z_1 \partial \overline{z}_1} \log \left(\sum_{j=0}^{\infty} \phi_j(z)\overline{\phi_j(z)} \right)\Bigg|_p \\ &= \frac{1}{2\left(\sum_{j=0}^{\infty} \phi_j(p)\overline{\phi_j(p)}\right)^2} \sum_{j,k=0}^{\infty} \left| \phi_j(p)\frac{\partial \phi_k}{\partial z_1} - \phi_k(p)\frac{\partial \phi_j}{\partial z_1} \right|^2 . \end{aligned}$$

Now

$$\left| \phi_j(p)\frac{\partial \phi_k}{\partial z_1} - \phi_k(p)\frac{\partial \phi_j}{\partial z_1} \right| \geq 0$$

for all j, k. Also, it is positive when $j = 0$ and $k = 1$ since $\phi_0(p) > 0$ and $\frac{\partial \phi_k}{\partial z_1} > 0$ while $\phi_1(p) = 0$. Thus $g_{1\overline{1}} > 0$.

For later applications (Chapter 10 as well as intrinsic interest), it is worth noting that the construction of ϕ_0 and ϕ_1 can be extended inductively to find an orthonormal basis for $A^2(\Omega)$ with special properties relative to a given point $p \in \Omega$: First, list all the z-coordinate derivative operators beginning with the value (no derivative), then the first z_1-derivative, the first z_2-derivative, ..., the first z_n-derivative, followed by the second derivatives in lexicographic order ($\partial^2/\partial z_1^2$, $\partial^2/\partial z_1 \partial z_2$, ..., $\partial^2/\partial z_n^2$), then the third derivatives in lexicographic order, and so forth. With ϕ_0, ϕ_1 as already defined, we describe the inductive step to define $\phi_{\ell+1}$ given ϕ_0, ϕ_1, ..., ϕ_ℓ: choose among the elements of $A^2(\Omega)$ orthogonal to ϕ_0, ..., ϕ_ℓ the element of $\psi_{\ell+1}$ which has maximum positive real value at p for the $(\ell+1)$-th differential (in its lexicographic order) in the derivative list. By the argument already given to show that $\phi_1(p) = 0$, it follows that $\psi_{\ell+1}(p) = 0$ and all derivative operators at or before the ℓ-th spot in the list equal 0 at p. Set $\phi_{\ell+1} = \psi_{n\ell+1}/\|\psi_{\ell+1}\|$. Then of course $\phi_{\ell+1}$ also has this vanishing-at-p property. This special basis is sometimes useful in terms of estimating the behavior of differential geometric invariants of the Bergman kernel and metric at p, since differentiations up to a given order k involve only the ϕ_j with j less than or equal to some computable upper bound: all higher j values give ϕ_j's which vanish at p and have derivatives of order k or less equaling 0 at p.

The preceding discussion shows that the Hermitian form defined by

$$\sum_{j,k=1}^{\infty} g_{j\overline{k}} dz_j \otimes d\overline{z}_k$$

is positive definite at each point of Ω. It is called the Bergman metric of Ω. It is a Kähler metric. It is an important and remarkable fact that this Kähler metric and its associated, real-part Riemannian metric, are Aut (Ω)-invariant. At first sight, there seems to be no reason why this invariance should occur. But this fact turns out to be a consequence of a familiar basic fact about holomorphic mappings, already noted in Chapter 1. Recall:

Jacobian Identity: If $f : U \to \mathbb{C}^n$ is a holomorphic mapping of an open set U in $\mathbb{C}^n$ into $\mathbb{C}^n$, then for each $p \in U$,

$$J_f^{\mathbb{R}} = |\mathcal{J}|^2,$$

where $J_f^{\mathbb{R}}$ is the Jacobian of f considered as a real mapping from an open subset of $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$ and $\mathcal{J}$ is the holomorphic Jacobian (determinant)

$$\det\left(\frac{\partial f_j}{\partial z_k}\right)_{j,k=1,\ldots,n}.$$

From this, the general transformation property of the Bergman kernel follows.

Proposition 3.1.1. *If Ω_1 and Ω_2 are bounded domains in $\mathbb{C}^n$ and if $F\colon \Omega_1 \to \Omega_2$ is a biholomorphic mapping, then*

$$K_{\Omega_1}(z,\zeta) = \mathcal{J}_F(z)\overline{\mathcal{J}_F(\zeta)}K_{\Omega_2}(F(z),F(\zeta)).$$

Proof. First note that, since the Jacobian of F as a real mapping is $|\mathcal{J}_F|^2$, it holds, for $f, g \in A^2(\Omega_2)$, that

$$\begin{aligned}
&\int_{z\in\Omega_1} \mathcal{J}_F(z)f(F(z))\overline{\mathcal{J}_F(z)g(F(z))}\,dV(z)\\
&\quad= \int_{z\in\Omega_1} f(F(z))\overline{g(F(z))}|\mathcal{J}_F(z)|^2\,dV(z)\\
&\quad= \int_{w\in\Omega_2} f(w)\overline{g(w)}\,dV(w).
\end{aligned}$$

Thus

$$f \xrightarrow{I} \mathcal{J}_F(z)f(F(z)), \quad z \in \Omega_1$$

is an isometry of $A^2(\Omega_2)$ into $A^2(\Omega_1)$. Similar considerations applied to F^{-1} give an inverse isometry of $A^2(\Omega_1)$ into $A^2(\Omega_2)$. In particular, I is bijective.

This argument shows that if $\{\phi_j\}$ is a complex orthonormal basis for $A^2(\Omega_2)$, then

$$\{I(\phi_j)\} = \{\mathcal{J}_F(z)\phi_j(F(z))\}$$

is a complete orthonormal basis for $A^2(\Omega_1)$. The formula of the proposition now follows from formula (3.3) for K_{Ω_1} and K_{Ω_2} in terms of orthonormal bases. $\square$

The transformation property just established of the Bergman kernel under biholomorphic mappings leads to the isometry property for the Bergman metric, by linear algebra. To express this fact precisely, we write $B_{\Omega_1}(\cdot,\cdot)\big|_p$ and $B_{\Omega_2}(\cdot,\cdot)\big|_q$ for the Bergman metrics of Ω_1 and Ω_2, respectively, at $p \in \Omega_1$ and at $q \in \Omega_2$. Thus, for example, with $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ in $\mathbb{C}^n$, we have

$$B_{\Omega_1}(\alpha, \beta)\bigg|_p = \sum_{j,k=1}^{n} \alpha_j \overline{\beta_k} \left(\frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log K_{\Omega_1} \right)\bigg|_p .$$

Again at $p \in \Omega_1$, and with $F : \Omega_1 \to \Omega_2$ given by $F(z) = (F_1(z), \dots, F_n(z))$, we see that

$$dF(\alpha) = \left(\sum_{j=1}^{n} \frac{\partial F_1}{\partial z_j} \alpha_j, \dots, \sum_{j=1}^{n} \frac{\partial F_n}{\partial z_j} \alpha_j \right).$$

Proposition 3.1.2. *If $F : \Omega_1 \to \Omega_2$ is a biholomorphic mapping, and if $p \in \Omega$, $\alpha, \beta \in \mathbb{C}^n$, then*

$$B_{\Omega_1}(\alpha, \beta)\Big|_p = B_{\Omega_2}\big(dF\big|_p(\alpha), dF\big|_p(\beta)\big)\Big|_{F(p)}.$$

Proof. By the previous proposition,

$$\begin{aligned}
B_{\Omega_1}(\alpha, \beta) &= \sum_{j,k} \alpha_j \overline{\beta_k} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log K_{\Omega_1}(z, z)\bigg|_p \\
&= \sum_{j,k} \alpha_j \overline{\beta_k} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log \mathcal{J}_F(z) \overline{\mathcal{J}_F(z)} K_{\Omega_2}\big(F(z), F(z)\big)\bigg|_p \\
&= \sum_{j,k} \alpha_j \overline{\beta_k} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \Big(\log \mathcal{J}_F(z) \\
&\qquad\qquad + \log \overline{\mathcal{J}_F(z)} + \log K_{\Omega_2}\big(F(z), F(z)\big)\Big)\bigg|_p \\
&= \sum_{j,k} \alpha_j \overline{\beta_k} \frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log K_{\Omega_2}\big(F(z), F(z)\big)\bigg|_p ,
\end{aligned}$$

where the logarithms in the third equality are taken locally. The last equality follows because $\log \mathcal{J}_F(z)$ is a holomorphic function and $\log \overline{\mathcal{J}_F(z)}$ is conjugate holomorphic, so that both are annihilated by $\partial^2/\partial z_j \partial \overline{z_k}$. Now, by the chain rule,

$$\begin{aligned}
&\sum_{j,k} \alpha_j \overline{\beta_k} \frac{\partial^2}{\partial z_j \overline{z_k}} \log K_{\Omega_2}\big(F(z), F(z)\big)\bigg|_p \\
&\quad = \sum_{j,k,\ell,m} \alpha_j \overline{\beta_k}\, \frac{\partial F_\ell}{\partial z_j} \overline{\left(\frac{\partial F_m}{\partial z_k}\right)} \frac{\partial^2}{\partial w_\ell \partial \overline{w_m}} \log K_{\Omega_2}(w,w)\bigg|_{F(p)} \\
&\quad = B_{\Omega_2}\big(dF(\alpha), dF(\beta)\big)\bigg|_{F(p)} . \quad \square
\end{aligned}$$

This straightforward calculation can be put in a more conceptual framework using differential forms. That approach is explained in the next section.

3.2 The Bergman Metric on Complex Manifolds

The construction of the Bergman metric given in the previous section makes overt use of Euclidean coordinates. Thus the fact that automorphisms act as isometries of the Bergman metric comes as a surprise. Since automorphisms do not in general have any particular relationship to Euclidean coordinates beyond holomorphicity and nonsingularity, it was really not expected in advance that a metric construction involving Euclidean coordinates would end up invariant under automorphisms.

In this section, we provide a different construction, originally introduced in [Kobayashi 1970], which explains the automorphism invariance conceptually and has the additional useful property of constructing automorphism-invariant metrics on complex manifolds in many cases.

The basic idea is to replace the space $A^2(\Omega)$ of functions by a space of differential forms. Specifically, suppose that M^n is a complex manifold of complex dimension n. Recall that a (complex) differential form of degree n is said to be of type $(n,0)$ if, in local holomorphic coordinates $(z_1,\ldots,z_n)$, it is expressible as $f(z_1,\ldots,z_n)\, dz_1 \wedge \cdots \wedge dz_n$. On an open subset of $\mathbb{C}^n$, there is an obvious association of functions to $(n,0)$ forms: $f \leftrightarrow f\, dz_1 \wedge \cdots \wedge dz_n$. But, for general complex manifolds, where global coordinates are not expected to exist, the $(n,0)$ forms have such a one-to-one correspondence with functions only locally.

If ω is a (complex) differential form of type $(n,0)$, then the form $\omega \wedge \overline{\omega}$, where $\overline{\omega}$ is the complex conjugate of ω, is a top-degree differential form on M. In particular, the value of

$$\int_M \omega \wedge \overline{\omega}$$

is well defined and does not depend on any metric or coordinate choices. Thus we can consider a (pre)-Hilbert space here analogous to $A^2(D)$, first the set of holomorphic differential forms ω of type $(n,0)$ such that

$$\left| \int_M \omega \wedge \overline{\omega} \right| < +\infty.$$

[Note that, for an $(n, 0)$ form, the concept of being holomorphic is well defined, corresponding to $\omega = f\, dz_1 \wedge \cdots \wedge dz_n$, f holomorphic in local coordinates.] In case M is a domain in $\mathbb{C}^n$, this new space, denoted by $A^2(M)$, is the same as our previous A^2 space: If $\omega = f\, dz_1 \wedge \cdots \wedge dz_n$, then

$$\omega \wedge \overline{\omega} = |f|^2\, dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$$

and $dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$ equals the Euclidean volume form $dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_n \wedge dy_n$ up to a constant factor. For convenience, we assimilate this constant factor once and for all: we define the inner product on $A^2(M)$ to be

$$\langle \omega, \phi \rangle = c \int_M \omega \wedge \overline{\phi},$$

where c is chosen so that $\langle \omega, \omega \rangle \geq 0$ is real and so that $\langle \omega, \phi \rangle$ coincides with the Euclidean volume integral $\int f\overline{g}$ if $\omega = f\, dz_1 \wedge \cdots \wedge dz_n$ and $\phi = g\, dz_1 \wedge \cdots \wedge dz_n$. [In fact, one computes that

$$dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n = (-2)^n i^{n^2} \left(dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \right),$$

which gives c, but this exact value is of no interest.]

Note that it is possible that $A^2(M)$ consists of the 0-form alone: for example, if $M = \mathbb{C}$, then $A^2(M)$ is $\{0\}$, since a holomorphic $(1, 0)$ form $f(z)\, dz$ on $\mathbb{C}$ cannot be L^2 unless it is 0. This follows easily from the mean value property for the holomorphic function f^2 on larger and larger discs around any point of $\mathbb{C}$.

In any case, Cauchy estimates show, as in the previous section, that $A^2(M)$ is a Hilbert space: sequences that are L^2 Cauchy have limits which are themselves L^2 holomorphic $(n, 0)$ forms.

Now choose a point $p \in M$. The evaluation map $\omega \mapsto \omega(p)$ is continuous from $A^2(M)$ to the possible values at p, as before. But, to apply the Riesz representation theorem, we need to have a map to $\mathbb{C}$.

For this, choose a holomorphic local coordinate system around p_0, say $(z_1, \ldots, z_n)$, and consider the map (for p near p_0)

$$\omega \longmapsto f_\omega(p) \in \mathbb{C},$$

where $f_\omega(p)$ is determined by the equation $\omega(p) = f_\omega(p)\, dz_1 \wedge \cdots \wedge dz_n$. By the Riesz representation theorem, there is a holomorphic $(n, 0)$ differential form, with values at each point $z \in M$ denoted by $k_p(z)$, such that

$$\langle \omega, k_p \rangle = f_\omega(p).$$

Define functions f_j by $\phi_j = f_j dz_1 \wedge \cdots \wedge dz_n$. Then, as before, one sees that

$$k_p(z) = \sum_j \overline{f_j(p)} \phi_j(z).$$

Here, of course, $\{\phi_j\}$ is a basis for $A^2(M)$ that is orthonormal relative to $\langle , \rangle$.

We are interested, for metric considerations, in $K_p(p)$ when p is near p_0. Notice that

$$k_p(z) = \left(\sum_j \overline{f_j(p)} f_j(z) \right) dz_1 \wedge \cdots \wedge dz_n.$$

The function $\sum_j \overline{f_j(p)} f_j(p)$ is a C^∞ function which is nonnegative. We call this function $\mathcal{B}(p)$. It depends on the choice of local coordinates $(z_1, \ldots, z_n)$ near p_0, but this dependence is of a simple sort: if one changes coordinates, to $(w_1, \ldots, w_n)$ say, then $\mathcal{B}(p)$ is transformed into a function $\mathcal{J}(p)\overline{\mathcal{J}(p)}\mathcal{B}(p)$, where $\mathcal{J}$ is holomorphic. This is an immediate algebraic consequence of the fact that

$$dz_1 \wedge \cdots \wedge dz_n = \mathcal{J} dw_1 \wedge \cdots \wedge dw_n,$$

where $\mathcal{J} = \det(\partial z_i / \partial w_j)$ is the holomorphic Jacobian (determinant) of the coordinate change.

Now recall that, for any C^∞ function ρ on a complex manifold M, the Levi form L_ρ of ρ defined in local coordinates $(z_1, \ldots, z_n)$ by

$$\sum_{i,j} \frac{\partial^2 \rho}{\partial z_i \partial \overline{z}_j} dz_i \otimes d\overline{z}_j$$

is a Hermitian form that is independent of the choice of local coordinates. The form L_ρ is a Hermitian-symmetric form on the holomorphic tangent space of M, i.e., on the span of $\partial/\partial z_1, \ldots, \partial/\partial z_n$ in the complexification of the real tangent space of M. The independence of coordinate choice is a straightforward and standard calculation using the holomorphic chain rule (cf. [Greene 1987] or [Krantz 2001]).

Now consider, in a neighborhood of p_0, $L_{\log \mathcal{B}}$, supposing that $\log \mathcal{B}(p_0) > 0$. This Hermitian form is independent of all coordinate choices. The possibilities for different $\log \mathcal{B}$ under change of coordinates are of the form

$$\log(\mathcal{B}\mathcal{J}\overline{\mathcal{J}}) = \log \mathcal{B} + \log \mathcal{J} + \log \overline{\mathcal{J}}$$

(local logarithms), and the Levi form of this function is the same as the Levi form of $\mathcal{B}$ itself, $\log \mathcal{J}$ being annihilated by $\partial/\partial \overline{z}_j$ and $\log \overline{\mathcal{J}}$ being annihilated by $\partial/\partial z_j$. The Levi form itself is, as already noted, coordinate-choice independent.

Thus we may set the Bergman metric of M equal to $L_{\log \mathcal{B}}$, wherever $\mathcal{B} > 0$, to get a coordinate-invariant nonnegative semidefinite Hermitian quadratic form. This leads of course algebraically to a nonnegative real metric on M, which is also independent of coordinate choices.

What is intriguing here is that this metric is obviously invariant under automorphisms of M. This invariance is indeed metamathematical: the construction depended only on the complex structure of M and the intrinsically defined space $A^2(M)$, which is invariant under automorphisms. None of the local coordinate choices turned out to matter. Thus a diffeomorphic map which preserves the complex structure has to act isometrically: in the metric construction, there is no way to distinguish the geometry after the application of the map from the geometry before the map is applied. The reader is invited to consider carefully how these philosophical remarks in fact constitute a proof of the invariance of the metric under automorphisms.

It is of course of interest to know when the $\mathcal{B}$ functions are in fact positive everywhere on a given manifold. This is so on the unit disc or, more generally, on any bounded domain in $\mathbb{C}^n$. But the $\mathcal{B}$ functions are identically 0 on $\mathbb{C}$ or on $\mathbb{C}^n$ for $n \geq 2$. Thus the positivity of $\mathcal{B}$ and the positive definiteness of the Bergman metric $L_{\log \mathcal{B}}$ are in a sense indicators of similarity to bounded domains as opposed to the whole of Euclidean space.

It is a longstanding idea of geometric function theory (see for instance [Greene/Wu 1977]) that similarity to a bounded domain versus all of $\mathbb{C}^n$ is associated to the existence of Hermitian or Kähler metrics of negative curvature. In the case of the manifold Bergman metric, this general expectation was given specific form by [Greene/Wu 1977]. In particular, the following were proved there, using L^2 methods of the $\overline{\partial}$ problem.

Theorem 3.2.1 (Greene/Wu, [Greene/Wu 1977], p. 144). *Let M be a complex Kähler manifold that is simply connected and has nonpositive Riemannian sectional curvature everywhere. Denote by r the distance from a fixed point $p \in M$. Then*

(1) *If there are positive constants ϵ and A such that the inequality*

$$\textit{(sectional curvature)} \leq \frac{-A}{r^2(\log r)^{1-\epsilon}}$$

holds outside some compact set containing p, then the Bergman kernel of M is nowhere 0 and the Bergman metric is positive definite.

(2) *If there are positive constants A and B such that, outside a compact set containing p,*

$$\frac{-B}{r^2} \leq \textit{(sectional curvature)} \leq \frac{-A}{r^2},$$

then there is a positive constant C_1 such that the Bergman metric is $\geq (C_1/[1+r^2]) \cdot G$, where G is the Kähler metric of M. In particular, the Bergman metric is complete.

(3) *If there are positive constants A and B such that*

$$-B \leq \textit{(sectional curvature)} \leq -A$$

everywhere on M, then there is a positive constant C_2 such that the Bergman metric is $\geq C_2 G$, where G is the Kähler metric of M, as in **(2)**. *In particular, the Bergman metric is complete. In this situation, there are positive constants A_1 and A_2 such that the Bergman kernel form $K(p,p)$ satisfies $A_1 \Omega \leq K \leq A_2 \Omega$, where Ω is the volume form determined by the Kähler metric G.*

This result gives that Aut (M) is a Lie group under the hypotheses given. Whenever the Bergman metric exists, then the biholomorphic mappings are a closed subgroup of the isometry group of the Bergman metric, and the isometry group is a Lie group. This latter is part of the circle of ideas surrounding the Bochner–Montgomery theorem (in Chapter 1, Theorem 1.3.11): the isometry group of any Riemannian metric is easily checked to be locally compact and the Bochner–Montgomery theorem gives that it is a Lie group. Since a closed subgroup of a Lie group is a Lie group, the automorphism group of a complex manifold with a positive definite Bergman metric is a Lie group. But weaker curvature conditions in Theorem 3.2.1 will imply this Lie group conclusion: these are obtained by considering the Kobayashi metric, rather than the Bergman metric. This matter will be discussed in Section 7.2.

3.3 Examples of Bergman Kernels and Metrics

In general, it is difficult to calculate the Bergman kernel of a domain or a complex manifold. Indeed, it is often impossible to do so in any reasonable sense with formulas, although it is always possible in principle via numerical analysis methods, in the sense that $K(z,p)$ can be found to any desired accuracy for z, p given in the domain.[1] This latter process can be carried out, for instance, using the characterization of $K(z,p)$ as a function of z in terms of a special basis. The holomorphic function ϕ_0 (constructed by extending the technique we used to show that the Bergman metric is positive definite) with positive, real value at p, that value being maximal among all holomorphic functions on the given domain with L^2 norm 1, is computable numerically. One chooses some countable collection of functions $\{f_j\}$ with

$$\bigcup_N \operatorname{span}\{f_1, \ldots, f_N\}$$

[1] This circle of ideas is particularly well developed in the one-variable setting. For a full development of some of this work, see Bell's book [Bell 1992] and the papers of Kerzman and Stein [Kerzman/Stein 1978]. There is very little explicit literature in the several variables setting.

dense in $A^2(\Omega)$ and then finds ϕ_1 as the limit of a succession of finite-dimensional maximization problems (much as in the finite element method). first, one maximizes $|\sum_1^N \alpha_j f_j|$ at p subject to the constraint $\|\sum_1^N \alpha_j f_j\| = 1$. Alternatively, one could apply the Gram–Schmidt orthogonalization process to $\{f_j\}$ and use formula (3.3) in Section 3.1.

That such a construction exists in principle is interesting. But to actually carry it out is usually computationally very tedious, and not particularly illuminating. When the domain has a great deal of symmetry, that is to say, a large automorphism group, then the whole picture is made much simpler by the transformation formula of Proposition 3.1.1 and often the Bergman kernel and metric are explicitly computable. For certain specific classes of domains (e.g., strongly pseudoconvex [Fefferman 1974] as already mentioned, or finite type in $\mathbb{C}^2$ [Nagel/Rosay/Stein/Wainger 1989]), an asymptotic expansion for the Bergman kernel may be calculated. In many applications, such an expansion is every bit as good as (and in some ways provides more structured information than) an explicit formula.

In this section we give some relatively simple but important instances of how to do the sorts of calculations indicated, for domains with "symmetry."

Kernel Functions

(1) The unit disc $D = \{\zeta \in \mathbb{C} : |z| < 1\}$.

The space $A^2(D)$ has an orthonormal basis that one can write explicitly:

$$\frac{1}{\sqrt{\pi}}, \frac{\sqrt{2}z}{\sqrt{\pi}}, \ldots, \frac{\sqrt{n+1}z^n}{\sqrt{\pi}}, \ldots .$$

The orthogonality is immediate by writing the integral in polar coordinates, and the fact that each has L^2 norm 1 also follows by integration in polar coordinates. Thus

$$K(z,\zeta) = \frac{1}{\pi} + \frac{2}{\pi}(z\bar{\zeta}) + \frac{3}{\pi}(z\bar{\zeta})^2 + \cdots$$

so that (summing the geometric series)

$$K_D(z,\zeta) = \frac{1}{\pi}\frac{1}{(1-z\bar{\zeta})^2}.$$

This formula can also be derived from the transformation law of Proposition 3.1.1 and the use of suitable Möbius transformations.

We can carry out the first of these programs as follows. First note that the largest absolute value at 0 for a holomorphic function of L^2 norm 1 is $1/\sqrt{\pi}$, attained by the constant function. For this, note that

$$f^2(0) = \frac{1}{\pi}\iint f^2(x+iy)\,dxdy$$

by the mean value property so that

$$|f(0)|^2 \leq \frac{1}{\pi}\|f\|^2$$

with equality only for constant functions. Hence, from the special basis construction for $K(z,z)$, we see that

$$K_D(0,0) = \frac{1}{\pi}.$$

To find $K_D(z_0,z_0)$, $z_0 \in D$, we apply Proposition 3.1.1 with $F(z) = (z + z_0)/(1+\overline{z_0}z)$, $F : D \to D$ being biholomorphic (conformal) with $F(0) = z_0$. Thus we have:

$$K_D(0,0) = \frac{1}{\pi} = \mathcal{J}_F(0)\overline{\mathcal{J}_F(0)}K_D(z_0,z_0).$$

Now $\mathcal{J}_F(0) = F'(0) = 1 - z_0\overline{z_0}$. Thus

$$K_D(z_0,z_0) = \frac{1}{\pi}(1 - z_0\overline{z_0})^{-2}.$$

The function $K_D(z,\zeta)$ is holomorphic in z and conjugate holomorphic in ζ. It must be that

$$K_D(z,\zeta) = \frac{1}{\pi}(1 - z\overline{\zeta})^{-2},$$

because the difference between these two functions is 0 on the maximal-dimensional totally real subspace $\{(z,\overline{z}) : z \in D\}$ in $D \times D$.

(2) The product $D \times D$.

If $\{\phi_j\}$ is an orthonormal basis for $A^2(D)$, then

$$\Phi_{j,k}(z_1,z_2) \equiv \phi_j(z_1)\phi_k(z_2), \qquad j,k = 1,2,\ldots$$

is an orthonormal basis for $A^2(D \times D)$. Thus

$$K_{D\times D}\big((z_1,z_2),(\zeta_1,\zeta_2)\big) = \frac{1}{\pi^2}\frac{1}{(1 - z_1\overline{\zeta_1})^2} \cdot \frac{1}{(1 - z_2\overline{\zeta_2})^2}.$$

More generally, the Bergman kernel for $\Omega_1 \times \Omega_2$ is the product of the kernels for Ω_1 and Ω_2 in an obvious sense, for any bounded domains Ω_1, Ω_2 in $\mathbb{C}^{n_1}$, $\mathbb{C}^{n_2}$ respectively. The proof is just the same.

(3) The unit ball in $\mathbb{C}^n$: $B = \{(z_1,\ldots,z_n) : \sum_j |z_j|^2 < 1\}$.

While it is possible to compute an explicit orthonormal basis of monomials in $z_1,\ldots,z_n$, the calculation involves tedious integrals (see [Krantz 2001]). While any two such monomials are clearly orthogonal (just by averaging considerations), it is not at all clear what their L^2 norms are. The one point that is obvious is, by the same logic as for the disc, that

$$K_B(0,0) = \frac{1}{\mathrm{vol}\,(B)} = \frac{n!}{\pi^n}.$$

Now, to find $K_B(z,z)$, note that there is a unitary rotation (i.e., an element of $U(n)$) which takes z to $((\sum_j |z_j|^2)^{1/2}, 0, \ldots, 0)$. If we set $r = (\sum_j |z_j|^2)^{1/2}$ then $r \geq 0$ and we have rotated z to $\mathbf{r} = (r, 0, \ldots, 0)$. This rotation is an automorphism of B with Jacobian of absolute value 1, so

$$K_B(z,z) = K_B(\mathbf{r},\mathbf{r}).$$

Also

$$T((z_1,\ldots,z_n)) = \left(\frac{z_1+r}{1+rz_1}, \frac{z_2\sqrt{1-r^2}}{1+rz_1}, \ldots, \frac{z_n\sqrt{1-r^2}}{1+rz_1}\right)$$

is an automorphism of B which takes 0 to $\mathbf{r}$. At 0, the holomorphic Jacobian

$$\mathcal{J}_T(0) = \det \begin{pmatrix} 1-r^2 & & & & \\ & \sqrt{1-r^2} & & 0 & \\ & & \ddots & & \\ & & & \ddots & \\ & 0 & & & \\ & & & & \sqrt{1-r^2} \end{pmatrix}.$$

So

$$\mathcal{J}_T(0)\overline{\mathcal{J}_T(0)} = (1-r^2)^{n+1}.$$

Thus

$$K_B(z,z) = \frac{n!}{\pi^n}\left(1-|z|^2\right)^{-(n+1)},$$

where $|z| = r = \left(\sum_{j=1}^n |z_j|^2\right)^{1/2}$. Finally, by the same logic as for D,

$$K_B(z,\zeta) = \frac{n!}{\pi^n}(1 - z\cdot\overline{\zeta})^{-(n+1)},$$

where

$$z\cdot\overline{\zeta} = \sum_{j=1}^n z_j\overline{\zeta_j}.$$

3.3.1 Bergman Metrics

(1) The unit disc.

By direct calculation,

$$\frac{\partial^2}{\partial z \partial \overline{z}} \log K_D(z,z) = \frac{2}{(1-z\overline{z})^2}.$$

Thus the Bergman metric is $[2/(1-z\overline{z})^2]\, dz\, d\overline{z}$. Since

$$\begin{aligned} dz\, d\overline{z} &= (dx + idy)(dx - idy) \\ &= dx^2 + dy^2 + i(dydx - dxdy), \end{aligned}$$

the real part, or equivalently, the symmetric part, of the Bergman metric is

$$\frac{2(dx^2 + dy^2)}{(1-z\overline{z})^2}.$$

This is of course (up to a constant factor) equal to the Poincaré metric.

For another point of view, the Bergman metric for D is determined up to a constant factor by Proposition 3.1.2 alone: since the Bergman metric is Hermitian, its real part must be a multiple of $dx^2 + dy^2$ at 0, say $c(dx^2 + dy^2)$. The image under dF_η with $F_\eta = (z+\eta)/(1+\overline{\eta}z)$ of the vector $(1,0)$ at 0 must have length^2 $= c$ since $F_\eta \in \operatorname{Aut}(D)$ must act as an isometry. But $(dF_\eta|_0)((1,0))$ has (Euclidean length)^2 $= (1-\eta\overline{\eta})^2$ by calculation. So the Bergman metric at η must be $c(1-\eta\overline{\eta})^{-2}(dx^2+dy^2)$.

Again, since the members of $\operatorname{Aut}(D)$ act as isometries and, since $\operatorname{Aut}(D)$ acts transitively on D, the (real part of the) Bergman metric of D must have constant Gaussian curvature. Moreover, this real Riemannian metric, which we shall also call the Bergman metric, is complete, because every homogeneous Riemannian manifold is complete (see [Kobayashi/Nomizu 1963] for example). We can now deduce that the Gauss curvature is constant negative as follows. If it were constant positive, then D would be compact by Bonnet's theorem (see [Petersen 2006] for example). If it were zero, then D with the Bergman metric would be isometric to $\mathbb{R}^2$ in the Euclidean metric, and hence D would be conformally equivalent to $\mathbb{C}$. Thus constant negativity is the only possibility.

Of course the negativity of the constant curvature of the Bergman metric on the disc can also be established by direct calculation. Recall that the Gaussian curvature of a metric of the form $e^{2u}(dx^2 + dy^2)$ is

$$-\frac{\triangle u}{e^{2u}}.$$

Thus the Gaussian curvature of $2(1 - z\overline{z})^{-2}(dx^2 + dy^2)$ is

$$\frac{\left[-4\frac{\partial^2}{\partial z \partial \overline{z}} \log(1 - z\overline{z})^{-1}\right]}{[2(1 - z\overline{z})^{-2}]} = -2.$$

In order to arrange for Gaussian curvature -1, the metric is sometimes taken to be 2 times the one we have given here: the metric $4(1 - z\overline{z})^{-2}(dx^2 + dy^2)$ has curvature $\equiv -1$.

Note for future reference that the formula above implies that the Gauss curvature of λg, λ a positive constant, is $\frac{1}{\lambda}$(Gauss curvature of g). This is a general fact, valid in all dimensions: the sectional curvatures of λg are $\frac{1}{\lambda}$(sectional curvature of g) for the same 2-plane (cf. [Kobayashi/Nomizu 1963] or [Petersen 2006]).

(2) Product domains of the form $\boldsymbol{\Omega_1 \times \Omega_2}$, and the bidisc in particular.

Since the Bergman kernel for a product domain is the product kernel, as already discussed, it follows immediately that the Bergman metric of $\Omega_1 \times \Omega_2$ is the product metric, in the usual sense of Riemannian and Hermitian geometry, of the Bergman metrics of Ω_1 and Ω_2, respectively.

(3) The unit ball $\boldsymbol{B^n \subseteq \mathbb{C}^n}$, $\boldsymbol{n \geq 2}$.

The only Hermitian inner products on $T_0 B^n = \mathbb{C}^n$ that are $U(n)$-invariant are the constant multiples of the standard Euclidean metric. Since $U(n) \subseteq \mathrm{Aut}\,(B^n)$, it follows that the Bergman metric of B^n at the origin is a multiple of the Euclidean metric. This is also apparent from the formula:

$$\frac{\partial^2}{\partial z_j \partial \overline{z_k}} \log(1 - |z|^2)^{-(n+1)}\bigg|_0 = (n+1)\delta_{jk}.$$

The metric elsewhere is easy enough to compute explicitly, but more insight is derived from the following reasoning: if $(z_1, \ldots, z_n) \in B^n$, $(z_1, \ldots, z_n) \neq 0$, then there is an element of $U(n)$ that takes $(z_1, \ldots, z_n)$ to $(r, 0, \ldots, 0)$ with $r = (\sum_j |z_j|^2)^{1/2}$ as before. Also, as before, the automorphism

$$(z_1, \ldots, z_n) \mapsto \left(\frac{z_1 + r}{1 + r z_1}, \frac{z_2\sqrt{1 - r^2}}{1 + r z_1}, \ldots, \frac{z_n\sqrt{1 - r^2}}{1 + r z_1}\right)$$

has holomorphic Jacobian at 0 given by the diagonal matrix

$$\begin{pmatrix} 1 - r^2 & & & \\ & \sqrt{1 - r^2} & & \\ & & \ddots & \\ & & & \sqrt{1 - r^2} \end{pmatrix}.$$

Thus its real Jacobian at 0 relative to the real coordinate system denoted by $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ is the $2n \times 2n$ diagonal matrix

$$\begin{pmatrix} 1-r^2 & & & & & \\ & 1-r^2 & & & & \\ & & \sqrt{1-r^2} & & & \\ & & & \sqrt{1-r^2} & & \\ & & & & \ddots & \\ & & & & & \sqrt{1-r^2} \end{pmatrix}.$$

The associated linear transformation maps any orthonormal basis at 0 (relative to the Bergman metric) to an orthonormal basis at $(r, 0, \ldots, 0)$. Now, using the identification

$$\frac{\partial}{\partial x_1} = (1, 0, \ldots, 0) \in \mathbb{R}^{2n}$$

and

$$\frac{\partial}{\partial y_1} = (0, 1, \ldots, 0) \in \mathbb{R}^{2n},$$

the vectors

$$\frac{1}{\sqrt{n+1}} \frac{\partial}{\partial x_1}, \frac{1}{\sqrt{n+1}} \frac{\partial}{\partial y_1}, \ldots, \frac{1}{\sqrt{n+1}} \frac{\partial}{\partial y_n}$$

are a Bergman-metric-orthonormal basis at 0. Thus

$$\begin{aligned} &\frac{1-r^2}{\sqrt{n+1}} \frac{\partial}{\partial x_1}, \frac{1-r^2}{\sqrt{n+1}} \frac{\partial}{\partial y_1}, \\ &\quad \frac{\sqrt{1-r^2}}{\sqrt{n+1}} \frac{\partial}{\partial x_2}, \frac{\sqrt{1-r^2}}{\sqrt{n+1}} \frac{\partial}{\partial y_2}, \ldots, \frac{\sqrt{1-r^2}}{\sqrt{n+1}} \frac{\partial}{\partial x_n}, \frac{\sqrt{1-r^2}}{\sqrt{n+1}} \frac{\partial}{\partial y_n} \end{aligned}$$

are Bergman-metric orthonormal at $(r, 0, \ldots, 0)$. This result gives the geometric picture that the Bergman measurement of the length squared of a vector in the radial direction and of one in the $i \times$ radial direction (or, more formally, the J(radial direction)) is equal to

$$\left(\frac{\sqrt{n+1}}{1-r^2} \right) \times (\text{Euclidean metric length squared}).$$

Also, the $(2n-2)$-dimensional (Euclidean) orthogonal complement of the span of the radial and the $i \times$ radial directions is also the Bergman orthogonal complement. And on this orthogonal complement, the Bergman metric length squared of a vector is equal to

$$\left(\frac{\sqrt{n+1}}{\sqrt{1-r^2}}\right) \times \text{(Euclidean metric length squared)}.$$

In particular, when r is close to 1, the Bergman metric on this orthogonal complement is much smaller than the radial metric, so that Bergman unit vectors are comparatively much larger. In Euclidean terms, as $r \to 1$, Bergman unit vectors get small, but radial (and i times them) vectors shrink faster than tangential ones. This is the distinction between complex normal and complex tangential directions as far as the Bergman metric is concerned.

Since $U(n) \subseteq \operatorname{Aut}(B^n)$ acts transitively on complex lines through the origin (i.e., real 2-dimensional J-invariant subspaces—see [Wells 1979]), it follows from Proposition 3.1.2 that the Riemannian sectional curvatures of such 2-planes in the Bergman metric are all equal to some constant c. That is, the holomorphic sectional curvatures[2] at 0 are all equal to c. Since $\operatorname{Aut}(B^n)$ acts on B^n transitively as a group of isometries, it follows that B^n has everywhere constant holomorphic sectional curvature c. And in the case of the unit disc, we can see without calculation that c must be negative: the Bergman metric is complete because $\operatorname{Aut}(B^n)$ is transitive and every homogeneous Riemannian manifold is complete. Also the Riemannian sectional curvatures lie between $c/4$ and c, inclusive. So, if c were positive, then the sectional curvatures would be bounded away from 0 by the positive constant c, and, by Myers's theorem, B^n would be compact. If c were 0, then B^n equipped with its Bergman metric would be holomorphically isometric to $\mathbb{C}^n$. This conclusion is of course impossible. Thus it must be that $c < 0$.

To evaluate c explicitly, note that $P = \{(z_1, 0, \ldots, 0) \in B^n\}$ is a totally geodesic 2-dimensional submanifold since P is the fixed point set of the isometric involution of B^n given by

$$(z_1, z_2, \ldots, z_n) \mapsto (z_1, -z_2, -z_3, \ldots, -z_n).$$

So the holomorphic sectional curvature c is equal to the Gaussian curvature of the Bergman metric restricted to P. This metric is clearly equal to

$$\frac{n+1}{2} \cdot \text{(the Bergman metric of } P \text{ considered as a disc in } \mathbb{C}).$$

Thus $c = -4/(n+1)$ from our discussion in **(1)** on metrics in one variable and their scaling of curvature when multiplied by a constant (cf. the end of (1)).

[2]The basic ideas for holomorphic sectional curvature and related matters can be found in, for example, [Greene 1987] or [Kobayashi/Nomizu 1963] as well as many other places. For many of our purposes, it will suffice simply to know that holomorphic sectional curvature is a "differential invariant" that attaches a number to each J-invariant 2-plane in the real tangent space of a complex manifold with a Kähler metric and that this number is preserved by holomorphic isometries of the Kähler metric.

3.4 The Bergman Metric on Strongly Pseudoconvex Domains

When a domain has a large automorphism group, the transformation properties of the Bergman kernel under automorphisms (Proposition 3.1.1) can make it easier to determine the Bergman kernel and metric than the definitions alone of the kernel and metric would suggest. This was discussed in the previous section, and there we saw this principle in action in the determination, for example, of the kernel and metric for the unit ball in $\mathbb{C}^n$. But, for domains which lack such symmetry, the explicit calculation of the Bergman kernel and metric is likely to be difficult.

However, in many cases, the asymptotic behavior of the kernel and metric as the boundary is approached can be estimated quite effectively by using "globalization" techniques obtained by solving the $\bar{\partial}$ problem, even when the kernel and the metric over the whole domain are not known explicitly.

There are several different ways to approach this issue. One natural one is to use this fact discussed in Section 3.1: $K(z, w)$, w fixed, is $\varphi_0(z)\overline{\varphi_0(w)}$ where φ_0 is a function which maximizes $|\varphi(z)|$ among all holomorphic functions φ with $\|\varphi\|_2 = 1$. Thus estimates on $K(z, w)$ and in particular $K(z, z)$ can be obtained by exhibiting candidates for φ_0, so to speak. first, $K(z, z) \geq |\varphi(z)|^2$ for any $\varphi : \Omega \to \mathbb{C}$ holomorphic with $\|\varphi\|_2 = 1$.

To see how this idea can be combined in practice with $\bar{\partial}$ techniques, it is instructive to look at a special example. Suppose that Ω is a bounded domain with C^∞ boundary and that $\partial\Omega$ coincides with the boundary of the unit ball in a neighborhood of $\mathbf{1} = (1, 0, \dots, 0)$, so that $(\Omega \cup \partial\Omega) \cap B^n(\mathbf{1}, \epsilon) = \mathrm{cl}(B^n(0, 1)) \cap B^n(\mathbf{1}, \epsilon)$ for some $\epsilon > 0$. For $z \in \Omega \cap B^n(\mathbf{1}, \epsilon)$, we have a possible "candidate" function φ, first, the Bergman kernel of the ball $K_{B^n}(z, w)$ restricted to $\Omega \cap B^n(\mathbf{1}, \epsilon)$, where w is a point of $\Omega \cap B^n(\mathbf{1}, \epsilon)$ which we consider to be approaching $\mathbf{1} = (1, 0, \dots, 0)$ along the real part of the z_1-axis. The function $K_{B^n}(z, w)$ is of course known explicitly to be $c_n(1 - z \cdot \bar{w})^{-(n+1)}$, $c_n = n!/\pi^n$: see Section 3.3.

Now the restriction of $K_{B^n}(z, w)$ described above is not really a qualified "candidate" function. First of all, it need not be holomorphic on all of Ω: Ω may contain points z where $z \cdot \bar{w} = 1$. While this does not happen for z near $(1, 0, \dots, 0)$, it can certainly happen at remote points of Ω. Secondly, even if $K_{B^n}(z, w)$ is (for a fixed w) holomorphic on all of Ω, it need not have L^2-norm equal to 1.

The first of these difficulties is more fundamental—the second one would just change the candidate by some estimatable constant factor. Dealing with the first difficulty, however, fits nicely into the standard ideas of $\bar{\partial}$ estimates.

For the purposes of our present example, we now assume that Ω is C^∞ strongly pseudoconvex: it is in this context that the strongest general results about $\bar{\partial}$ solvability hold.

The function $K_{B^n}(z, w)$, with w near $(1, 0, \dots, 0)$ on the Re z_1-axis, can be extended as a C^∞ function to all of Ω by the usual method of multiplying by a "bump function." Specifically, let $\rho : B^n(\mathbf{1}, \epsilon) \to \mathbb{R}$ be a C^∞ function with values in $[0, 1]$, identically 1 on $B^n(\mathbf{1}, \epsilon/2)$ and identically 0 on $B^n(\mathbf{1}, \epsilon) \setminus B^n(\mathbf{1}, (3\epsilon)/4)$. Then $\rho(z)K_{B^n}(z, w)$ extended by 0 on $\Omega \setminus B^n(\mathbf{1}, \epsilon)$ is C^∞ and is identical with $K_{B^n}(z, w)$ on $B^n(\mathbf{1}, \epsilon/2)$. Call this extended function $F_w(z)$. Now $F_w(z)$ is clearly not in general holomorphic. However, for each $k = 1, 2, 3, \ldots$, F_w is bounded in real C^k-norm as a function of z, uniformly as w approaches $(1, 0, \ldots, 0)$ along the Re z_1-axis. This is easily checked from the fact that $1 - z \cdot \bar{w}$ is bounded away from 0 on the set where $\bar\partial F_w(z)$ is nonzero: note for this that $\bar\partial F_w(z) \equiv 0$ for $z \in \Omega \cap B^n(\mathbf{1}, \epsilon/2)$.

If u_w is a $\mathcal{C}^\infty$ function on Ω with $\bar\partial u_w(z) = \bar\partial F_w(z)$, then of course $F_w(z) - u_w(z)$ is holomorphic. And if u_w of this sort can be chosen to be in a suitable sense small compared to $F_w(z)$ (when w is close to $(1, 0, \ldots, 0)$ and z is close to w), then $F_w(z) - u_w(z)$ would have the same kind of growth behavior as $K_{B^n}(z, w)$. Since $(F_w(z) - u_w(z))/\|F_w - u_w\|_2$ has L^2-norm 1, it is a "candidate" function to estimate $K_\Omega(w, w)$ from below:

$$K_\Omega(w, w) \geq \left| \frac{F_w(w) - u_w(w)}{\|F_w - u_w\|_2} \right|^2 .$$

Thus if a suitable u_w can be found, suitable in terms of having L^2-norm bounded and having $u_w(w)$ small compared to $K_B(w, w)$ where w is near $(1, 0, \ldots, 0)$ on the Re z_1-axis, then indeed we see that $K_\Omega(w, w)$ grows as w approaches $(1, 0, \ldots, 0)$ with the same order of growth as $K_B(w, w)$.

The solution of the $\bar\partial$-Neumann problem obtained by J.J. Kohn (see [Folland/Kohn 1972]) provides the function u_w needed. Since $\bar\partial F_w$ is bounded uniformly in C^2-norm on the closure of Ω, uniformly as w goes to $(1, 0, \ldots, 0)$, the "$\bar\partial$-Neumann solution" of $\bar\partial u_w = \bar\partial F_w$ has the needed boundedness properties. first, according to Kohn's results, there is a unique solution u_w of $\bar\partial u_w = \bar\partial F_w$ such that u_w is perpendicular to $A^2(\Omega)$ and this solution u_w has the desired properties; in particular, its C^0-norm (i.e., the supremum norm) on Ω is bounded uniformly as w approaches $(1, 0, \ldots, 0)$.

The construction just given provides only a lower bound on $K_\Omega(w, w)$. But the roles of Ω and B^n can be interchanged, with K_Ω transferred to B^n and "corrected" by $\bar\partial$ on the unit ball B^n. This will give, clearly, an estimate from above on $K_\Omega(w, w)$. It follows that $K_\Omega(w, w)$ is bounded above and below by constant multiples of $K_{B^n}(w, w)$ as w approaches $(1, 0, \ldots, 0)$ along the Re z_1-axis.

The literal coincidence of the boundary $\partial\Omega$ of Ω and the boundary ∂B^n of the ball B^n near the point around which estimation is occurring was not really required in the discussion just given. It would have sufficed to have $\partial\Omega$ making a sufficiently high order of contact with a biholomorphic image of (part of) $\partial\Omega$. The "transfer" of K_{B^n} to an open subset of Ω would then have been by a map which was not quite holomorphic, since one would need to move the image of ∂B^n to coincide with $\partial\Omega$ locally. But this would involve only estimable errors: one would use $\bar\partial$ estimates simultaneously to extend the function F_w and to make it holomorphic near the distinguished point.

The details of this assertion would need checking, but the possibility clearly will be realized if there is sufficiently high order of contact between $\partial\Omega$ and some biholomorphic image of ∂B^n.

In fact, it turns out that, when Ω is C^∞ strongly pseudoconvex, then every point $p \in \partial\Omega$ has the property that a sufficiently high order of contact can be obtained at a strongly psuedoconvex point—actually up to fourth order—by a biholomorphic image of the ball (cf. [Fefferman 1974], for example). Carrying out the extension of the reasoning of our example yields this:

If Ω is a bounded, C^∞ strongly pseudoconvex domain, then there are positive constants C_1 and C_2 such that

$$C_1[dis\ (w, \partial\Omega)]^{-(n+1)} \le K_\Omega(w,w) \le C_2[dis\ (w, \partial\Omega)]^{-(n+1)}$$

for all $w \in \Omega$.

This is of significance, of course, only for w near $\partial\Omega$.

This kind of estimation began with L. Hörmander ([Hörmander 1965]; see also [Hörmander 1990] for example) and reached its culmination with the complete asymptotic expansion of K_Ω obtained by C. Fefferman [Fefferman 1974], which we shall discuss in more detail momentarily. Important intermediate results were obtained by K. Diederich in [Diederich 1973]. There has also been extensive work on domains of "finite type," a condition less restrictive than strong pseudoconvexity. Our concern here is primarily with the strongly pseudoconvex case; the interested reader should consult, e.g., [McNeal 1992] for information on the finite type situation.

At first sight, the "globalization" method we have been discussing might not seem to readily yield information about the Bergman metric, since the metric involves differentiation of the kernel function. However, a closer connection arises than might at first appear between the metric and holomorphic function globalization as follows. Choose the first two elements of an orthonormal basis of $A^2(\Omega)$ by setting $\varphi_0 = 1/\sqrt{\mathrm{Vol}(\Omega)}$, φ_1 a holomorphic function orthogonal to φ_0 with L^2-norm 1, and then complete φ_0, φ_1 to an orthonormal basis $\varphi_0, \varphi_1, \varphi_2, \ldots$. Since $K_\Omega(z,w) = \sum \varphi_j(z)\overline{\varphi_j(w)}$, direct computation yields

$$\begin{aligned}\text{Bergman length squared of } \frac{\partial}{\partial z_1} &= \frac{\partial^2}{\partial z_1 \partial \bar{z}_1} \log K_\Omega(z,z) \\ &\ge \frac{1}{2K_\Omega(z,z)^2} \left| \varphi_0(z) \frac{\partial \varphi_1}{\partial z_1} \right|^2 .\end{aligned}$$

(Here, we use the fact that $\frac{\partial \varphi_0}{\partial z_1} \equiv 0$.) Thus estimating from below the Bergman length of the vector $\frac{\partial}{\partial z_1}$ at $w \in \Omega$ can be accomplished by finding a holomorphic function $\varphi_w(z)$ with L^2-norm 1 and with $\frac{\partial}{\partial z_1}\varphi_w\big|_w$ bounded from below in absolute value, compared to the size of $K_\Omega(z,z)$, which we have already estimated. This falls into the same pattern as our previous globalization approach to estimating $K_\Omega(w,w)$.

From the viewpoint of applying the usual methods of Riemannian geometry, the crucial question is the completeness of the Bergman metric. Recall that if M is a connected Riemannian manifold and $p, q \in M$, then the distance dis (p, q) from p to q is defined to be the infimum of the length of piecewise C^1 curves from p to q. This definition of distance makes M a metric space with the metric space topology being the same as the manifold topology. The given Riemannian metric on M is said to be complete if the metric space $(M, \text{dis}\,(\cdot, \cdot))$ is complete in the usual Cauchy sense: Cauchy sequences converge. It is a well-known result in Riemannian geometry (the Hopf–Rinow theorem) that completeness of M is equivalent to closed bounded subsets of M being compact and either of these conditions is equivalent to all geodesics on M being infinitely extendible.

In the context of the Bergman metric on a bounded domain in $\mathbb{C}^n$, completeness is implied by "distance to the boundary being infinite" in the sense that any piecewise C^1 curve $\gamma : [0, 1) \to \Omega$ with finite total length must lie in a compact subset of Ω. (A suitable version of this implication holds for any Riemannian manifold, as does a suitable converse.) So one can deduce that Ω is complete in its Bergman metric if one can show that the Bergman length of vectors of unit Euclidean length goes to infinity sufficiently rapidly as the boundary of Ω is approached. Indeed one needs to show this only for vectors that are essentially normal to the boundary in a suitable sense. In this context, one now sees how the "globalization" technique already described could be used to establish completeness of the Bergman metric for C^∞ strongly pseudoconvex domains. This was carried out in detail in [Diederich 1973], where indeed detailed estimates of both kernel and metric were obtained:

Theorem 3.4.1 (Diederich). *The Bergman metric of a bounded, C^∞ strongly pseudoconvex domain in $\mathbb{C}^n$ is complete.*

This result was extended to domains satisfying successively weaker hypotheses over a period of years. This development culminated in the following result of Ohsawa, based on earlier work of Pflug ([Ohsawa 1981], [Pflug 1975]).

Theorem 3.4.2 (Ohsawa). *If Ω is a bounded, pseudoconvex domain in $\mathbb{C}^n$ with C^1 boundary, then the Bergman metric of Ω is complete.*

The "globalization" technique using $\overline{\partial}$ theory that we have been discussing is conceptually appealing, but it requires rather *ad hoc* specifics in each case. A more systematic and unified approach to the relationship between boundary shape at a point and the behavior of the Bergman kernel and metric near the point will be presented in Chapter 10. This "scaling method" is clearly related to the $\overline{\partial}$ results we have been discussing, but it considerably simplifies the technical details involved. We shall prove the preceding theorem (Theorem 3.4.1) of Diederich by this method there.

For strongly pseudoconvex domains, the idea that the behavior of the Bergman kernel near the boundary is predictable via "globalization" and

$\overline{\partial}$ methods achieved its ultimate form in an asymptotic expansion obtained by Fefferman ([Fefferman 1974]). The singular part of this expansion at a given point p in the boundary is determined by differential invariants of the boundary at p. The global structure of Ω thus appears only in the smooth, nonsingular part. To express this remarkable result precisely, we need some preliminary notation as follows.

Let Ω_0 be a bounded, strongly pseudoconvex domain in $\mathbb{C}^n$ having C^∞ boundary given by

$$\Omega_0 = \{z \in \mathbb{C}^n : \rho_{\Omega_0}(z) < 0\}.$$

Here ρ_{Ω_0} is a "C^∞ defining function" on $\mathbb{C}^n$ in the sense that $\nabla \rho_{\Omega_0} \neq 0$ on $\partial\Omega_0$. For a specific and useful instance of a defining function, let 'dis' denote Euclidean distance. Then the function

$$\rho(z) = \begin{cases} -\mathrm{dis}(z, \partial\Omega) & \text{if } z \in \Omega_0 \\ \mathrm{dis}(z, \partial\Omega) & \text{if } z \in \mathbb{C}^n \setminus \Omega_0 \end{cases}$$

is C^∞ near $\partial\Omega$ and satisfies $\nabla\rho \neq 0$ on $\partial\Omega$. One can obtain a defining function by extending ρ to all of $\mathbb{C}^n$ to be smooth everywhere, negative on Ω and positive on the complement of the closure of Ω, leaving ρ as defined near $\partial\Omega$.

Now, for $\delta > 0$ small, define

$$E^\delta_{\Omega_0} = \{(z, w) \in \mathrm{cl}(\Omega_0) \times \mathrm{cl}(\Omega_0) : \mathrm{dis}(z, w) + \mathrm{dis}(z, \partial\Omega_0) + \mathrm{dis}(w, \partial\Omega_0) < \delta\}.$$

Set

$$\begin{aligned} X_{\rho_0}(z, \zeta) = \rho_{\Omega_0}(\zeta) &+ \sum_\ell \left(\left. \frac{\partial \rho_{\Omega_0}}{\partial z_\ell} \right|_\zeta \right) \cdot (z_\ell - \zeta_\ell) \\ &+ \frac{1}{2} \sum_{\ell, m} \left(\left. \frac{\partial^2 \rho_{\Omega_0}}{\partial z_\ell \partial z_m} \right|_\zeta \right) \cdot (z_\ell - \zeta_\ell)(z_m - \zeta_m). \end{aligned}$$

The function X is commonly called the *Levi polynomial* of Ω.

In this notation, Fefferman's asymptotic expansion ([Fefferman 1974]) can be expressed as follows. There exist smooth functions ϕ_{Ω_0} and $\widetilde{\phi}_{\Omega_0}$ on $\mathrm{cl}(\Omega_0) \times \mathrm{cl}(\Omega_0)$—where $\mathrm{cl}(\Omega_0)$ = the closure of Ω_0 as usual—such that, for all $(z, \zeta) \in E^\delta_{\Omega_0}$,

$$K_{\Omega_0}(z, \zeta) = \frac{\phi_{\Omega_0}(z, \zeta)}{[-X_{\rho_{\Omega_0}}(z, \zeta)]^{n+1}} + \widetilde{\phi}_{\Omega_0}(z, \zeta) \cdot \log[-X_{\rho_{\Omega_0}}](z, \zeta). \tag{3.4}$$

Here, as usual, functions are considered to be smooth on the closed set $\mathrm{cl}(\Omega_0) \times \mathrm{cl}(\Omega_0)$ if they are restrictions to the closed set of a smooth function on some open neighborhood of $\mathrm{cl}(\Omega_0) \times \mathrm{cl}(\Omega_0)$ in $\mathbb{C}^n \times \mathbb{C}^n$.

The asymptotic expansion implies by direct calculation that the Bergman metric "blows up" at the boundary of Ω in essentially the same way as it does for the unit ball. Calculation with this expansion will recover the result that the Bergman metric of Ω_0 is complete: the metric grows near $\partial\Omega_0$ in such a way that curves of finite Bergman length must remain in compact subsets of Ω_0 (cf. our earlier discussion of completeness in this section).

More surprisingly, the asymptotic expansion (3.4) is also sufficient to establish a strong result about the curvature of the Bergman metric of a strongly pseudoconvex domain.[3]

Theorem 3.4.3 ([Klembeck 1978]). *Let Ω be a smoothly bounded, strongly pseudoconvex domain in $\mathbb{C}^n$. Then the holomorphic sectional curvatures of the Bergman metric of Ω converge uniformly to the constant $-4/(n+1)$ as the boundary is approached.*

More formally, one could express the conclusion as follows. Given $\epsilon > 0$, there is a $\delta > 0$ such that, if $x \in \Omega$ and $\mathrm{dis}(x, \mathbb{C}^n \setminus \Omega) < \delta$ and if P is a J-invariant 2-plane at x, then the (holomorphic) sectional curvature $\kappa(P)$ of the Bergman metric for the 2-plane P satisfies

$$-\delta - \frac{4}{n+1} < \kappa(P) < +\delta - \frac{4}{n+1}.$$

We recall that holomorphic sectional curvature has a very intuitively appealing geometric meaning. In the notation we have been using, given a J-invariant 2-plane P at $x \in \Omega$ (or in a complex manifold), consider all nonsingular holomorphic images $F(u)$ with F mapping a neighborhood U in $\mathbb{C}$ into Ω. Here, with $F(0) = x$ and with $F(U)$ tangent to P at x, i.e., $dF\big|_0(\text{tangent space of } \mathbb{C}) = P$, we think of $F(U)$ as a piece of Riemann surface through x. Such a "local surface" has a metric induced from the Bergman metric of Ω. Let κ_F be the Gauss curvature of this induced metric at x. Then the holomorphic sectional curvature $\kappa(P)$ is the maximum possible value of κ_F. This maximum is in fact always attained by any such surface which has second fundamental form 0 at x.

The asymptotic constancy result (Theorem 3.4.3) was obtained originally as part of P. Klembeck's Ph.D. dissertation research under the direction of one of the authors (Greene). We shall give a proof by the "scaling method" (to a more general statement, in fact) later in Chapter 10.

The asymptotic constancy result as stated actually implies the seemingly stronger result that the whole curvature tensor of the Bergman metric converges uniformly at the boundary to the standard curvature tensor of constant holomorphic sectional curvature $-4/(n+1)$. This and related matters of Kähler geometry are discussed in Section 3.6. But we have stated the result in terms of holomorphic sectional curvature rather than in terms of curvature tensor convergence because of the appealing directly geometric interpretation

[3]It is enlightening to consider this in the special instance of formula (3.4) when the domain in question is the unit ball. In that case the asymptotic formula (3.2) reduces to the standard formula for the Bergman kernel of the ball.

of holomorphic sectional curvature just noted. For almost all our later purposes, the Kähler geometric formalities of Section 3.6 can be ignored or, when needed, taken for granted, if desired.

At first sight, this asymptotic constancy result appears surprising. But actually it arises very logically. Let $\delta_{B^n}(z)$ denote the distance of the point z to the boundary of the unit ball. Now, on the unit ball, the Levi form of $\log(\delta_{B^n}(z))$ on the unit ball has asymptotically constant sectional curvature. This is not surprising at all: up to the constant $n+1$, the Bergman metric is the Levi form of $-\log(1-\sum_j |z_j|^2)$, and

$$-\log\left(1-\sum_j |z_j|^2\right) = -\log\left(1-\sqrt{\sum_j |z_j|^2}\right) - \log\left(1+\sqrt{\sum_j |z_j|^2}\right).$$

The first term is $-\log(\delta_{B^n}(z))$ while the second term is C^∞ at the boundary and hence could be expected to be negligible compared to the first term, which blows up at the boundary. Since the Bergman metric of the unit ball has constant holomorphic sectional curvature, the metric (near the boundary) arising from the Levi form of $-\log(\delta_{B^n}(z))$ is thus expected to have—and does have—asymptotically constant holomorphic sectional curvature as the boundary is approached. Of course, in this case of the unit ball, this assertion can be verified by explicit computation.

Now any strongly pseudoconvex domain matches the boundary of a biholomorphic image of the ball quite well in a neighborhood of a given boundary point (cf. [Fefferman 1974]). So it is natural to expect that the Levi form of $-\log(\delta_{B^n}(z))$ again has asymptotically constant sectional curvature. After all, the situation looks like the ball as far as the highest order terms are concerned. This is in fact true: this Levi form, which is positive definite, does give a metric of asymptotically constant sectional curvature as the boundary is approached.

Finally, according to the Fefferman expansion restricted to the case $z = \zeta$, the Bergman metric is obtained as the Levi form of the logarithm of a (positive) function, the highest order term of which is the $(n+1)$-st power of $\delta_{B^n}(z)$. Write δ_Ω for "distance to the boundary" for domain Ω in general. Then the Levi form of $\log K(z,z)$ is

$$\begin{aligned} L_{\log K(z,z)} = (n+1)\cdot (&\text{Levi form of a function having } -\log\delta_\Omega \text{ as} \\ &\text{the highest order term of its asymptotic expansion}). \end{aligned}$$

The holomorphic sectional curvature of $-L_{\log\delta_\Omega}$ is asymptotically constant, by our previous discussion of approximation by the unit ball. It is in fact asymptotically -4, as one sees from comparison to the Bergman metric of the ball and the usual curvature scaling under constant factors. So, to check that the Bergman metric has asymptotically constant sectional curvature $-4/(n+1)$, one need only check that the lower order terms of the Fefferman expansion do not disturb the situation asymptotically. Again, this is not surprising. [To follow the constant factors, recall that the curvature of λg, g a metric, is $(1/\lambda)$(curvature of g).]

While this all sounds plausible, one has to check it. In principle, higher order terms might be canceling out in the curvature calculation, making lower order terms significant! This in fact does not happen, and the outline given becomes a precise proof. The reader is referred to [Klembeck 1978] for the details of this (which we omit here), since we shall give a separate proof in Section 10.1 as noted above. This proof will actually give the result where the boundary is only assumed to be C^2 (so that the Fefferman expansion does not apply).

The asymptotic sectional curvature of the Bergman metric gives an interesting geometric view of the C^∞ case of the well-known result of Bun Wong [Wong 1977] that a C^2 strongly pseudoconvex bounded domain with noncompact automorphism group is necessarily biholomorphic to the unit ball. (This result was later extended by Rosay to remove the hypothesis of global strong pseudoconvexity; cf. [Rosay 1979].) In particular, the following corollary follows from the asymptotic constancy result:

Corollary 3.4.4. *Let Ω be a C^∞ strongly pseudoconvex domain. Suppose that* $\mathrm{Aut}\,(\Omega)$ *is noncompact. Then Ω is biholomorphic to the unit ball.*

We restrict our attention for the moment to the C^∞ case of this result, since we want to apply Theorem 3.4.3, which is for C^∞ domains only. Later Theorem 3.4.3 itself will be extended to the C^2 case (Theorem 10.1.1). The curvature proof that follows will then yield the C^2 version of Corollary 3.4.4.

To deduce this corollary, we first show that the Bergman metric of Ω has constant holomorphic sectional curvature. For this, suppose that H is a holomorphic (J-invariant) 2-plane in $T_p\Omega$. By Proposition 1.3.10 and the remarks immediately following it, the noncompactness of $\mathrm{Aut}\,(\Omega)$ implies that there is a sequence of automorphisms ϕ_j such that $\{\phi_j(p)\}$ diverges to the boundary, that is, it does not have a subsequence converging to an interior point of Ω. Passing to a subsequence, we can assume that $\phi_j(p)$ converges to a boundary point q. Since the automorphisms of Ω act as isometries of the Bergman metric, the holomorphic sectional curvature $\kappa(H)$ is equal to the holomorphic sectional curvature $\kappa(d\phi_j|_p(H))$ of the 2-plane at $\phi_j(p)$ that is the image under $d\phi_j|_p$ of H. Since $\phi_j(p)$ converges to q, Theorem 3.4.3 implies that $\kappa(d\phi_j|_p(H))$ converges to $-4/(n+1)$ as $j \to +\infty$. Thus $\kappa(H) = -4/(n+1)$.

Recall that (by Theorem 3.4.1) the Bergman metric is complete. So Ω has a complete (Bergman) metric of constant holomorphic sectional curvature $-4/(n+1)$.

By standard Riemannian geometry, the metric universal cover of Ω is the unit ball with its Bergman metric. But it remains to be seen that Ω is simply connected. This can be established by one of several methods.

First, one could appeal to the theorem of Lu Qi-Keng (Theorem 4.2.2) that a domain with complete Bergman metric of constant holomorphic sectional curvature is automatically simply connected, and hence biholomorphic to the ball (the constant being necessarily negative). This result is proved using

Bergman representative coordinates. It will be discussed in relation to that topic in Section 4.2.

Second, the information in Section 3.4 on the boundary behavior of the Bergman metric will show that, for any fixed $R > 0$, point q in the boundary of Ω, and Euclidean neighborhood U of q, there is an $\epsilon > 0$ such that if the Euclidean distance from q to a point $p \in \Omega$ is less than ϵ, then the Bergman metric ball of radius R around p is contained in U. This is an aspect of the Bergman metric "blowing up" at the boundary of Ω. [Note: This is not implied by the completeness of the Bergman metric as such. It involves the fact that the metric also blows up in directions parallel to the boundary, while completeness is guaranteed if one has sufficient blow-up in directions normal to the boundary.] Now suppose that γ is a loop (i.e., a closed curve) at p, with p, ϕ_j, and q as earlier. Then, if ϵ is chosen sufficiently small as above, it follows that, for j large, $\phi_j(\gamma)$ lies in a simply connected open set. This is because the smoothness of the boundary of Ω implies that, for $\epsilon > 0$ sufficiently small, the ϵ-radius Euclidean ball around q intersects Ω in a simply connected set, while if R is large enough, then the Bergman metric ball of radius R around p contains γ (by completeness of the Bergman metric). Since $\phi_j(\gamma)$ is freely homotopic to a constant loop, and since ϕ_j is a homeomorphism of Ω, γ is freely homotopic to a constant in Ω. Thus every loop in Ω is freely homotopic to a constant and Ω is simply connected.

A third, final, possible argument for simple connectivity is to use a peak-point argument to show that $\phi_j(\gamma)$ lies in a small Euclidean neighborhood of q, with the argument concluding then as in the previous paragraph (cf. [Rosay 1979] for the peak-point argument).

All three methods have been discussed here because each has further utility in other contexts.

Thus the simple connectivity of Ω follows from any one of these three methods, and hence Ω is biholomorphic to the unit ball in $\mathbb{C}^n$. This completes the proof of Corollary 3.4.4.

3.5 Stability of the Geometry of the Bergman Metric

Strong pseudoconvexity is an open condition: a C^2 small perturbation of a strongly pseudoconvex domain yields a domain that is again strongly pseudoconvex. This makes it natural to consider the question of stability under small perturbations of the geometry of the Bergman metrics of strongly pseudoconvex domains. Such a consideration turns out to produce interesting results about automorphism groups in particular.

The first step in making such general ideas precise is to define exactly what one means by a "small" perturbation. To do this, one needs in effect to define a topology on bounded domains. We shall be especially interested in the C^∞ topology on bounded domains with C^∞ boundary. The corresponding ideas for C^k, k finite, topologies are analogous and will not be discussed here explicitly.

To define the C^∞ topology, let Ω_0 be a bounded domain with C^∞ boundary $\partial\Omega_0$. In this setting, $\Omega_0 \cup \partial\Omega_0$ is a C^∞ manifold-with-boundary in the usual sense (see [Hirsch 1976] for instance). We define a neighborhood of Ω_0 in the C^∞ topology to be the set of all bounded domains Ω obtained as $F(\Omega_0)$ where $F : \Omega_0 \cup \partial\Omega_0 \to \mathbb{C}^n$ is restricted to lie in some C^∞ neighborhood of the injection map $\iota : \Omega_0 \cup \partial\Omega_0 \to \mathbb{C}^n$. Here we are taking for granted the C^∞ topology of C^∞ mappings of compact manifolds-with-boundary into $\mathbb{C}^n \cong \mathbb{R}^{2n}$. This topology in turn arises from the C^∞ topology on C^∞ functions since a map into $\mathbb{R}^{2n}$ is simply a $2n$-tuple of C^∞ functions.

The C^∞ topology on bounded domains with C^∞ boundary thus defined is equivalent to various other perhaps more easily envisioned possible definitions. In fact, all reasonable ideas of "C^∞ close" are equivalent. For example, one could say that Ω was C^∞ close to Ω_0 if Ω admits a defining function close to a given defining function for Ω_0 (close on some fixed neighborhood of $\partial\Omega_0$). Or one could consider Ω to be C^∞ close to Ω_0 if $\partial\Omega$ arises in the form $\{\mathbf{p} + f(p)\mathbf{N}(p) : p \in \partial\Omega_0\}$, where $\mathbf{N}$ is the (C^∞) exterior unit normal to $\partial\Omega_0$ and $f : \partial\Omega_0 \to \mathbb{R}$ is a C^∞ function which lies in some specified C^∞ neighborhood of the 0-function: Thus we say that f is "C^∞ small".

Note that in all cases we consider only Ω which have the property that $\Omega \cup \partial\Omega$ is diffeomorphic to $\Omega_0 \cup \partial\Omega_0$. This is surely a natural requirement for Ω to be C^∞ close to Ω_0.

Our first "stability" result about the Bergman geometry of strongly pseudoconvex domains is a version of asymptotic constancy of holomorphic sectional curvature that is locally uniform over variation of the domain:

Theorem 3.5.1 ([Greene/Krantz 1982]). *Let Ω_0 be a C^∞ strongly pseudoconvex, bounded domain in $\mathbb{C}^n$. For each $\epsilon > 0$, there is a $\delta > 0$ and a neighborhood $\mathcal{U}$ of Ω_0 in the C^∞ topology of bounded domains such that if $\Omega \in \mathcal{U}$ and $x \in \Omega$ with $\mathrm{dis}\,(x, \mathbb{C}^n \setminus \Omega) < \delta$, then, for every J-invariant 2-plane P at x, the holomorphic sectional curvature $\kappa(P)$ of P in the Bergman metric of Ω satisfies*

$$-\frac{4}{n+1} - \epsilon < \kappa(P) < -\frac{4}{n+1} + \epsilon.$$

As we have already explored in our discussion of Theorem 3.4.3, the asymptotic constancy of holomorphic sectional curvature provides a geometric way of distinguishing points near the boundary from at least some of the more interior points, when the domain is not the ball. To make this distinction "stable," we need the following result on the interior stability to supplement the boundary stability in the theorem just stated.

Theorem 3.5.2 ([Greene/Krantz 1982]). *Let Ω_0 be a C^∞ bounded, strongly pseudoconvex domain in $\mathbb{C}^n$ and let A be a compact subset of Ω_0. Then, for each $\epsilon > 0$, there is a neighborhood $\mathcal{U}$ of Ω_0 in the C^∞ topology such that*

(a) *For each $\Omega \in \mathcal{U}$, $A \subset \Omega$;*
and

(b) *For each $z \in A$ and each $\Omega \in \mathcal{U}$, and for each J-invariant 2-plane P at z, the holomorphic sectional curvatures $\kappa_\Omega(P)$ and $\kappa_{\Omega_0}(P)$ satisfy*

$$|\kappa_\Omega(P) - \kappa_{\Omega_0}(P)| < \epsilon.$$

Here $\kappa_\Omega(P)$ (resp. $\kappa_{\Omega_0}(P)$) are the holomorphic sectional curvatures attached to P in the Bergman metric of Ω (resp. Ω_0).

We have stated this result specifically for holomorphic sectional curvatures to emphasize its relationship to Theorem 3.4.3. But in fact the theorem just stated is a special case of a much more general fact: It is actually the case that the Bergman kernel and all its derivatives at points of A are stable under small perturbations of Ω_0. Not just the curvature, but the Bergman metric itself and all its derivatives at points of A are continuous functions as the domain Ω is varied (near Ω_0) in the C^∞ topology.

This latter stability can be put in a philosophically convincing context. Choose $\lambda > 0$ such that the distance of each point of A to any point of $\partial\Omega$ is at least 10λ, let us say. Choose a real-valued, C^∞, nonnegative "bump" function φ on $\mathbb{C}^n$ with support contained in a ball of radius λ around the origin such that $\varphi(z)$ depends only on $|z|$, $z \in \mathbb{C}^n$, and such that $\int_{\mathbb{C}^n} \varphi(z)\, dz = 1$.

By the mean value theorem, $f(z) = \int \varphi(w - z) f(w)\, dV(w)$ for each $f \in A^2(\Omega_0)$, or indeed for any analytic function on Ω_0, whether the function is L^2 or not.

Now define $\varphi_z(w) = \varphi(w - z)$. Let u_z be the C^∞ solution of the equation $\bar\partial \overline{u_z} = \bar\partial \varphi_z$ (in w-variables only) which has $u_z \in L^2$ and $\overline{u_z}$ orthogonal to the space $A^2(\Omega_0)$ in $L^2(\Omega_0)$. That is to say, $\overline{u_z}$ is the "canonical" solution of the $\bar\partial$ equation $\bar\partial v = \bar\partial \varphi_z$ so that $\int_{\Omega_0} f u_z = \int_{\Omega_0} f \overline{\overline{u_z}} = 0$ if $f \in A^2(\Omega_0)$. Clearly, if $f \in A^2(\Omega_0)$, then

$$f(z) = \int_{\Omega_0} f(w)\varphi_z(w) = \int_{\Omega_0} f(w)[\varphi_z(w) - u_z(w)].$$

Thus the function of w given by $\varphi_z(w) - u_z(w)$ has the "reproducing property" for the point z. Since $\partial_w(\varphi_z - u_z) = 0$ and $\varphi_z - u_z$ is L^2, it follows that $\varphi_z - u_z$ is in fact the Bergman kernel $K(z, w)$.

Thus the Bergman kernel is obtained from the canonical solution of the $\bar\partial$ equation. [This idea was the starting point for the development in [Fefferman 1974].] In particular, the interior stability results for perturbations of the domain Ω_0 can be established by proving C^∞ stability of the canonical solution. This stability is perhaps not surprising, and indeed seems almost to be taken for granted in [Folland/Kohn 1972], e.g., in the discussion of how to prove the Newlander–Nirenberg theorem by $\bar\partial$ methods. In any case, it is checked in detail in [Greene/Krantz 1982].

The boundary stability result Theorem 3.5.1 is an altogether more difficult matter. Here one needs stability of the "coefficient functions" in the Fefferman expansion with respect to the perturbation of the boundary. Then it is not hard to see that the computations of [Klembeck 1978] will give the theorem as stated. The details of the stability of the Fefferman expansion are lengthy and intricate. The sufficiently determined reader is invited to consult [Greene/Krantz 1982]. But it does not seem worthwhile to repeat those arguments here.

The C^∞ interior stability of the Bergman kernel can be established under quite general conditions not involving strong, or even weak, pseudoconvexity. For this, note first that the issue is really only one of C^0 stability: Think of $K(z, \overline{w})$ as a holomorphic function of variables z, w on $\Omega_0 \times \Omega_0$. If A is a compact set in Ω_0, then Cauchy estimates control the Bergman kernel K on $A \times A$ in the C^∞ sense if C^0 control is known.

On the other hand, the function $K(z, w)$ is known rather explicitly: if the holomorphic function $\phi : \Omega \to \mathbb{C}$ is such that its L^2-norm $\|\phi\|_2$ is minimal among all holomorphic functions in $A^2(\Omega)$ and that $\phi(z) = 1$ for $z \in \Omega$ (momentarily fixed), then

$$K(z, w) = \frac{\phi(z)\overline{\phi(w)}}{\|\phi\|^2}.$$

In the case of an expanding sequence of domains, this suffices to analyze the limiting behavior.

Theorem 3.5.3 ([Ramadanov 1967]). *Suppose that Ω_j, $j = 1, 2, \ldots$, is a sequence of bounded domains such that the closure of Ω_j is contained in Ω_{j+1} for all $j = 1, 2, \ldots$ and suppose that $\Omega_0 = \cup_j \Omega_j$ is bounded. Then, the Bergman kernel $K_{\Omega_j}(z, w)$ converges uniformly for z, w in any given compact subset of $\Omega_0 \times \Omega_0$ to $K_{\Omega_0}(z, w)$. Consequently, K_{Ω_j} converges C^∞ uniformly to K_{Ω_0} on compact subsets of $\Omega_0 \times \Omega_0$.*

For the proof, we refer to Sections 10.1.6. In fact, a slightly more general theorem is introduced there. See Theorem 10.1.4.

3.6 Further Observations on the Geometric Stability of the Bergman Curvature

> **Note to the Reader:** *The differential geometric details in this section are not needed in the remainder of the book except for the statement of Theorem 3.6.2. The remainder of the section can be omitted if desired as far as the rest of the book is concerned, if Theorem 3.6.2 is accepted as stated.*

The stability of curvature behavior near the boundary for the Bergman metrics of strongly pseudoconvex domains was presented in the previous section in terms of holomorphic sectional curvature. But, for some geometric applications, one really wants information on Riemannian sectional curvatures in general, not just the sectional curvatures attached to J-invariant 2-planes.

In the case of holomorphic sectional curvature close to a (negative) constant, information about sectional curvature follows on strictly algebraic grounds, from the algebra of Kähler curvature tensors.

In particular, we can apply the following result:

Proposition 3.6.1. *Suppose that $A > 0$ and that, at a point x in a Kähler manifold, there is a $\delta > 2/3$ such that the holomorphic sectional curvatures κ_h all satisfy $-A \leq \kappa_h \leq -\delta A$. Then all sectional curvatures κ at x satisfy*

$$-A \leq \kappa \leq -\frac{1}{4}(3\delta - 2)A.$$

References for this result include [Bishop/Goldberg 1963], [Berger 1967] and [Kobayashi/Nomizu 1963] vol. II, note 23, p. 369. This shows in particular that, if a Kähler manifold has at a point all holomorphic sectional curvatures sufficiently close to, say, $-4/(n+1)$, then at that point all sectional curvatures are negative and lie nearly in the interval $[-4/(n+1), -1/(n+1)]$. We shall use this result to establish the following.

Theorem 3.6.2. *There is a neighborhood $\mathcal{U}$ of the unit ball in $\mathbb{C}^n$ in the C^∞ topology of bounded domains in $\mathbb{C}^n$ such that, if $\Omega \in \mathcal{U}$, then the Bergman metric of Ω has all sectional curvatures negative at every point. More precisely, given $\epsilon > 0$, there is a neighborhood $\mathcal{U}_\epsilon$ such that if $\Omega \in \mathcal{U}_\epsilon$, $x \in \Omega$, and P is a 2-plane at x, then the sectional curvature $\kappa(P)$ satisfies*

$$-\frac{4}{n+1} - \epsilon < \kappa(P) < -\frac{1}{n+1} + \epsilon.$$

Proof. It suffices to prove the second statement. For this, choose a C^∞ neighborhood $\mathcal{V}$ of the unit ball and a number $\lambda > 0$ such that, if $\Omega \in \mathcal{V}$ and $\operatorname{dis}(x, \partial\Omega) < \lambda$, $x \in \Omega$, then the sectional curvatures of the Bergman metric of Ω at x lie in the interval $(-\epsilon - 4/(n+1), \epsilon - 1/(n+1))$. This is possible by Theorem 3.5.1 and the "δ-pinching" remarks just prior to the statement of the theorem. Now set $C = \{z\colon |z| \leq 1 - \lambda/3\}$. By Theorem 3.5.2, there is a neighborhood $\mathcal{W}$ (in the C^∞ topology of domains) of the unit ball such that if $\Omega \in \mathcal{W}$, then $C \subset \Omega$, and the sectional curvatures of the Bergman metric of Ω at points of C lie in the interval $(-\epsilon - 4/(n+1), \epsilon - 1/(n+1))$. We can and will also assume that if $\Omega \in \mathcal{W}$ then Ω lies in $\{z\colon 1 - \lambda/3 < |z| < 1 + \lambda/3\}$. Then $\mathcal{U} = \mathcal{V} \cap \mathcal{W}$ satisfies the conditions required in the theorem since if $x \in \Omega \in \mathcal{U}$, then either $x \in C$ or $\operatorname{dis}(x, \partial\Omega) < \lambda$. $\square$

The Bishop/Goldberg/Berger result on "δ-holomorphic-pinching" is a quantitative aspect of a general convergence statement:

(∗) *If x is a point of a Kähler manifold and c is a positive number then, given $\epsilon > 0$, there is a $\delta > 0$ with the following property. If all holomorphic sectional curvatures at x belong to the interval $(-c - \delta, -c + \delta)$, then*

$$\|R - R_c\| < \epsilon,$$

where R is the Riemann curvature tensor at x and R_c is the unique curvature tensor of holomorphic sectional curvature $-c$, and the norm is the usual Riemannian norm on tensors.

[Note: R_c can of course be given explicitly. It is

$$\begin{aligned}R_c(X,Y,Z,W) = -\frac{c}{4}\Big\{&g(X,Y)g(Z,W) - g(X,W)g(Y,Z)\\ &+g(X,JZ)g(Y,JW) - g(X,JW)g(Y,JZ)\\ &+2g(X,JY)g(Z,JW)\Big\}.\end{aligned}$$

See [Kobayashi/Nomizu 1963], vol. II, pp. 166–167.]

The convergence statement $(*)$ is established by making two polarization arguments quantitative. The first is the argument that if $R - R_c$ vanishes on 4-tuples X, JX, X, JX then it vanishes on 4-tuples X, Y, X, Y (see [Kobayashi/Nomizu 1963], p. 166). Second is the standard polarization expressing $R(X,Y,Z,W)$ in terms of 4-tuples $R(U,V,U,V)$. See for instance [Kobayashi/Nomizu 1963], vol. I. We omit straightforward details of making these arguments for the constant holomorphic sectional curvature case (implying that $R - R_c = 0$) work to show that nearly constant holomorphic sectional curvature implies $\|R - R_c\|$ small.

It is now clear that Theorem 3.6.2 implies a version dealing with curvature tensors.

Theorem 3.6.3. *Let Ω be a C^∞, strongly pseudoconvex, bounded domain in $\mathbb{C}^n$. Then given $\epsilon > 0$, there is a $\delta > 0$ such that, if $x \in \Omega$, dis$(x, \mathbb{C}^n \setminus \Omega) < \delta$, and X, Y, Z, W are vectors at x of unit length in the Bergman metric of Ω, then*

$$|R(X,Y,Z,W) - R_{-4/(n+1)}(X,Y,Z,W)| < \epsilon,$$

where $R_{-4/(n+1)}$ is the curvature-like tensor of constant holomorphic sectional curvature $-4/(n+1)$ defined earlier and R is the Riemann curvature tensor of the Bergman metric of Ω at x.

Similarly, Theorem 3.6.2 implies a stable result about curvature tensor convergence at the boundary.

Theorem 3.6.4. *Let Ω_0 be a C^∞ strongly pseudoconvex, bounded domain in $\mathbb{C}^n$. Then, given $\epsilon > 0$, there is a neighborhood $\mathcal{U}$ of Ω_0 in the C^∞ topology on domains and a $\delta > 0$ such that, if $\Omega \in \mathcal{U}$ and $x \in \Omega$ with dis$(x, \mathbb{C}^n \setminus \Omega) < \delta$, and if X, Y, Z, W are vectors at x with unit length in the Bergman metric of Ω at x, then*

$$|R_\Omega(X,Y,Z,W) - R_{-4/(n+1)}(X,Y,Z,W)| < \epsilon,$$

where R_Ω *is the Riemann curvature tensor of the Bergman metric of* Ω *at* x *and* $R_{-4/(n+1)}$ *is again the standard curvature tensor of holomorphic sectional curvature* $-4/(n+1)$*.*

Theorem 3.6.2 again implies Theorem 3.6.4 without appeal to the Berger and/or Bishop–Goldberg pinching results: the convergence statement $(*)$ on the previous page applies, as for Theorem 3.6.3.

4

Applications of Bergman Geometry

In this chapter, results will be presented that arise by combining geometric arguments with the asymptotic curvature constancy at the boundary (discussed in the previous chapter) and other aspects of the geometry of the Bergman metric. The completeness of the Bergman metric of strongly pseudoconvex domains (Theorem 3.4.2) fits the whole situation into the framework of global Riemannian geometry, the basic idea of which is that the global geometry of a complete Riemannian manifold is controlled by curvature. Without completeness, this property fails entirely (cf. [Gromov 1969]). But, with completeness in hand, one expects curvature information to control the geometry in many respects.

4.1 Applications of Stability near the Boundary

The first result to be discussed has to do with small perturbations of the unit ball in $\mathbb{C}^n$, $n \geq 2$. A perturbation of the unit disc in $\mathbb{C}$ that is small in the C^∞ sense produces a domain that is still biholomorphic to the unit disc, by the Riemann mapping theorem. But in $\mathbb{C}^n$, $n \geq 2$, perturbations of the unit ball are generically not biholomorphic to the unit ball. This can be seen from Tanaka-Chern–Moser theory, but it can also be established by using more elementary arguments involving only counting the parameters in biholomorphic mappings and in representations of the boundary. There are, at high jet levels, more parameters in boundary choice than in germs of biholomorphic mappings. Details of this idea, which goes back to Poincaré, can be found in [Fefferman 1974] or [Greene/Krantz 1981].

Theorem 4.1.1. *There is a neighborhood $\mathcal{U}$ of the unit ball in $\mathbb{C}^n$ in the C^∞ topology on domains such that every $\Omega \in \mathcal{U}$ is either*

(1) *biholomorphic to the unit ball*

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DOI 10.1007/978-0-8176-4622-6_4,

or else

(2) *Aut(Ω) has a fixed point, which is to say, there is an $x \in \Omega$ such that $\gamma(x) = x$ for every $\gamma \in Aut(\Omega)$.*

Proof. To begin with, choose $\mathcal{U}$ so that if $\Omega \in \mathcal{U}$, then Ω is (C^∞) strongly pseudoconvex. By Corollary 3.4.4, Ω is biholomorphic to the unit ball if $\mathrm{Aut}(\Omega)$ is noncompact. Now impose on $\mathcal{U}$ the additional conditions (via Theorem 3.6.2) that, if $\Omega \in \mathcal{U}$, then the Bergman metric has negative sectional curvatures and that, if $\Omega \in \mathcal{U}$, then Ω is diffeomorphic to the ball and hence simply connected. [This latter condition is taken for granted in general by our discussion of C^∞ topology. We reiterate it here for emphasis.]

With $\mathcal{U}$ satisfying these conditions, if $\Omega \in \mathcal{U}$ and Ω is not biholomorphic to the unit ball, then $\mathrm{Aut}(\Omega)$ is a compact group of isometries of a complete, simply connected manifold of everywhere negative sectional curvature—first, Ω with its Bergman metric. It is a standard theorem of E. Cartan (cf. [Klingenberg 1982], for example) that a compact group of isometries of a complete manifold of nonpositive sectional curvature has a fixed point. [The fixed point is obtained as the "center of gravity" of the orbit of any arbitrary point.] □

The fixed point theorem of E. Cartan that was applied to establish Theorem 4.1.1 is usually proved using the strict convexity of the square of the distance function. first, on a complete, simply connected Riemannian manifold with all sectional curvatures nonpositive, the function $\mathrm{dis}^2(\cdot, p_0)$ is C^∞, strongly convex for each point $p_0 \in M$. Indeed, its second derivative along each arclength-parameter geodesic is ≥ 2. This is an aspect of the Hessian comparison ideas developed in a more general context in [Greene/Wu 1977]. [It is also related to H. Karcher's proof ([Karcher 1989]; see also, e.g., [Klingenberg 1982], p. 226 ff) of the Toponogov comparison theorem ([Toponogov 1959]). But there the inequalities go the other way: nonnegative sectional curvature implies second derivatives ≤ 2.] But in the specific instance at hand, a direct proof by the second variation Formula is easy and standard.

With this convexity in mind, one establishes the existence of a fixed point for a compact group G of isometries of M as follows. Choose p_0 in M arbitrarily. Define $F : M \to \mathbb{R}$ by, for each $p \in M$, $F(p) = \int_{g \in G} \mathrm{dis}^2(g(p), p_0)\, dg$, where dg is the invariant measure on G. The function F is C^∞ and strongly convex; indeed, its second derivative along each arclength-parameter geodesic is ≥ 2, as one sees by differentiation under the integral sign. Moreover, completeness, the compactness of G, and the triangle inequality combine to show that F is proper. If p is far from p_0, then $F(p)$ is large because p is far from the compact set $\{g(p_0) : g \in G\}$. So F goes to infinity as p tends to infinity. Thus F has a unique minimum, say at the point q_0. But, because the function F is G-invariant—$F(g(x)) = F(x)$ for all $x \in M$, $g \in G$—this unique minimum is fixed by the elements of G. [Note that there is no claim that q_0 is the unique fixed point of the G action. A different choice of p_0 could potentially yield a different fixed point, and indeed the G action might have many fixed points.]

This argument admits a variant in which differentiability is brought less to the fore. This is a considerable digression, but it will make possible in a moment an equally considerable generalization of Theorem 4.1.1. In this variant, one considers, instead of the function F, convex sets associated to the situation.

Each closed ball $\mathrm{cl}(B(p_0, r)) \equiv \{q : \mathrm{dis}(q, p_0) \leq r\}$, $p_0 \in M$, is convex, because $\mathrm{dis}^2(\cdot, p_0)$ is a convex function. [The notion of convexity is unambiguous here since geodesic connections are unique on such manifolds.] Now, if G is not the one-element group, then, for small r, the set $\bigcap_{g \in G} \mathrm{cl}(B(g(p_0), r))$ is empty. On the other hand, if r is large, then, since G is compact, this intersection is definitely nonempty. Thus there is an $r_0 > 0$ such that the intersection is empty for $r < r_0$ and nonempty for $r > r_0$. One sees easily that $\bigcap_{g \in G} \mathrm{cl}(B(g(p_0), r_0))$ is nonempty but has empty interior. This set is clearly G-invariant.

At this point, one can bring into play a familiar "trick" of Riemannian geometry (cf. [Cheeger/Gromoll 1971]): a closed, convex set with empty interior (as a subset with possibly nonempty boundary) lies in a totally geodesic submanifold of M of lower dimension, which dimension can be chosen to be minimal. The group G acts on this unique, minimal-dimensional submanifold, so the argument can be repeated. Repetition yields eventually (since dimension drops at each stage) a compact, G-invariant, totally geodesic submanifold of M. But, for our particular M, such a submanifold must be a point: This follows from the strong convexity of $\mathrm{dis}^2(\cdot, q)$ for any point q chosen arbitrarily in M. [*Detail:* If N is a compact, totally geodesic submanifold of M with no boundary, then, for any $q \in M$, $\mathrm{dis}^2(\cdot, q)$ has a maximum value on N, say at $x \in N$. But then $\mathrm{dis}^2(\cdot, q)$ has a maximum at x along each geodesic through x. Thus $\mathrm{dis}^2(\cdot, q)$ is constant along such geodesics, contradicting strong convexity of $\mathrm{dis}^2(\cdot, q)$. This contradiction can be averted only if N consists of the point x alone.] We have gone into this matter in some detail because in fact this alternative line of reasoning enables Theorem 4.1.1 to be extended considerably. first, L. Lempert has proved the following (personal communication to the third author).

Theorem 4.1.2 (Lempert). *If G is a compact group of automorphisms of a convex, bounded, open domain Ω (convex in the usual Euclidean sense of the word), then G has a fixed point.*

The proof of this result is obtained by first showing that the balls in Ω relative to the Kobayashi metric are convex in the Euclidean sense ([Lempert 1981]). Then one can apply the geometric reasoning just discussed. In more detail: On a strongly convex domain with C^6 boundary, consider the convex sum of two extremal discs for the Kobayashi metric. The sum defines a holomorphic disc contained in the domain due to convexity. From this follows the Euclidean convexity of the Kobayashi distance ball for the strongly convex domain. Then the exhaustion of a bounded convex domain by strongly convex domains implies the Euclidean convexity for the Kobayashi distance ball for general convex domains. To obtain a fixed point of the compact subgroup G, consider the G-orbit of a point. As before in the Riemannian case, for a positive number r, the intersection, say S_r, of the closed balls of radius r centered at a point in the orbit is nonempty for some sufficiently large r. Take the smallest r for which S_r is nonempty. Then this S_r is convex and has empty interior. Thus it has dimension strictly less than that of the original domain. Equip S_r with the restricted Kobayashi distance. Then continue this process with S_r. This ends with a G-invariant 0-dimensional set which is convex and hence a single point. This is a fixed point of G.

To put Theorems 4.1.1 and 4.1.2 into context, one needs to know that, in general, a compact group of automorphisms of a C^∞ strongly pseudoconvex domain can be free of fixed points, even when the domain is homeomorphic or diffeomorphic to the ball. This is *not* obvious! Most compact topological group actions on balls that come to mind are conjugate to linear actions and hence have fixed points. And, *a fortiori*, examples of compact automorphism groups of domains homeomorphic to balls without fixed points are even less accessible.

Here, however, is a way to produce examples:

There is a finite group, say Γ, acting smoothly on S^7 with exactly one fixed point ([Stein 1976]; see also, for more on the general situation, [Petrie 1982]). This action can in fact be taken to be real analytic: this possibility is a general feature, once the existence of such a smooth action is known ([Illman 1994]). For any such (real analytic) action by Γ, a Γ-invariant Riemannian metric g_0 can be found by the usual averaging process. Then the complement in S^7 of every sufficiently small closed g_0-ball around the fixed point is real analytically diffeomorphic to a (standard) ball in $\mathbb{R}^7$ on which the finite group Γ acts real analytically and acts without fixed point. In this way, one obtains a bounded domain W in $\mathbb{R}^7$, diffeomorphic to the ball, such that W is real analytically acted upon by the finite group, without fixed points, and the closure of W is contained in a larger bounded domain V to which the group action extends real analytically, also without fixed points. The domain W (as well as V at the same time) can be taken to be real analytically equivalent to a standard ball. In fact, W can be taken to be a standard ball in $\mathbb{R}^7$.

By averaging, there is a group-invariant function $F : V \to \mathbb{R}$ such that F is real analytic and $W = \{p \in V \colon F(p) < 1\}$ and such that dF is nowhere zero on $\{p \in V \colon F(p) = 1\}$.

Now each element γ of the finite group Γ extends to be a biholomorphic map of some neighborhood V_γ of the closure of W in $\mathbb{C}^7$ into some open neighborhood of the closure of V. The intersection $\widehat{W} := \bigcap_{\gamma \in \Gamma} V_\gamma$ is a neighborhood in $\mathbb{C}^7$ of the closure of W.

Consider the function $y_1^2 + y_2^2 + \cdots + y_7^2$ on $\mathbb{C}^7$, where $z_j = x_j + \sqrt{-1} y_j$. By averaging and shrinking $\widehat{W}$ if necessary (while still keeping it a neighborhood of the closure of W), we obtain a group-invariant C^∞ function $\varphi \colon \widehat{W} \to \mathbb{R}$, say, such that $\varphi \geq 0$ and $\{p \in \widehat{W} \colon \varphi(p) = 0\}$ is the set where $y_j = 0$ for all

$j = 1, \ldots, 7$ and such that φ is strictly plurisubharmonic (since $y_1^2+y_2^2+\cdots+y_7^2$ is). Here, "group-invariant" does not mean that the set $\widehat{W}$ is invariant under the action by Γ but only that $\varphi(p) = \varphi(\gamma(p))$ for each $\gamma \in \Gamma$ and each $p \in \widehat{W}$.

Next, note that we can also average the function

$$(z_1, \ldots, z_7) \mapsto F(x_1, \ldots, x_7)$$

over the Γ-action, when $z = (z_1, \ldots, z_7)$ is in a neighborhood in $\mathbb{C}^7$ of the closure of W. This yields a group-invariant function $\widehat{F}$ on a small enough such neighborhood in $\mathbb{R}^7 \subset \mathbb{C}^7$.

Now consider $\widehat{F} + M\varphi$, where M is a (large) positive constant to be determined and let

$$\widetilde{W}_M := \{p \colon \widehat{F}(p) + M\varphi(p) < 1\}.$$

Then $W \subset \widetilde{W}_M$, since $F = \widehat{F} < 1$ on W and $\varphi = 0$ on W. Moreover, for M large enough, $\widetilde{W}_M$ is C^∞ strongly pseudoconvex because φ is C^∞ strictly plurisubharmonic. The nonvanishing of the gradient of $\widehat{F} + M\varphi$ at the boundary of $\widetilde{W}_M$ is easily checked. Finally, the domain $\widetilde{W}_M$ is group-invariant—the group Γ acts on it—because the defining function is group-invariant.

When M again is large enough, the group action on $\widetilde{W}_M$ is without fixed point. For, otherwise a limiting argument would produce a fixed point for the group action on W, since, as $M \to +\infty$, the domains $\widetilde{W}_M$ collapse to W.

This construction is of course quite general. It would apply to any finite group acting smoothly on a sphere with exactly one fixed point: the specific reference to S^7 is only an historical tribute to [Stein 1976]. Indeed, one could similarly deal with compact groups in general acting smooth on a sphere with one fixed point. Note also that the domain $\widetilde{W}_M$ cannot be biholomorphic to the ball, since every finite (or indeed compact) subgroup of automorphism group of the ball has a fixed point. Thus Aut $(\widetilde{W}_M)$ is a compact group (see Corollary 3.4.4) acting without fixed points on $\widetilde{W}_M$.

Now we explore results from the paper [Greene/Krantz 1981] that are based on Theorem 3.5.1, on the stability of Bergman metric curvature near the boundary of a C^∞ strongly pseudoconvex domain.

The following lemma will be pivotal to the considerations in this subsection.

Lemma 4.1.3. *Let Ω_0 be a fixed strongly pseudoconvex domain with C^∞ boundary that is not biholomorphic to the ball. Then there are a neighborhood $\mathcal{U}$ of Ω_0 in the C^∞ topology on domains, a number $\delta > 0$, and a point $p \in \Omega_0$ such that if $\Omega \in \mathcal{U}$ then $p \in \Omega$ and*

$$dis(f(p), \partial\Omega) \geq \delta$$

for all $f \in$ Aut (Ω).

Proof. According to Theorem 4.2.2, the holomorphic sectional curvature of the Bergman metric of Ω_0 is not constant. (Theorem 4.2.2 will be proved later by an argument independent of the present Lemma 4.1.3.) In particular, there is a constant $\lambda > 0$, a point $p \in \Omega_0$ and a J-invariant 2-plane P such that the sectional curvature $\kappa(P)$ of the Bergman metric of Ω_0 at p satisfies

$$\left|\kappa(P) + \frac{4}{n+1}\right| > \lambda.$$

From the stability result Theorem 3.5.2, there is a neighborhood $\mathcal{U}_1$ of Ω_0 in the C^∞ topology on domains such that $p \in \Omega$ if $\Omega \in \mathcal{U}_1$ and

$$\left|\kappa_\Omega(P) + \frac{4}{n+1}\right| > \frac{\lambda}{2}$$

for all $\Omega \in \mathcal{U}_1$, where $\kappa(P) =$ the sectional curvature of the 2-plane P at p for the Bergman metric of Ω. By Theorem 3.5.1, there is a C^∞ neighborhood $\mathcal{U}_2$ of Ω_0 and a constant $\delta > 0$ such that if $\Omega \in \mathcal{U}_2$, if $q \in \Omega$ with dis $(q, \mathbb{C}^n \setminus \Omega) < \delta$, and if Q is a J-invariant 2-plane at q, then

$$\left|\kappa_\Omega(Q) + \frac{4}{n+1}\right| > \frac{\lambda}{2}.$$

Now sectional curvature is invariant under isometries, and hence sectional curvatures of a Bergman metric are invariant under biholomorphic maps. Moreover, (the differentials of) biholomorphic maps take J-invariant 2-planes to J-invariant 2-planes. It follows that if $\Omega \in \mathcal{U}_1 \cap \mathcal{U}_2$, then the orbit of the point p under Aut (Ω) contains no points x with dis $(x, \partial\Omega) < \delta$. $\square$

Let $\Omega \subseteq \mathbb{C}^n$ be a domain. We say that Ω is *rigid* if Aut $(\Omega) = \{\text{id}\}$. In other words, Ω is rigid if the only biholomorphic mapping of Ω to itself is the identity mapping.

Theorem 4.1.4. *Let Ω_0 be a smoothly bounded, strongly pseudoconvex domain that is rigid. Then any sufficiently small C^∞ perturbation of Ω_0 is also rigid. In other words, the set of rigid, strongly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary is open in the C^∞ topology of domains.*

Remark. It follows from [Burns/Shnider/Wells 1978] (which uses the theory of Tanaka/Chern/Moser invariants [Chern/Moser 1974], [Tanaka 1965]) that the collection of all smoothly bounded, rigid, strongly pseudoconvex domains is dense in the collection of all smoothly bounded, strongly pseudoconvex domains. Actually, this density can be established without the use of invariant theory, just by parameter counting, by using systematically that the number of parameters at a given jet level for a hypersurface is larger than the number of parameters for local biholomorphic maps, as already discussed. Coupled with the result of the theorem, this implies that the collection of smoothly bounded strongly pseudoconvex domains with nontrivial automorphism group is residual—in the sense of the Baire category theory. The rigid domains are an open dense set (in the C^∞ topology on domains). Rigidity is "generic."

Proof of Theorem 4.1.4. The proof will be by contradiction: Suppose there is a sequence $\{\Omega_j\}_{j=1}^{\infty}$ of C^∞ strongly pseudoconvex domains converging in the C^∞ topology to a C^∞ strongly pseudoconvex domain Ω_0 with $\operatorname{Aut}(\Omega_0) = \{\mathrm{id}\}$ but such that, for each $j \geq 1$, $\operatorname{Aut}(\Omega_j) \neq \{\mathrm{id}\}$. Observe that if $\alpha_j : \Omega_j \to \Omega_j$ is a sequence of holomorphic mappings then, by standard normal families arguments, there is a subsequential limit mapping $\alpha_0 : \Omega_0 \to \mathrm{cl}(\Omega_0)$. Choose, for each j, $\alpha_j \in \operatorname{Aut}(\Omega_j)$, $\alpha_j \neq \mathrm{id}_{\Omega_j}$.

The domain Ω_0 is certainly not biholomorphic to the ball. So Lemma 4.1.3 tells us that there is a point $p \in \Omega_0$ and a number $\delta > 0$ such that the points $\{\alpha_j(p)\}$ lie in $\{z \in \Omega_j : \mathrm{dis}(z, \partial\Omega_j) > \delta\}$ for all sufficiently large j. In particular, we can be sure that $\{\alpha_j(p)\}$ lie in $\{z \in \Omega_0 : \mathrm{dis}(z, \partial\Omega_0) > \delta\}$ as long as j is sufficiently large. As a result, the mapping $\alpha_0 : \Omega_0 \to \mathrm{cl}(\Omega_0)$ must itself be an automorphism. (See Theorem 1.3.4.)

Since $\operatorname{Aut}(\Omega_0) = \{\mathrm{id}\}$, we conclude that $\alpha_0 = \mathrm{id}$. In order for us to obtain a contradiction, it suffices to show that the sequence $\{\alpha_j\}$ could have been selected to be bounded away from the identity, for all large j, on some compact subset of Ω_0. In so constructing the sequence α_j, we will (discarding a finite number of domains and mappings if necessary) take $p \in \Omega_j$ and $\mathrm{dis}(p, \partial\Omega_j) > \delta$ for all j.

We first claim that there is an $\epsilon > 0$ such that, if the orbit of p under $\operatorname{Aut}(\Omega_j)$ is contained in the Bergman metric ball on Ω_j of size ϵ around p, then there is a fixed point of $\operatorname{Aut}(\Omega_j)$ contained in this ball. To prove this claim, notice that the group $\operatorname{Aut}(\Omega_j)$ will be compact if the orbit of p is bounded in the Bergman metric; and if the orbit of p is contained in a sufficiently small ball about p, then that compact orbit will also have a unique Riemannian center of mass in the ball, which will be a fixed point of the group action. The required smallness of this ball is stable under C^∞ perturbation of the metric, hence under C^∞ perturbation of the domain. Hence that smallness can be chosen uniformly in j. This stability and consequent uniformity comes from the C^∞ interior stability of the Bergman metric and the usual conditions for existence of a unique Riemannian center of mass (cf. [Grove/Karcher 1973]).

Now, suppose that it is not possible to select $\alpha_j \in \operatorname{Aut}(\Omega_j)$ which are bounded away from the identity on the Euclidean ball of radius $\delta/4$ around p. Passing to a subsequence if necessary, we may assume that $\operatorname{Aut}(\Omega_j)$ restricted to this ball converges to the identity. Then, as we have previously noted, for all large j there will be a fixed point—call it p_j—for $\operatorname{Aut}(\Omega_j)$ with p_j in the Bergman metric ball of radius ϵ about p. [Here we are assuming, without loss of generality, that the Bergman metric balls of radius 2ϵ around p for the Bergman metrics of the Ω_j are all contained in the Euclidean ball of radius $\delta/4$ about p.]

Thus, for all large j, $\operatorname{Aut}(\Omega_j)$ is isomorphic to a subgroup H_j of the unitary group via the mapping $\alpha \mapsto d\alpha|_{p_j}$, as usual. Now here is the crux of the argument: since the unitary group does not contain arbitrarily small nontrivial subgroups, there is a positive constant $\eta > 0$ such that, for each sufficiently large j, there is an element $\beta_j \in \operatorname{Aut}(\Omega_j)$ with the distance of $d\beta_j\big|_{p_j}$ to the identity exceeding η (where distance is relative to some fixed bi-invariant metric on the unitary group). But this fact, together with the facts that the Bergman metrics of the Ω_j converge C^∞ to that of Ω_0 uniformly on the Euclidean ball of radius $3\delta/8$ about p and that the p_j lie in the fixed compact closed ball of Euclidean radius $\delta/4$ about p, implies that the action of the elements β_j does not converge to the identity on the Euclidean ball about p of radius $3\delta/8$. This contradiction completes the proof. □

A similar, but simpler, argument establishes the following result. We refer the reader to [Greene/Krantz 1981] for the details.

Theorem 4.1.5. *Each biholomorphic equivalence class is closed in the C^∞ topology on the set of C^∞ strongly pseudoconvex domains.*

4.2 Bergman Representative Coordinates

The Bergman kernel function gives rise not only to the Bergman metric, as already discussed, but also to some special local holomorphic coordinate systems which play a significant role in the study of biholomorphic mappings and in particular will be heavily used here. These local coordinate systems, known as Bergman representative coordinates, share certain properties with the geodesic normal coordinates of Riemannian geometry. In particular, biholomorphic mappings are linear when expressed in representative coordinates, in analogy with isometries being linear in geodesic normal coordinates. But geodesic normal coordinates are never holomorphic unless the (Kähler) metric is flat, that is, locally isometric to $\mathbb{C}^n$, while the Bergman representative coordinates are holomorphic in all cases where they are defined.

As we shall see, the Bergman representative coordinates provide a natural way to analyze, among other things, smoothness to the boundary of biholomorphic mappings. But this possibility was overlooked for some time by the mathematical community as a whole. Bergman himself suggested this use for representative coordinates at the 1975 AMS Summer Institute on Several Complex Variables in Williamstown, Massachusetts. This suggestion was treated with respect by the several hundred people who heard it there, as befitted Bergman's venerable age and his stature in the field. But the remark was almost, it seems, completely misunderstood. This is somewhat surprising in view of the great interest at that time in simplifying the latter part of Fefferman's then new paper [Fefferman 1974], in which the asymptotic expansion for the Bergman kernel obtained in the first part is shown by an intricate argument involving geodesics to imply boundary smoothness. As we shall see below, Bergman's suggested use of representative coordinates was exactly *a propos*: these coordinates provide precisely the right tool to obviate the analysis of geodesics and to go directly to smoothness to the boundary. [The later paper [Webster 1979] gives one method for implementing Bergman's idea, though without attribution to Bergman and hence, one supposes, independently.]

Bergman's representative coordinates are also involved in the proof of Lu Qi-Keng's theorem (Theorem 4.2.2) on bounded domains with Bergman metrics of constant holomorphic sectional curvature. This result will be stated in detail and proved in the present section.

We turn first to the definition of Bergman representative coordinates.

Let Ω be a bounded domain in $\mathbb{C}^n$ and let q be a point of Ω. The "diagonal" Bergman kernel $K_\Omega(q,q)$ is of course real and positive so that there is a neighborhood of q such that, for all z, w in the neighborhood, $K_\Omega(z,w) \neq 0$. Then for all z, w in that neighborhood, we define

$$b_j(z) = b_{j,q}(z) = \frac{\partial}{\partial \overline{w}_j} \log \frac{K(z,w)}{K(w,w)} \bigg|_{w=q}.$$

It is actually certain constant-coefficient linear combinations of these that will be the ultimate "Bergman representative coordinates," but we begin with the functions just defined. Note that these coordinates are well defined, independent of the choice of logarithmic "branches." Each $b_j(z)$ is clearly a holomorphic function of z.

Notice that some restriction on z to be in a neighborhood of q may be actually necessary, since it may be that $K_\Omega(z,w)$ vanishes for some pairs $(z,w) \in \Omega \times \Omega$.[1] In any event, the mapping

$$z \longmapsto \big(b_1(z), \ldots, b_n(z)\big) \in \mathbb{C}^n$$

is defined and holomorphic in a neighborhood of the point q. Note also that $(b_1(q), \ldots, b_n(q)) = (0, \ldots, 0)$.

We are hoping to use these functions, and later certain special linear combinations of them, as holomorphic local coordinates in a neighborhood of q. By the holomorphic inverse function theorem, these functions give local coordinates if the holomorphic Jacobian

$$\det \left(\frac{\partial b_j}{\partial z_k} \right)_{j,k=1,\ldots,n}$$

is nonzero at q.

But in fact the nonvanishing of this determinant at q is an immediate consequence of a fact that we have established already, first, that the Bergman metric is positive definite. To see this relationship, notice that

$$\begin{aligned} \frac{\partial b_j}{\partial z_k}\bigg|_{z=q} &= \frac{\partial}{\partial z_k} \left(\frac{\partial}{\partial \overline{w}_j} \log K(z,w) \right) \bigg|_{z=w=q} \\ &= \frac{\partial^2}{\partial z_k \partial \overline{z}_j} \log K(z,z) \bigg|_{z=q}. \end{aligned}$$

[1]The point w is involved only very near q, but variation of z over all of Ω might lead to zeros of $K(z,w)$. In fact the zeros of $K_\Omega(z,w)$ do actually arise, even when Ω is required to be topologically a ball; see, e.g., [Boas 1986].

This last term is of course the Hermitian inner product $\left\langle \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_j} \right\rangle\Big|_q$ with respect to the Bergman metric. Thus the expression

$$\det\left(\frac{\partial b_j}{\partial z_k}\right)\Bigg|_q$$

is the determinant of the inner product matrix of a positive definite Hermitian inner product. Hence this determinant is positive and, in particular, nonzero.

The utility of the new coordinates in studying biholomorphic mappings comes from the following lemma.

Lemma 4.2.1. *Let Ω_1 and Ω_2 be two bounded domains in $\mathbb{C}^n$ with $q_1 \in \Omega_1$ and $q_2 \in \Omega_2$. Denote by $b_1^1, \ldots, b_n^1$ the Bergman coordinates as defined near q_1 in Ω_1 (using the Bergman kernel for Ω_1) and $b_1^2, \ldots, b_n^2$ the Bergman coordinates defined in the same way near q_2 in Ω_2 (using the Bergman kernel for Ω_2). Suppose that there is a biholomorphic mapping $F : \Omega_1 \to \Omega_2$ with $F(q_1) = q_2$. Then the function defined near $0 \in \mathbb{C}^n$ by*

$$\begin{aligned}(\alpha_1, \ldots, \alpha_n) \longmapsto {} & \text{the } b^2\text{-coordinates of the } F\text{-image of the point} \\ & \text{of } \Omega_1 \text{ with } b^1\text{-coordinates } (\alpha_1, \ldots, \alpha_n)\end{aligned}$$

is a $\mathbb{C}$-linear transformation.

In short form, we say that biholomorphic mappings are linear when expressed in the Bergman representative coordinates b^j.

Proof of the lemma. To avoid confusion, we write $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ for the $\mathbb{C}^n$-coordinates in Ω_1 and $(Z_1, \ldots, Z_n)$ and $(W_1, \ldots, W_n)$ for the $\mathbb{C}^n$-coordinates in Ω_2. In addition, we write K_1 for K_{Ω_1} and K_2 for K_{Ω_2}. Now observe that, for each $j = 1, \ldots, n$,

$$\frac{\partial}{\partial \overline{w}_j} \log \frac{K_2(F(z), F(w))}{K_2(F(w), F(w))} = \frac{\partial}{\partial \overline{w}_j} \log \frac{K_1(z, w)}{K_1(w, w)}.$$

The reason for this identity is

$$\begin{aligned}\frac{K_2(F(z), F(w))}{K_2(F(w), F(w))} = {} & \frac{K_1(z, w)}{K_1(w, w)} \times (\text{a holomorphic function of } z) \\ & \times (\text{a holomorphic function of } w).\end{aligned}$$

This last follows from the transformation law—the factors that are conjugate holomorphic in w cancel out, since they are the same in numerator and denominator. Thus we obtain (from the complex chain rule) that

$$\begin{aligned}b_j^1(z) &\stackrel{\text{def}}{=} \frac{\partial}{\partial \overline{w}_j} \log \frac{K_1(z, w)}{K_1(w, w)}\Bigg|_{w=q_1} \\ &= \frac{\partial}{\partial \overline{w}_j} \log K_2(F(z), F(w)) - \log K_2(F(w), F(w))\Bigg|_{w=q_1} \\ &= \sum_k \left[\frac{\partial \overline{F}^k}{\partial \overline{w}_j} \cdot \frac{\partial}{\partial \overline{W}_k} \cdot \log \frac{K_2(F(z), W)}{K_2(W, W)}\right]\Bigg|_{W=F(q_1)},\end{aligned}$$

where F^k is the k-th coordinate of $F(w_1, \ldots, w_k)$. But this last expression is exactly

$$\sum_k \frac{\partial \overline{F}^k}{\partial \overline{w}_j}\bigg|_{w=q_1} \cdot b_k^2(F(z)).$$

Hence

$$b_j^1(z) = \sum_k \frac{\partial \overline{F}^k}{\partial \overline{w}_j}\bigg|_{w=q_1} \cdot b_k^2(F(z)).$$

Since the Jacobian matrix $(\partial F^k/\partial w_j)$ of F is invertible at q, it follows that the $b_k^2(F(w))$ are linear functions of the $b_j^1(z)$ coordinates, as required. $\square$

The lemma is sufficiently surprising to justify looking at an explicit example. Let $\Omega_1 = \Omega_2 =$ the unit disc in $\mathbb{C}$. Set $q_1 = a$ in the disc, and take $q_2 = 0$. Define

$$F(z) = \lambda \cdot \frac{z-a}{1-\overline{a}z}$$

for some complex λ of unit modulus. Then the b^1-coordinates at $q = a$ are the evaluation at $w = a$ of

$$\begin{aligned}\frac{\partial}{\partial \overline{w}} \log \frac{1/(1-z\overline{w})^2}{1/(1-w\overline{w})^2} &= -2\frac{\partial}{\partial \overline{w}}\left[\log(1-z\overline{w}) - \log(1-w\overline{w})\right] \\ &= -2\left(\frac{-z}{1-z\overline{w}} + \frac{w}{1-w\overline{w}}\right).\end{aligned}$$

Therefore

$$\begin{aligned}b^1(z) &= 2\left(\frac{z}{1-z\overline{a}} - \frac{a}{1-a\overline{a}}\right) \\ &= 2\left(\frac{z - za\overline{a} - a + za\overline{a}}{(1-z\overline{a})(1-a\overline{a})}\right) \\ &= \left(\frac{z-a}{1-\overline{a}z}\right) \cdot \frac{2}{1-a\overline{a}}.\end{aligned}$$

To get b^2-coordinates, we do the same calculations, but evaluate at 0 to obtain

$$b^2(z) = 2z.$$

Thus the biholomorphic map F takes the point z with b^1-coordinate α (equaling $2(z-a)/[(1-\overline{a}z)(1-\overline{a}a)]$) to the point with z-coordinate $\lambda(z-a)/(1-\overline{a}z)$ and hence with b^2-coordinate $2\lambda(z-a)/(1-\overline{a}z) = (1-\overline{a}a)\lambda \cdot b^1(z)$.

The mapping is indeed linear. The computationally inclined reader is invited now to see how the b^1-, b^2-coordinate setup enables one to regenerate the formula for the automorphisms (found in Section 3.3) of the ball in $\mathbb{C}^n$ that take, e.g., $(a, 0, \ldots, 0)$ to $(0, \ldots, 0)$; one need only be armed with the knowledge that the Bergman kernel for the unit ball in $\mathbb{C}^n$ is $c_n(1 - z \cdot \overline{w})^{-(n+1)}$. Of course, in practice, we used these biholomorphic mappings originally to actually compute the Bergman kernel, but it is still a matter of some interest to watch this regeneration of the maps in action.

The coordinates we have been discussing can be pushed one step further towards being truly "canonical," that is, dependent only on the complex structure. Let $q \in \Omega$ and let $V_1, \ldots, V_n$ denote holomorphic vector fields defined in an open neighborhood of q satisfying

$$\left\langle V_j, V_k \right\rangle \Big|_q = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

Then for each fixed $z \in \Omega$, we define

$$\beta_j = \overline{V_j} \log \frac{K_\Omega(z, w)}{K_\Omega(w, w)} \Big|_{w=q}.$$

[Hence the V_js act as differential operators only on the w-variables.] The proof that $(b_1, \ldots, b_n)$ defines a coordinate system at q can be easily modified to show that this map $z \mapsto (\beta_1(z), \ldots, \beta_n(z))$ is a well-defined holomorphic coordinate system at q.

Again, given a biholomorphic mapping $F : \Omega \to \widetilde{\Omega}$ and the respective Bergman representative coordinate systems $(\beta_1, \ldots, \beta_n)$ at $q \in \Omega$ and $(\widetilde{\beta}_1, \ldots, \widetilde{\beta}_n)$ at $F(q) \in \widetilde{\Omega}$, the map F takes expression in these coordinate systems as follows: there is a nonsingular complex matrix A_{jk} such that

$$\widetilde{\beta}_j = \sum_{k=1}^{n} A_{jk} \beta_k.$$

There are further properties. At the "center" q, the β-coordinate vector fields are orthonormal relative to the Bergman metric. (The same holds, of course, for $\widetilde{\beta}$-coordinates at $F(q)$.) It is these coordinates that we shall hereinafter call the *Bergman representative coordinates* of Ω at q. It of course remains true that biholomorphic mappings are linear in these coordinates. But in addition they are *unitary* linear mappings![2]

Notice that these coordinates themselves are unique up to a unitary rotation. Generally, one could not expect any further canonical aspect than that: Since unitary rotations act as biholomorphic maps on the unit ball, one cannot expect coordinates that are canonical beyond up-to-a-unitary-rotation. The Bergman representative coordinates are as canonical as holomorphic coordinates could be.

[2] On the other hand, the ordinary holomorphic (but nonbiholomorphic) mappings do not show any particular characteristic property in this coordinate system.

The Bergman representative coordinates are, as already noted, in some ways similar to geodesic normal coordinates, but with the additional property of being holomorphic. Further extraordinary properties will develop as we continue our discussion. Note, meanwhile, that the whole concept of representative coordinates extends essentially automatically to complex manifolds for which the Bergman metric construction for $(n, 0)$ forms already discussed above yields a positive definite metric. The construction can still be done locally, using general local holomorphic coordinates, and it remains true that the Bergman coordinates linearize holomorphic mappings. And again, the coordinates can be made more nearly canonical by using a basis for the differentiation that is orthonormal relative to the Bergman metric. A new point arises in that the quotient $K(z, w)/K(w, w)$ is not defined as such: it becomes defined only after a local coordinate choice around w and separately around z, if z is far from w. This turns out not to matter: this whole matter is discussed further in Chapter 11.

Our first application of Bergman representative coordinates is to the proof of a remarkable theorem of Lu Qi-Keng on domains with a Bergman metric of constant holomorphic sectional curvature.

Theorem 4.2.2 (Lu Qi-Keng). *If Ω is a bounded domain in $\mathbb{C}^n$, the Bergman metric of which is complete and has constant holomorphic sectional curvature, then Ω is biholomorphic to the unit ball.*

Notice that this result is certainly specific to the Bergman metric. For example, the annulus $\{\zeta \in \mathbb{C} : 1 < |\zeta| < R\}$, $R > 1$, admits a complete metric of constant (holomorphic) sectional curvature (see Section 2.3). But it is not even homeomorphic to the unit disc, much less biholomorphic to it.

This theorem has a complex manifold generalization: this is presented in Chapter 11.

Proof of Theorem 4.2.2. We first observe that the holomorphic sectional curvature, say c, must be negative. For, if c were positive, then Ω would be a complete Riemannian manifold with all sectional curvatures greater than or equal to $c/4 > 0$ (see Section 3.5).[3] Hence Ω would be compact by standard Riemannian geometry. [This is Myers's theorem: A complete Riemannian manifold with sectional curvature everywhere $\geq \epsilon > 0$ has diameter $\leq \pi/\sqrt{\epsilon}$ and is hence compact (cf. e.g., [Petersen 2006]).]

If c were zero, then the universal cover of Ω would be a complete, simply connected Kähler manifold of sectional curvature 0 and hence would be biholomorphically isometric to $\mathbb{C}^n$. But then, since Ω is bounded, the covering map into Ω would be constant by Liouville's theorem. This would contradict surjectivity of the covering map (to say the least!).

[3]This follows by the formula for Riemannian sectional curvature in case the holomorphic sectional curvature is constant. See Section 3.6 for the negative case: the positive case is the same up to the sign change.

It remains to discuss the case $c < 0$. If g_Ω is the Bergman metric of Ω (with constant negative holomorphic sectional curvature c), then the metric

$$g := -\frac{c(n+1)}{4} g_\Omega$$

has constant (negative) holomorphic sectional curvature $-4/(n+1)$ (cf. the remarks on scaling by constant factors at the end of Subsection 3.3.1). Thus the simply connected covering space $\widehat{\Omega}$ of Ω with the pullback $\widehat{g}$ of the metric g is a complete simply connected Kähler manifold with constant holomorphic sectional curvature $-4/(n+1)$. By standard Kähler geometry (cf. [Kobayashi/Nomizu 1963]), $(\widehat{\Omega},\ \widehat{g})$ is biholomorphically isometric to B^n with its Bergman metric. Thus we obtain a holomorphic covering map $F : B^n \to \Omega$ which is locally isometric for the Bergman metric on B^n and g on Ω, respectively.

To prove the theorem, we need only show that F is in fact injective.

For this let $q = F(0)$. Since F is a covering map, it is locally invertible. first, there exists an open neighborhood U of q and a neighborhood V of 0 such that $F|_V : V \to U$ is a biholomorphism. Denote by H_0 the inverse of $F|_V$.

With $z, w \in U$, let

$$K_0(z, w) := K_{B^n}(H_0(z), H_0(w)).$$

Then

$$\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K_0(z, z) = g_{j\bar{k}} = \lambda\ g_{\Omega j\bar{k}}$$

by the condition on F above, where $\lambda = -\frac{c(n+1)}{4}$. This implies that

$$\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K_0(z, z) - \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \lambda \log K_\Omega(z, z) = 0$$

for every $z \in U$, and furthermore that

$$\log K_0(z, w) - \lambda \log K_\Omega(z, w) = \varphi(z) + \overline{\varphi(w)}$$

for every $z, w \in U$, for some holomorphic function $\varphi : U \to \mathbb{C}$. Actually for this one may need to replace U by a smaller, simply connected neighborhood; but that can be done without loss of generality, here and in what follows. Consequently one obtains

$$\frac{\partial}{\partial \bar{w}_j} \log \frac{K_0(z, w)}{K_0(w, w)} - \frac{\partial}{\partial \bar{w}_j} \lambda \log \frac{K_\Omega(z, w)}{K_\Omega(w, w)} = 0$$

for every $z, w \in U$.

This last gives rise to the direct computation with Bergman's representative coordinate systems $b^1 : V \to \mathbb{C}^n$ and $b^2 : U \to \mathbb{C}^n$. As in the introduction for Bergman's representative coordinate systems, one obtains that

$$H_0(\zeta) = (F|_V)^{-1} = (b^1)^{-1} \circ A \circ b^2(\zeta) \qquad (\star)$$

for every $\zeta \in U$. Here, of course, A is the linear map represented by the matrix with the (j,k)-th entry

$$\lambda \overline{\frac{\partial F_k}{\partial z_j}\Big|_0}.$$

Now look at the expressions in $(\star)$. The map b^1 is in fact a constant multiple of the Euclidean coordinate system. Therefore it extends to all of $\mathbb{C}^n$ holomorphically, needless to say. So does the linear map A. The map $\zeta \to b^2(\zeta)$ extends to a holomorphic mapping of $\Omega \setminus Z_q$, where

$$Z_q = \{\zeta \in \Omega \mid K_\Omega(\zeta, q) = 0\}.$$

Since $K_\Omega(\cdot, q)$ is a holomorphic function on Ω with $K_\Omega(q,q) \neq 0$, the set Z_q is an analytic variety whose complex codimension in Ω is 1. Hence $\Omega \setminus Z_q$ is a connected, dense, and open subset of Ω. Therefore, using the expression of H_0 in $(\star)$, the map H_0 extends to a holomorphic mapping of $\Omega \setminus Z_q$ into $\mathbb{C}^n$. Let H denote this extension.

Now, let $X := F^{-1}(Z_q)$. Then one immediately sees that

$$X = \{z \in B^n \mid K_\Omega(F(z), q) = 0\}.$$

Since $K_\Omega(F(0), q) = K_\Omega(q,q) \neq 0$, we see that X is again a complex analytic subvariety of B^n with complex codimension 1. Thus $B^n \setminus X$ is a connected, dense, and open subset of B^n. Furthermore, $H \circ F : B^n \setminus X \to \mathbb{C}^n$ is holomorphic with $H \circ F(z) = z$ for every $z \in V$, as $H = H_0$ on V. This means that $H \circ F(z) = z$ for every $z \in B^n \setminus X$. Now, for every $\zeta \in \Omega \setminus Z_q$, choose $x \in B^n$ such that $F(x) = \zeta$. Then

$$H(\zeta) = H(F(x)) = x.$$

This implies that $H(\Omega \setminus Z_q) \subset B^n$.

We see that H is holomorphic on $\Omega \setminus Z_q$. The removable singularity theorem for bounded holomorphic maps (the Riemann extension theorem) yields that H extends to a holomorphic mapping of Ω into $\mathbb{C}^n$. Since H continues to play the role of left inverse of F, it follows easily that F has to be injective. This completes the proof. □

It is worthwhile to look back to see the exact role of completeness in this proof. Completeness in fact played no role in the construction of the local inverse which turned out to be a global, one-sided inverse. But completeness was used to get the holomorphic, locally isometric covering map from B to Ω in the first place. Without completeness, one would have only a locally defined covering map, and the subsequent arguments would not apply to inverting this map, it not being defined on all of B.

4.3 Equivariant Embedding and Concrete Realization of Abstract Complex Structures

Suppose that Ω is a bounded domain in $\mathbb{C}^n$ that contains the origin 0. There may be (nonidentity) elements of $\mathrm{Aut}(\Omega)$ that act on Ω as the restrictions to Ω of unitary linear transformations of $\mathbb{C}^n$, that is, as elements of $U(n)$. The set of such elements of $\mathrm{Aut}(\Omega)$ is clearly the set of restrictions to Ω of those elements α of $U(n)$ such that $\alpha(\Omega) = \Omega$. If every element of the isotropy group $I_0 = \{f \in \mathrm{Aut}(\Omega) : f(0) = 0\}$ arises in this fashion, then we say that Ω is *equivariantly embedded* in $\mathbb{C}^n$ at 0.

In this case, the mapping of I_0 into $U(n)$ defined by $f \mapsto df|_0$, with $f \in I_0$, is an injective, continuous isomorphism of I_0 into a compact subgroup of $U(n)$, with each element of this subgroup mapping Ω to itself. Thus the isomorphism of Corollary 1.3.7 becomes a concrete matter: the group of differentials, always isomorphic for any bounded Ω to the isotropy at a point, is literally the group of mappings itself. The obvious examples of this kind of behavior are balls and polydiscs centered at the origin. In fact, by Corollary 1.3.2, any complete circular domain has this equivariant embedding property. The following surprising result gives in effect an equivariant "re-embedding" of any domain close to the ball.

Theorem 4.3.1 (Greene–Krantz). *There is a neighborhood $\mathcal{U}$, in the C^∞ topology on domains, of the unit ball in $\mathbb{C}^n$ such that, for every $\Omega \in \mathcal{U}$, there is a biholomorphic map $F : \Omega \to \mathbb{C}^n$ with $F(0) = 0$ and with $F(\Omega)$ equivariantly embedded at 0.*

In the case $n = 1$, this result expresses the familiar fact (the Riemann mapping theorem) that a domain that is C^∞ close to the disc is biholomorphic to the disc via a biholomorphic mapping taking 0 to 0. The disc itself is of course equivariantly embedding at 0. But, for $n \geq 2$, the theorem is startling, just because the Riemann mapping theorem fails entirely even for domains C^∞ close to the ball. In general, Ω will definitely not be biholomorphic to the ball; also $F(\Omega)$ will be *not* the ball but some other domain that somehow exhibits the "abstract" symmetries of Ω around 0 as concrete geometric symmetries of $F(\Omega)$ that extend to be unitary rotations of $\mathbb{C}^n$ itself.

Proof of Theorem 4.3.1. It has already been observed that the expression of an automorphism in Bergman representative coordinates (around a point and its image) is a unitary linear transformation. Thus, taking F to be the Bergman representative coordinate map at 0 of Ω will do the job for the theorem, provided that the Bergman representative map is defined on all of Ω and is injective and nonsingular everywhere. These properties are *not* automatic; for example, for general bounded domains Ω, $K_\Omega(z, w)$ can have zeros even in cases where Ω is homeomorphic to the ball ([Boas 1986]). However, it turns out that the Bergman representative coordinate map $F_\Omega : \Omega \to \mathbb{C}^n$ at $0 \in \Omega$ is in fact an everywhere-defined holomorphic diffeomorphism onto a bounded, open set in $\mathbb{C}^n$ for all Ω that are sufficiently close in the C^∞ sense to the unit ball B^n.

To establish this last fact, note first that F_{B^n} is indeed a diffeomorphism. Indeed, it is the identity map of the ball to the ball (up to a dilation constant). This one checks by direct calculation. In particular, F_{B^n} extends to be a diffeomorphism of the closed ball $\mathrm{cl}(B^n)$ into $\mathbb{C}^n$, in the sense that it extends to the closure to be an injective C^∞ map with everywhere nonzero (real) Jacobian determinant.

The next step of the proof is to recall from basic differential topology (cf. [Munkres 1966]) that the property of being a diffeomorphism of a compact manifold-with-boundary into a Euclidean space is stable in the C^1 topology. In particular, a C^∞ mapping of the closed unit ball that is C^1 close to the identity will be such a diffeomorphism.

In our case, we are interested in a C^∞ mapping, the mapping via Bergman representative coordinates, not of the ball but of a domain Ω that is C^∞ close to the ball. But, following the usual terminology of differential topology, we say that a map $F : \mathrm{cl}(\Omega) \to \mathbb{C}^n$ is C^1 (or C^∞) close to a map $G : \mathrm{cl}(B^n) \to \mathbb{C}^n$ if there is a diffeomorphism $H : \mathrm{cl}(B^n) \to \mathrm{cl}(\Omega)$, H itself close to the identity, with $F \circ H$ close to the map G on B^n. Then it remains true that if F is C^1 close to a diffeomorphism in this sense, then it is itself a diffeomorphism (of $\mathrm{cl}(\Omega)$) into $\mathbb{C}^n$.

Thus the question of $F : \Omega \to \mathbb{C}^n$ being a diffeomorphism can be dealt with by showing that F extends C^∞ to $\mathrm{cl}(\Omega)$ and that $F : \mathrm{cl}(\Omega) \to \mathbb{C}^n$ is C^1 close to the G on B^n in the sense indicated.

At first sight this might seem difficult to establish: There are two direct approaches to the Bergman kernel. One is by its definition via the "reproducing property", that inner product with $K(z, w)$ gives the value at w for elements of $A^2(\Omega)$. The other is the formula for $K(z, w)$ in terms of an orthonormal basis for $A^2(\Omega)$. But neither of these seems amenable to producing information on the behavior of $K(z, w)$ with w fixed, z approaching the boundary. Interior behavior is more reasonably expected to be stable. (See Theorem 3.5.3, as well as Theorem 10.1.4.) But it turns out that the behavior of $K_\Omega(z, w)$, with w fixed in Ω, and z going to the boundary, can be effectively analyzed via the solution of the $\overline{\partial}$ problem as follows.

With $w \in \Omega$ fixed, let r be a positive number that is less than the distance of w to $\mathbb{C}^n \setminus \Omega$. Choose a nonnegative function $\rho : \mathbb{C}^n \to \mathbb{R}$ with $\rho(z)$ depending on $\|z\|$ only, and with $\rho(z) = 0$ if $\|z\| \geq r$ and finally with $\int_{\mathbb{C}^n} \rho(z)\, dV(z) = 1$. Then by the mean value property for each $f \in A^2(\Omega)$ this formula holds:

$$f(w) = \int_\Omega f(z)\rho(z - w)\, dV(z).$$

In particular, the reproducing (Bergman) kernel $K(z, w)$ with defining property

$$f(w) = \int_\Omega f(z)\overline{K(z, w)}\, dV(z)$$

is the L^2 projection of $\rho(z-w)$ onto $A^2(\Omega)$, with z being the variable and w fixed. This projection can be thought of as obtained via the solution of a $\overline{\partial}$ problem. first, let u be the solution (in $L^2(\Omega)$) of $\overline{\partial}u(z) = \overline{\partial}(\rho(z-w))$ which is orthogonal in $L^2(\Omega)$ to $A^2(\Omega)$. Then $K(z,w) = \rho(z-w) - u(z)$.

The solutions of $\overline{\partial}u = f$, where $\overline{\partial}f = 0$, with u orthogonal in $L^2(\Omega)$ to $A^2(\Omega)$, are of course the standard topics in the study of the $\overline{\partial}$-Neumann problem. In particular, in our case, when Ω is C^∞ close to B and hence strongly pseudoconvex, the indicated solution u of $\overline{\partial}u(z) = \overline{\partial}(\rho(z-w))$ is C^∞ on $\mathrm{cl}(\Omega)$. This is the usual smoothness-to-the-boundary result ([Folland/Kohn 1972]): note that $\overline{\partial}(\rho(z-w))$ is compactly supported in Ω and hence is itself obviously smooth on $\mathrm{cl}(\Omega)$.

Of course this method of finding $K(z,w)$ applies when $\Omega = B$ in particular. Thus the kind of C^1 closeness of $K_\Omega(z,w)$ to $K_B(z,w)$ that we are looking for can be considered from the viewpoint of the stability of the solution for the $\overline{\partial}$-Neumann problem under variation of the domain on which the solution is occurring. This stability seems eminently plausible. Indeed, it is assumed without further comment in Kohn's classic work on the $\overline{\partial}$-Neumann problem [Folland/Kohn 1972], where it is used to deduce the Newlander–Nirenberg theorem on integrable almost complex structures. But a completely explicit discussion of the stability issue can be found in [Greene/Krantz 1981], as part of the general discussion of the stability of the nondiagonal Bergman kernel and of the asymptotic expansion of the diagonal kernel function at the boundary.

There it is shown that, if Ω is sufficiently C^∞ close to a fixed, strongly pseudoconvex domain Ω_0, and if a $(0,1)$ form ω on $\mathrm{cl}(\Omega)$ with $\overline{\partial}\omega = 0$ is sufficiently C^∞ close to a (fixed) $(0,1)$ form ω_0 on $\mathrm{cl}(\Omega_0)$ with $\overline{\partial}\omega_0 = 0$, then the $\overline{\partial}$-Neumann solution of $\overline{\partial}u = \omega$ on Ω is C^∞ close on $\mathrm{cl}(\Omega)$ (i.e., in a given C^∞ neighborhood of) to the $\overline{\partial}$-Neumann solution of $\overline{\partial}u_0 = \omega_0$ on $\mathrm{cl}(\Omega_0)$. This is established via a detailed study of the standard proof of the regularity of the $\overline{\partial}$-Neumann problem.

This result implies the needed C^1 stability of Bergman representative coordinates to show that the Bergman map $F : \Omega \to \mathbb{C}^n$ via representative coordinates is a diffeomorphism. For Ω close to the unit ball and w close to 0, the $(0,1)$ form $\overline{\partial}_\Omega(\rho(z-w))$, w fixed, $\overline{\partial}$ calculated relative to z, is C^∞ close to $\overline{\partial}_{B^n}(\rho(z))$ if w is sufficiently close to 0. Our previous observation on the relationship between the $\overline{\partial}$ solution and the Bergman kernel implies that $K_\Omega(z,w)$ is uniformly C^∞ close to $K_{B^n}(z,w)$ for Ω which is C^∞ close to B^n and w in some fixed neighborhood of 0. Since $K_\Omega(z,w)$ is conjugate holomorphic in w, Cauchy estimates give that

$$\frac{\partial}{\partial \overline{w}} \log K_\Omega(w,w)\bigg|_{w=0} \quad \text{is uniformly close to} \quad \frac{\partial}{\partial \overline{w}} \log K_{B^n}(w,w)\bigg|_{w=0}$$

and that

$$\frac{\partial}{\partial \overline{w}} \log K_\Omega(z,w)\bigg|_{w=0} \quad \text{is } C^\infty \text{ close to} \quad \frac{\partial}{\partial \overline{w}} \log K_{B^n}(z,w)\bigg|_{w=0} \quad \text{on } \mathrm{cl}(\Omega).$$

Thus the Bergman representative coordinate map F_Ω for Ω at 0 is C^1 close to the Bergman representative coordinate map for the ball B^n, which is the identity (up to a constant factor). Thus the Bergman representative coordinate map F_Ω is a holomorphic diffeomorphism of $\mathrm{cl}(\Omega)$ into $\mathbb{C}^n$, and the proof of the theorem is complete. $\square$

The stability of the $\bar\partial$-Neumann solution under perturbation of the boundary of a strongly pseudoconvex bounded domain is a special case of a more general situation: Suppose that $\Omega_0 \cup \partial\Omega_0$ is a C^∞ manifold-with-boundary and that J_0 is an almost complex structure that is C^∞ on $\Omega_0 \cup \partial\Omega_0$ and integrable on Ω_0. In this situation, it makes sense to take as an hypothesis that $\partial\Omega_0$ is strongly pseudoconvex (cf. [Folland/Kohn 1972])—assume now that $\partial\Omega_0$ is indeed C^∞ strongly pseudoconvex. Suppose also that $\Omega_0 \cup \partial\Omega_0$ is given a C^∞ Hermitian metric. Then, if f is a C^∞ function on $\Omega_0 \cup \partial\Omega_0$, we may conclude that there is a unique function $u : \Omega_0 \to \mathbb{C}$ with $\bar\partial u = \bar\partial f$ on Ω and with u orthogonal to $A^2(\Omega)$ (in the inner product relative to the given Hermitian metric). Also u is C^∞ on $\Omega_0 \cup \partial\Omega_0$. [One can in fact so solve $\bar\partial u = \omega$, where ω is a $(0,1)$ form satisfying $\bar\partial\omega = 0$ and with ω having 0 harmonic representative. But the special situation where $\omega = \bar\partial f$, as indicated, suffices for our purposes, the harmonic representative being 0 following automatically in this instance.]

This setup has, as shown in [Greene/Krantz 1981] (and implied already in [Folland/Kohn 1972]), a stability similar to the stability associated to the stability under perturbation of a strongly pseudoconvex domain in $\mathbb{C}^n$ already discussed. first, let J be another almost complex structure on $\Omega_0 \cup \partial\Omega_0$ and let f be a C^∞ function on $\Omega_0 \cup \partial\Omega_0$ and J an almost complex structure tensor that is C^∞ close to J_0. If now f is C^∞ close to f_0 on $\Omega_0 \cup \partial\Omega_0$, then the $\bar\partial$-Neumann solution of $\bar\partial_J u = \bar\partial_J f$ is C^∞ close to the $\bar\partial$-Neumann solution of $\bar\partial_{J_0} u_0 = \bar\partial_{J_0} f_0$, provided that the $\bar\partial_J$ solution is determined for a J-Hermitian metric which is C^∞ close to the given J_0-Hermitian metric on $\Omega_0 \cup \partial\Omega_0$. This latter condition can always be arranged by setting $h =$ the J-symmetrization of h_0, i.e.,

$$h(\cdot,\cdot\cdot) = \frac{1}{2}\left(h_0(\cdot,\cdot\cdot) + h_0\left(J(\cdot), J(\cdot\cdot)\right)\right).$$

One could add into this picture the C^∞ perturbation of $\Omega_0 \cup \partial\Omega_0$ itself, but this would not actually increase the generality, since such a perturbation could be absorbed into perturbation of J_0 and f_0.

This more abstract form of $\bar\partial$-stability has an important application: it yields a proof of the perturbation result of Hamilton asserting that all perturbations of the complex structure of a bounded, strongly pseudoconvex domain can be realized by embedding ([Hamilton 1977]). This result was originally established by Hamilton using the Nash–Moser implicit function theorem. But the proof based on $\bar\partial$-stability in [Greene/Krantz 1981] is easier and more natural, and is also rather brief.

Theorem 4.3.2 ([Hamilton 1977]; cf. [Greene/Krantz 1981]). *If Ω_0 is a C^∞ bounded domain in $\mathbb{C}^n$ with strongly pseudoconvex boundary and if J is an almost complex structure defined and C^∞ on cl(Ω_0) which is integrable on Ω_0 and C^∞ close to the almost complex structure J_0 of $\mathbb{C}^n$ on $\Omega_0 \cup \partial\Omega_0$, then there is a domain Ω, C^∞ close to Ω_0 in the C^∞ topology on domains, such that (Ω_0, J) is biholomorphic to (Ω, J_0).*

In particular, every "abstract" perturbation of the ball is realized by a perturbation of the ball as a geometric object in $\mathbb{C}^n$.

Proof of Theorem 4.3.2. Let $f_1, \ldots, f_n$ be the coordinate functions on Ω_0, i.e.,

$$f_j(z) = \text{the } z_j \text{ coordinate function in } \mathbb{C}^n \text{ evaluated at the point } z.$$

Then $\overline{\partial}_{J_0} f_j \equiv 0$ for each $j = 1, \ldots, n$. If J is C^∞ close to J_0, then $\overline{\partial}_J f_j$ is C^∞ small on Ω_0. The stable $\overline{\partial}$ estimates then give that, if $\overline{\partial}_J u_j = \overline{\partial}_J f_j$ and u_j is the $\overline{\partial}_J$-Neumann solution of this equation, then each u_j is C^1 small in particular. [Here we use the construction described earlier for the automatic manufacture of a stably varying Hermitian metric for (Ω_0, J).] In particular, the n-tuple of functions $f_j - u_j$, $j = 1, \ldots, n$, gives a mapping which is C^1 close on $\Omega_0 \cup \partial\Omega_0$ to the mapping given by the f_js themselves, first the identity injection of Ω_0 into $\mathbb{C}^n$. In particular, the $f_j - u_j$, $j = 1, \ldots, n$, are coordinates of a diffeomorphism of $\Omega_0 \cup \partial\Omega_0$ onto an open set with smooth boundary in $\mathbb{C}^n$, by the C^1 stability of diffeomorphisms.

But the function $f_j - u_j$, each j, is J-holomorphic since $\overline{\partial}_J(f_j - u_j) = \overline{\partial}_J f_j - \overline{\partial}_J u_j \equiv 0$ on Ω_0. □

The idea of this last proof was originally proposed by M. Kuranishi and communicated to the first author (Greene) by J. Eells (private communication).

The uniqueness of the $\overline{\partial}$-Neumann solution, once a Hermitian metric is chosen, together with the proof method just used, makes possible an equivariant extension of Hamilton's embedding theorem. This result generalizes Theorem 4.3.1 to cases where equivariant embedding via Bergman representative coordinates cannot in general be obtained.

Theorem 4.3.3 ([Greene/Krantz 1982]). *Suppose that Ω_0 is a C^∞ strongly pseudoconvex domain in $\mathbb{C}^n$ and that G is a compact subgroup of* Aut (Ω_0). *Suppose further that Ω_0 is equivariantly embedded for G in the sense that G acts on Ω_0 as the restrictions of holomorphic isometries of $\mathbb{C}^n$. Let J be an almost complex structure on $\Omega_0 \cup \partial\Omega_0$ which is integrable on Ω_0 and is C^∞ close to the $\mathbb{C}^n$ complex structure J_0 on $\Omega_0 \cup \partial\Omega_0$ and let $\Gamma : G \times \Omega_0 \to \Omega_0$ be a G-action on Ω_0 which is J-holomorphic and C^∞ close to the original G-action on Ω_0. Then there is a diffeomorphism $F : \Omega_0 \cup \partial\Omega_0 \to \mathbb{C}^n$ such that:*

(1) *The mapping F is holomorphic as a map from (Ω_0, J) to $(\mathbb{C}^n, J_0)$.*
(2) *The mapping F is C^∞ close to the injection of Ω_0 into $\mathbb{C}^n$.*
(3) *The mapping $F \circ \Gamma \circ F^{-1}$, which is the G-action on $F(\Omega_0)$, is the restriction to $F(\Omega_0)$ of a G-action on $\mathbb{C}^n$ by holomorphic isometries of $\mathbb{C}^n$.*

(4) *The G-action on $\mathbb{C}^n$ given in* **(3)** *is C^∞ close to the original G-action on $\mathbb{C}^n$ attached to that equivariant embedding of Ω_0.*

Proof (outline). Let h_0 be the $\mathbb{C}^n$ Hermitian metric restricted to Ω_0 so that h_0 is invariant under the original G-action, say $\Gamma_0 \times \Omega_0 \to \mathbb{C}^n$, on Ω_0. Since Γ is C^∞ close to this original G-action, the average $\widehat{h}$ of h_0 with respect to the Γ-action is C^∞ close to h_0. Note that this is also C^∞ close to h_0 since Γ is C^∞ close to an action isometric for h_0. Observe further that $\widehat{h}$ may not be J-Hermitian, even though Γ acts by J-holomorphic maps, since h_0 is likely not J-Hermitian. but the J-symmetrization of $\widehat{h}$ already discussed, call it h, *is* J-Hermitian, and it is C^∞ close to h_0 since J is C^∞ close to J_0. This metric h is thus Γ-invariant, J-Hermitian, and C^∞ close to h_0.

Now let $f_1, \dots, f_n$ be the coordinate functions on Ω_0 so that G acts linearly on them, if we choose a suitable new origin in $\mathbb{C}^n$ (a compact group of isometries of $\mathbb{C}^n$ has a fixed point and we choose such a fixed point as origin). Let u_j be the $\overline{\partial}$-Neumann solution of $\overline{\partial}_J u_j = \overline{\partial}_J f_j$ determined by the Γ-invariant metric h. Since Γ acts almost linearly on the f_js, the mapping Γ acts almost linearly on the u_j s as well, because the $\overline{\partial}_J$ solution process is Γ-invariant. So Γ acts almost linearly on the holomorphic functions $f_j - u_j$ which, moreover, determine an embedding of $\Omega_0 \cup \partial\Omega_0$.

A standard process of making an almost-linear action linear, which will preserve J-holomorphicity, completes the construction of the desired equivariant J-holomorphic embedding. [The process involves replacing the functions $F_j = f_j - u_j$ by functions, which are C^∞ close, defined by

$$\big(\widehat{F}_1(z), \dots, \widehat{F}_n(z)\big) = \int_G \Gamma_0\big(g^{-1}, (F_1(gz), \dots, F_n(gz))\big)\, dg,$$

where $\int_G$ is the invariant (Haar) integral with total measure 1.][4] □

4.4 Semicontinuity of Automorphism Groups

Symmetry is easily destroyed but not so easily created. To make the straight crooked requires only an arbitrarily small effort, while to make the crooked straight requires a definite action.

These intuitions, that symmetry is unstable but an increase in symmetry requires a substantial change, holds with precision in a variety of circumstances. The goal of this section is a result of this type for the automorphism groups of C^∞ strongly pseudoconvex domains. This result will depend for its

[4] The reader unfamiliar with this process of converting close-to-linear to actually linear actions by way of re-embedding might find it instructive to consider the example in which G is the two-element group $\{1, g\}$ and $F(g(z))$ is close to $-F(z)$. Then the map $\widehat{F}$ defined by $z \mapsto [1/2](F(z) - F(g(z)))$ satisfies precisely $\widehat{F}(g(z)) = -F(z)$ so that G acts linearly indeed on the $\widehat{F}$ embedding, which really is an embedding since $\widehat{F}$ is in fact close to F.

proof on a theorem similar in spirit concerning compact Riemannian manifolds ([Ebin 1968]).

Theorem 4.4.1 (Ebin). *If (M, g_0) is a C^∞ compact Riemannian manifold, then there is a neighborhood $\mathcal{G}$ of g_0 in the C^∞ topology on C^∞ Riemannian metrics such that: If $g \in \mathcal{G}$ then there is a diffeomorphism $F : M \to M$ (C^∞ close to the identity) such that the set*

$$\left\{F \circ \alpha \circ F^{-1} : \ \alpha : M \to M \text{ is an isometry for } g\right\}$$

is a subset of, and hence a subgroup of

$$\{\beta : \ \beta : M \to M \text{ is an isometry for } g_0\} .$$

In particular, the group of isometries of M relative to g is isomorphic to a subgroup of the group of isometries of g_0.

Ebin's original proof of the theorem just stated involved infinite-dimensional manifolds and the construction of "slices" in the Lie group sense for the action of the diffeomorphism group on the manifold M. However, the result can in fact be established by finite-dimensional methods and ordinary Lie group theory. We outline the argument now.

Let

$$\begin{aligned} V_\Lambda = {} & \text{the finite-dimensional linear span of all eigenfunctions of the} \\ & \text{Laplacian for } g_0 \text{with eigenvalues} < \Lambda. \end{aligned}$$

[We use here the differential geometer's Laplacian $-\sum_j \partial^2/\partial x_j^2$ at the center of a geodesic normal coordinate system, so that the spectrum of the Laplacian is nonnegative.] If we equip V_Λ with the standard L^2 inner product on functions determined by the measure M for g_0, then the compact group of isometries for g_0 acts on V_Λ orthogonally. Moreover, if we choose an orthonormal basis $f_1, \ldots, f_N$ for V_Λ, then the map $E_0 : M \to \mathbb{R}^N$ defined by

$$M \ni p \mapsto (f_1(p), \ldots f_N(p))$$

is an embedding if Λ is chosen sufficiently large. This is an historic theorem of S. Bochner ([Bochner 1937], cf. [Greene/Wu 1975a] and [Greene/Wu 1975b] for a contemporary context and the noncompact manifold situation). With Λ so chosen, the embedding E_0 is equivariant in the sense that there is an injective homomorphism H_0 : [Isometry group of g_0] $\to O(N)$ such that, for each isometry α of g_0 and $p \in M$, $H_0(\alpha)$ applied to $E_0(p)$ equals $E_0(\alpha(p))$.

Now assume further that Λ is not in the spectrum of the Laplacian Δ_0 of g_0: this choice of course is possible consistently with the sufficient largeness of Λ of the previous paragraph, since the spectrum of Δ_0 is discrete. With Λ thus chosen, both sufficiently large and not in the spectrum of Δ_0, there is a "spectral stability" property of the equivariant embedding situation as follows.

Let g_j, $j = 1, 2, 3, \ldots$ be a sequence of C^∞ Riemannian metrics converging to g_0 in the C^∞ topology. Let $V_{\Lambda,j} =$ (the span of the eigenfunctions for the g_j-Laplacian Δ_j with eigenvalues $< \Lambda$). Then, for all j sufficiently large, the dimension of the finite-dimensional vector space $V_{\Lambda,j} =$ the dimension N of the space V_Λ defined earlier. Moreover, again for each j sufficiently large, there is a basis $(f_1^j, \ldots, f_N^j)$ for $V_{\Lambda,j}$, orthogonal with respect to the g_j-measure on M. These bases can be chosen so that, for each fixed $k \in \{1, \ldots, N\}$, the function f_k^j, $j = 1, 2, 3, \ldots$ converges to the function f_k in the C^∞ topology. This "spectral stability" result is part of the perturbation theory of linear operators; it is proved in detail in Kato's well-known book [Kato 1966] on that subject. [At first sight, these spectral stability results seem not only appealing but almost obvious, since the eigenfunctions of Δ_j are competitors, after suitable correction, for the minimization of Dirichlet integrals—the Rayleigh method—that gives eigenfunctions of Δ. But subtleties arise in any attempt to reason in the opposite direction, to control the eigenfunctions of Δ_j from those of Δ. These difficulties are treated in [Kato 1966] by the method of resolvents.]

From this we obtain embeddings $E_j : M \to \mathbb{R}^N$, for each j sufficiently large, which are equivariant for the isometry group of g_j. Moreover, the E_j's as constructed converge to E_0 in the C^∞ topology.

Let G_0 (the isometry group of g_0) be equal to the subgroup of $O(N)$ obtained by the equivariant embedding E_0, and $G_j =$ the subgroup arising in the same way from the isometry group of g_j and the equivariant embedding E_j.

Now, for any sequence $\{\alpha_j : M \to M\}$ such that α_j is an isometry of g_j for each $j = 1, 2, 3, \ldots$, there is a subsequence $\{\alpha_{j_k}\}$ which converges in the C^∞ topology to an isometry of g_0: this follows from a standard normal families argument. [Convergence to a g_0-distance-preserving map is immediate, and the limit must be a C^∞ isometry for g_0 by the Myers–Steenrod theorem [Myers/Steenrod 1939]. That the convergence is then in the C^∞ topology is a matter of standard differential geometry, using the facts that the isometries are determined by a single point image and differential at that point and that geodesics, which are preserved, depend C^∞ on the metric.] Thus, combining this with the C^∞ convergence of the E_j to E_0, we obtain the following.

If $\mathcal{U}$ is a neighborhood in $O(N)$ of G_0, then $G_j \subset \mathcal{U}$ for all j sufficiently large. By a standard result in Lie group theory ([Montgomery/Zippin 1942]), G_j is isomorphic to a subgroup of G_0 for each j sufficiently large, and this isomorphism is given by conjugation by an element A_j of $O(N)$. Here the A_j's can be taken to converge to the identity. Modifying the E_j's themselves by conjugation, we can assume that the A_j's are equal to the identity and $G_j \subset G_0$. Since E_j and E_0 are equivariant embeddings into $O(N)$, the desired diffeomorphism of M to M (to conjugate isometries of g_j into isometries of g_0) can be obtained by sending $p \in M$ to the $\mathbb{R}^N$-closest point to $E_j(p)$ in $E_0(M)$. □

The possibility of averaging over compact groups gives a useful corollary about group actions as such. For the statement of the corollary, we say that a sequence of C^∞ group actions $G_j \times M \to M$ *sub-converges* in the C^∞ topology to an action $G_0 \times M \to M$ if every sequence α_j of G_j-action elements has a subsequence α_{j_k} which converges in the C^∞ topology to a G_0-action element.

Corollary 4.4.2. *If $G_j \times M \to M$ is a sequence of actions on a compact manifold M by compact Lie groups G_j and if the G_j-actions sub-converge in the C^∞ topology to a compact Lie group action $G_0 \times M \to M$, then for all j sufficiently large, there is a diffeomorphism $F_j : M \to M$ such that the conjugation by F_j of the G_j-action is a subgroup of the G_0-action. Moreover, the F_j may be chosen to converge to the identity map of M in the C^∞ topology.*

This corollary follows from the proof of Ebin's theorem (Theorem 4.4.1) by averaging a fixed Riemannian metric over the group actions to produce G_j-invariant metrics g_j converging in C^∞ topology to a G_0-invariant metric g_0.

Generically, that is for a dense open set of metrics, the isometry group is in fact the identity alone (see [Ebin 1968]). Our interest here, however, is in the metrics which have a nontrivial isometry group.

The main goal of this section is to prove the statement analogous to Ebin's theorem (Theorem 4.4.1) for C^∞, strongly pseudoconvex domains.

Theorem 4.4.3 ([Greene/Krantz 1982a]). *If Ω_0 is a bounded, C^∞, strongly pseudoconvex domain in $\mathbb{C}^n$ that is not biholomorphic to the ball, then there is a neighborhood $\mathcal{U}$ of Ω_0 in the C^∞ topology (on bounded domains with C^∞ boundary) such that, if $\Omega \in \mathcal{U}$, then there is a real diffeomorphism $F : \Omega \to \Omega_0$ such that F is C^∞ close to the identity and*

$$\left\{F \circ \alpha \circ F^{-1} : \alpha \in Aut(\Omega)\right\} \subset \mathrm{Aut}\,(\Omega_0).$$

In particular, $Aut(\Omega)$ is isomorphic to a subgroup of $Aut(\Omega_0)$.

The essential idea of the proof of this theorem is to note, from Lu Qi-Keng's theorem (Theorem 4.2.2), that the Bergman metric of Ω_0 does not have constant holomorphic sectional curvature, while at the same time the holomorphic sectional curvature is asymptotically constant at the boundary. So far, this is just a recapitulation of the curvature proof of Bun Wong's theorem (Corollary 3.4.4, Theorem 9.2.1). Noting further that these curvature estimates are stable under C^∞ perturbations of $\partial\Omega_0$, one expects to find that the smooth extension to the closure $\mathrm{cl}(\Omega_0)$ of $\mathrm{Aut}\,(\Omega_0)$, guaranteed by Fefferman's result on smoothness to the boundary ([Fefferman 1974]) will also be stable under perturbation of $\partial\Omega_0$ in the following sense: If Ω is C^∞ close to Ω_0, then $\mathrm{Aut}\,(\Omega)$ on $\mathrm{cl}(\Omega)$ is C^∞ close to $\mathrm{Aut}\,(\Omega_0)$ on $\mathrm{cl}(\Omega_0)$ in the sense that each element of $\mathrm{Aut}\,(\Omega)$ belongs to some pre-chosen C^∞ neighborhood of $\mathrm{Aut}\,(\Omega)$ on $\mathrm{cl}(\Omega_0)$. Of course $\mathrm{cl}(\Omega_0)$ is a compact manifold with boundary so that Ebin's theorem (Theorem 4.4.1) as just stated and proved (for manifolds without boundary) does not apply as such. But, by passing to the "metric double" and introducing suitable automorphism-invariant metrics, we can apply Ebin's theorem on manifolds without boundary. We now turn to a more detailed version of the outline just given.

The detailed proof will be based on two propositions:

Proposition 4.4.4. *If Ω_0 is a C^∞ strongly pseudoconvex domain and if Ω_0 is not biholomorphic to the ball, then there are a point p in Ω_0, a compact set $K_0 \subset \Omega_0$, and a C^∞ neighborhood $\mathcal{V}$ of Ω_0 in the C^∞ topology on domains such that, if $\Omega \in \mathcal{V}$, then $\Omega \supset K_0 \cup \{p\}$ and the* $\mathrm{Aut}\,(\Omega)$*-orbit of p lies in K_0.*

This proposition has already been in effect established and is restated here only for convenience and clarity.

Proposition 4.4.5. *If Ω_0 is a C^∞ strongly pseudoconvex domain not biholomorphic to the unit ball then, for each $\ell = 1, 2, \ldots$, there are a C^∞ neighborhood $\mathcal{V}$ of Ω_0 and a positive constant C_ℓ such that, for each $\Omega \in \mathcal{V}$ and each $f \in \mathrm{Aut}\,(\Omega)$, the Euclidean derivatives of order $\leq \ell$ of f at points $p \in \Omega$ have absolute value $\leq C_\ell$.*

For brevity, we shall summarize this last statement by saying that

> The derivatives of order $\leq \ell$ of elements in $\mathrm{Aut}\,(\Omega)$ are stably uniformly bounded.

(where "stably" refers to variation of Ω near Ω_0 and "uniformly" refers to variation over the points of the domain Ω).

This proposition, which is in effect a stable version of the smoothness-to-the-boundary theorem by Fefferman, will be established later.

Armed with these propositions, we can now establish the following lemma of normal families type.

Lemma 4.4.6. *If Ω_j, $j = 1, 2, \ldots$, converge in the C^∞ topology to Ω_0 (with Ω_0 being C^∞, strongly pseudoconvex, and not biholomorphic to the ball), and if $g_j \in \mathrm{Aut}\,(\Omega_j)$, then there are subsequences Ω_{j_k}, g_{j_k}, $k = 1, 2, \ldots$, such that g_{j_k} converges in the C^∞ topology to an element $g_0 \in \mathrm{Aut}\,(\Omega_0)$.*

See the definition in Section 3.5 for C^∞ topology on the collection of domains in $\mathbb{C}^n$. Hereinafter, we write $G_j = \mathrm{Aut}\,(\Omega_j)$ and $G_0 = \mathrm{Aut}\,(\Omega_0)$. The lemma then says in effect that, for j large, the action of each element of G_j is close to the action of an element of G_0.

Proof of the lemma. Fix a point p and a compact set K_0 as in Proposition 4.4.4. Then, for j large, $g_j(p) \in K_0 \subset \Omega_j$. By normal families, there is a subsequence g_{j_k} which converges uniformly on each compact subset of Ω_0, and the limit of this subsequence is an element g_0 of G_0 (this follows from a straightforward modification of Theorem 1.3.4). Proposition 4.4.5 then implies the C^∞ convergence of $\{g_{j_k}\}$ on $\mathrm{cl}(\Omega_{j_k})$ (respectively to g_0 on $\mathrm{cl}(\Omega_0)$).

To check this last assertion in detail, it suffices to show that $\{g_{j_k}\}$ on $\mathrm{cl}(\Omega_{j_k})$ is a Cauchy sequence in the $C^{\ell+1}$ norm for each fixed $\ell = 1, 2, \ldots$. For this, suppose that $\epsilon > 0$ is given. Choose a compact set $K \subset \Omega_0$ such that, for all Ω which are C^∞ close enough to Ω_0 and $x \in \partial\Omega$, there is a polygonal arc in Ω, of length not exceeding $\epsilon/[3C_{\ell+1}]$, from some point $s \in K$ to the point x. [Here $C_{\ell+1}$ is the constant from Proposition 4.4.5.] The possibility of choosing K in this fashion is elementary: Simply let the set K be the $\epsilon/[4C_\ell]$ normal "push-in" of Ω_0.

Now choose k_0 so large that (from Cauchy estimates), $g_{j_{k_1}} - g_0$ and $g_{j_{k_2}} - g_0$ have C^ℓ-norm on K bounded above by $\epsilon/3$ if $k_1, k_2 \geq k_0$. For such k_1, k_2, the C^ℓ-norm of the difference $g_{j_{k_1}} - g_{j_{k_2}}$ is $\leq \epsilon$ on $\mathrm{cl}(\Omega_{k_1})$, $\mathrm{cl}(\Omega_{k_2})$ provided that k_1, k_2 are also required to be so large that Ω_{k_1}, Ω_{k_2} are sufficiently C^∞ close to Ω_0 and hence to each other. □

Lemma 4.4.7. *There is a neighborhood $\mathcal{V}$ of Ω_0 in the C^∞ topology on domains and a family g_Ω, $\Omega \in \mathcal{V}$, with g_Ω a C^∞ Riemannian metric on cl(Ω) such that,* **(1)** *if* $\mathrm{Aut}\,(\Omega)$ *acts isometrically on g_Ω and* **(2)** *if $\{\Omega_j\}$ is a sequence in $\mathcal{V}$ converging C^∞ to Ω_0, then $\{g_{\Omega_j}\}$ converges C^∞ to g_{Ω_0}.*

Proof. Set g_{Ω_0} equal to the average with respect to $\mathrm{Aut}\,(\Omega_0)$ of the Euclidean metric on $\mathrm{cl}(\Omega_0)$. For each $\Omega \neq \Omega_0$, choose diffeomorphisms $F_\Omega : \mathrm{cl}(\Omega) \to \mathrm{cl}(\Omega_0)$ such that F_Ω converges as Ω tends to Ω_0 in the C^∞ topology. Set g_Ω equal to the average over the compact (for $\mathcal{V}$ small enough) group $\mathrm{Aut}\,(\Omega)$ of the pullback metric $F_\Omega^* g_{\Omega_0}$. By Lemma 4.4.6, each element of $\mathrm{Aut}\,(\Omega)$ acts nearly isometrically on $F_\Omega^* g_{\Omega_0}$, in the C^∞ sense of "nearly," on $\mathrm{cl}(\Omega)$. This is because g_{Ω_0} is $\mathrm{Aut}\,(\Omega_0)$-invariant and each element of $\mathrm{Aut}\,(\Omega)$ is C^∞ close to an element of $\mathrm{Aut}\,(\Omega_0)$. The conclusion of the lemma concerning convergence follows. □

Lemma 4.4.8. *The metrics g_Ω in Lemma 4.4.7 can be chosen to be product metrics near the boundary.*

Here "the product metric" near the boundary of Ω means precisely that, for each boundary point x of $\mathrm{cl}(\Omega)$, there is a real local coordinate system $(x_1, x_2, \ldots, x_{2n})$ in a neighborhood of x with

- *the boundary* $\mathrm{cl}(\Omega) \setminus \Omega$ equaling $\{(x_1, x_2, \ldots, x_{2n-1}, 0)\}$;
- the points of Ω in the neighborhood of x satisfying $x_{2n} < 0$ (and vice versa);
- the metric in the given neighborhood having at $(x_1, x_2, \ldots, x_{2n})$ the form

$$dx_{2n}^2 + \big(\text{a positive definite quadratic formin } dx_1, dx_2, \ldots, dx_{2n-1} \\ \text{with coefficients depending only on } (x_1, x_2, \ldots, x_{2n-1})\big).$$

Proof of the lemma. An $\mathrm{Aut}\,(\Omega)$ product metric of this sort at and near the boundary is easily obtained using the map

$$\partial\Omega \times [0, \delta) \to \Omega$$

defined by

$$(b, t) \mapsto \exp_p(tN),$$

where N is the inward-pointing normal at b relative to the previous g_Ω-metric and $\exp_p$ is the g_Ω-exponential map. Choose δ so small that the map is a diffeomorphism and define the metric by declaring this diffeomorphism to be isometric for (the metric on $\partial\Omega$)$+dt^2$. This construction is $\mathrm{Aut}\,(\Omega)$-invariant. Using an $\mathrm{Aut}\,(\Omega)$-invariant partition of unity to make a transition to the previous g_Ω will provide all properties: the partition of unity function is taken to depend only on the t variable. $\square$

The proof of Theorem 4.4.3 can now be completed as follows: With the metrics g_Ω chosen as in Lemma 4.4.8, in particular as product metrics near the boundary, we form compact Riemannian manifolds $(\widehat{\Omega}, \widehat{g}_\Omega)$ by taking $\widehat{\Omega}$ to be the manifold "double" of Ω and $\widehat{g}_\Omega$ to be the natural metric on $\widehat{\Omega}$, equal to g_Ω on each copy of Ω and fitting together to form a C^∞ metric across the (one copy of) $\partial\Omega$ on account of the product metric. Let G_Ω be the group generated by $\mathrm{Aut}\,(\Omega)$ and the interchange operation I_Ω that interchanges the two copies of Ω that are "glued" to form $\widehat{\Omega}$. We now apply Ebin's theorem (Theorem 4.4.1) to deduce that the isometry group of $\widehat{\Omega}$ is diffeomorphism-conjugate (via a diffeomorphism close to the identity) to a subgroup H_Ω of the isometry group of $\widehat{\Omega}_0$. Now, by our previous analysis via normal families, H_Ω lies in a small neighborhood of G_{Ω_0} in the isometry group of $\widehat{\Omega}_0$. This isometry group is a compact Lie group and G_{Ω_0} is a compact, hence closed, subgroup and H_Ω is also compact and therefore closed. Standard Lie group theory yields that H_Ω is conjugate to a subgroup of $G_{\widehat{\Omega}_0}$ by way of an isometry of $\widehat{\Omega}_0$ close to the identity. Thus the diffeomorphism conjugation together with this second conjugation gives a close-to-the-identity diffeomorphism $F : \Omega \to \Omega_0$ conjugating $G_{\widehat{\Omega}}$ to $G_{\widehat{\Omega}_0}$.

Now $G_{\widehat{\Omega}_0}$ contains I_{Ω_0}. Also, the only possible fixed points of an element of $G_{\widehat{\Omega}}$ that is not preserving each copy of Ω are lying in $\partial\Omega$. It follows that F in fact maps $\partial\Omega$ diffeomorphically to $\partial\Omega$, and thus F, being close to the identity, maps Ω to Ω_0. As a result,

$$F\Big|_{\mathrm{cl}(\Omega)} : \mathrm{cl}(\Omega) \to \mathrm{cl}(\Omega_0)$$

is the conjugating diffeomorphism called for in the theorem. $\square$

The reader with a mind towards maximum generality will have noticed that complex analysis really played no role in the latter part of this proof. In particular, the proof technique gives rise to the following results.

Theorem 4.4.9 (Ebin's Theorem for Manifolds with Boundary). *If (M, g_0) is a compact, C^∞ Riemannian manifold with boundary, then there is a neighborhood $\mathcal{U}$ of g_0 in the C^∞ topology on Riemannian metrics such that, for each $g \in \mathcal{U}$, there is a diffeomorphism $F : M \to M$ (which can be chosen to be C^∞ close to the identity) such that, for each g-isometry $f : M \to M$, the mapping $F^{-1} \circ f \circ F$ is a g_0-isometry.*

Theorem 4.4.10. *If G_0 is a compact subgroup of the diffeomorphism group of a compact manifold (possibly with boundary), then there is a neighborhood $\mathcal{V}$ of G_0 in the C^∞ topology on the diffeomorphism group such that every compact subgroup G of the diffeomorphism group, with $G \subset \mathcal{V}$, is conjugate to* *a subgroup of* G_0 *via a diffeomorphism (which may be taken* C^∞ *close to the identity).*

The proofs of these results are obtained by extracting suitable portions of the proof of Theorem 4.4.3.

4.5 Obtaining a Stable Extension

Let K be a compact subset of Ω_0. Let ℓ be a positive integer. The Cauchy estimates then imply that there is a constant $C > 0$ such that

$$\left|\nabla^j \alpha(z)\right| \leq C$$

for all $\alpha \in \mathrm{Aut}\,(\Omega_0)$ and all $z \in K$. Thus the essential point in establishing Proposition 4.4.4 is to consider points near the boundary of Ω_0.

Lemma 4.5.1. *Let* $\epsilon > 0$ *be a positive number. Then*

$$\inf\{\mathit{dis}(\alpha(q), \partial\Omega_0) : \alpha \in \mathrm{Aut}\,(\Omega_0), q \in \Omega_0, \mathit{dis}(q, \partial\Omega_0) \geq \epsilon\}$$

is a positive number. [*Here, as usual, dis denotes Euclidean distance.*]

Proof. Suppose the contrary. Then there are a sequence $\{q_j\}$ of points in Ω_0 with $\mathrm{dis}(q_j, \partial\Omega_0) \geq \epsilon$ and a sequence of automorphisms $\alpha_j \in \mathrm{Aut}\,(\Omega_0)$ with

$$\lim_{j\to\infty} \mathrm{dis}(\alpha_j(q_j), \partial\Omega_0) = 0.$$

The sequence $\{\alpha_j\}$ is a normal family. By reasoning that has already been explained in detail, there is a subsequence $\{\alpha_{j_k}\}$ that converges normally to an automorphism $\alpha_0 \in \mathrm{Aut}\,(\Omega_0)$. Passing again to a subsequence, we may assume that $\{q_{j_k}\}$ converges to a point $q_0 \in \overline{\Omega_0}$.

But clearly $\mathrm{dis}(q_0, \partial\Omega_0) \geq \epsilon$, so q_0 actually lies in Ω_0 itself. As a result, $\alpha_0(q_0)$ is in Ω_0. But $\alpha_0(q_0)$ is the limit of the sequence $\alpha_{j_k}(q_{j_k})$ and also $\lim_{k\to\infty} \mathrm{dis}(\alpha_{j_k}(q_{j_k}), \partial\Omega_0) = 0$. In conclusion, $\mathrm{dis}(\alpha_0(q_0), \partial\Omega_0) = 0$ (since the distance function is continuous). This last statement contradicts the fact that $\alpha_0(q_0)$ lies in the interior of Ω_0. That is a contradiction. □

Lemma 4.5.2. *If* ϵ *is a positive number, then there is a* $\delta > 0$ *such that*

$$\sup\{\mathit{dis}(\alpha(q), \partial\Omega_0) : \alpha \in \mathrm{Aut}\,(\Omega_0), q \in \Omega_0, \mathit{dis}(q, \partial\Omega_0) \leq \delta\} < \epsilon.$$

Proof. The proof is similar to that of the last lemma, with a normal families argument now being applied to the inverses of the automorphisms. The details are left to the reader. □

Lemma 4.5.3. *Let* Ω_0 *be a strongly pseudoconvex domain with* C^∞ *boundary. Fix a point* $p_0 \in \partial\Omega_0$*. Then there are numbers* $\epsilon, \eta > 0$ *such that if* $z, w \in \Omega_0$*,* $\mathit{dis}(z, w) < \epsilon$*, and* $\mathit{dis}(w, p_0) < \epsilon$*, then* $|K_{\Omega_0}(z, w)| \geq \eta$*.*

Proof. This is an immediate consequence of the Fefferman asymptotic expansion (3.4) in Section 3.4. The details are again left to the reader. □

In the next lemma $\mathcal{J}_\Phi(z)$ denotes the complex Jacobian determinant of the mapping Φ at the point z.

Lemma 4.5.4. *If Ω_0 is a smoothly bounded, strongly pseudoconvex domain in $\mathbb{C}^n$, then there is a constant $C > 0$ such that*

$$\sup\{|\mathcal{J}_\alpha(z)| : \alpha \in \mathrm{Aut}\,(\Omega_0), z \in \Omega_0\} \leq C$$

and

$$\inf\{|\mathcal{J}_\alpha(z)| : \alpha \in \mathrm{Aut}\,(\Omega_0), z \in \Omega_0\} \geq C^{-1}.$$

Proof. The first estimate follows from the second by applying the result to α^{-1}. So we concentrate on the second.

Suppose that no such C exists. Then there are a sequence of automorphisms $\alpha_j \in \mathrm{Aut}\,(\Omega_0)$ and a sequence of points $q_j \in \Omega_0$ such that $\lim_{j\to\infty} \mathcal{J}_{\alpha_j}(q_j) = 0$. Passing to a subsequence if necessary, we may assume that the q_j converge to a point $q_0 \in \overline{\Omega_0}$.

We claim that $q_0 \in \partial\Omega_0$. For, if it were the case that $q_0 \in \Omega_0$, then Lemma 4.5.1 tells us that $\{\alpha_j(q_j)\}$ is bounded away from $\partial\Omega_0$. Hence, by the Cauchy estimates, $\{|\mathcal{J}_{\alpha_j^{-1}}(\alpha_j(q_j))|\}$ is bounded as $j \to +\infty$. This last is impossible since $\mathcal{J}_{\alpha_j^{-1}}(\alpha_j(q_j)) = 1/\mathcal{J}_{\alpha_j}(q_j)$ and $\lim \mathcal{J}_{\alpha_j}(q_j) = 0$.

So $q_0 \in \partial\Omega_0$, and there are, by Lemma 4.5.3, positive numbers ϵ and η such that $|K_{\Omega_0}(z,w)| \geq \eta$ if $z, w \in \Omega_0$ are within distance η of q_0. Therefore $|K_{\Omega_0}(q_0, r_0)| \geq \eta$ for any $r_0 \in \Omega_0$ with $\mathrm{dis}(q_0, r_0) < \epsilon$. Choose a fixed such r_0. It follow from Lemma 4.5.1 that $\liminf_{j\to\infty} \mathrm{dis}(\alpha_j(r_0), \partial\Omega_0) > 0$. Then, by the Cauchy estimates, it follows that $\limsup_{j\to\infty} |\mathcal{J}_{\alpha_j}(r_0)|$ is finite. But we also know that $\limsup_{j\to\infty} |K_{\Omega_0}(\alpha_j(q_j), \alpha_j(r_0))|$ is finite.

Now $K_{\Omega_0}(q_j, r_0) = \mathcal{J}_{\alpha_j}(q_j)\overline{\mathcal{J}_{\alpha_j}(r_0)} K_{\Omega_0}(\alpha_j(q_j), \alpha_j(r_0))$. Since $\lim_{j\to\infty} \mathcal{J}_{\alpha_j}(q_j) = 0$, the finiteness of the two limits-suprema just established now implies that $\lim K_{\Omega_0}(q_j, r_0) = 0$. But $\lim_{j\to\infty} K_{\Omega_0}(q_j, r_0) = K_{\Omega_0}(q_0, r_0) \neq 0$. This contradiction completes the proof of the lemma. □

Lemma 4.5.5. *If Ω_0 is a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary, then there exist $\epsilon, \eta > 0$ such that: If $w \in \Omega_0$ and $dis(w, \partial\Omega_0) < \epsilon$ and if $z \in \Omega_0$ and $dis(z,w) < [3/2]\, dis(w, \partial\Omega_0)$, then $|K_{\Omega_0}(z,w)| \geq \eta$ and $|det(\partial b_{i,w}/\partial z_j)| \geq \eta$, where the determinant is that of the complex Jacobian of the Bergman representative coordinate map $(b_{1,w}, \ldots, b_{n,w})$ at w.*

Proof. The basic bound $|K_{\Omega_0}(z,w)| \geq \eta$ can be deduced from Lemma 4.5.3 by a compactness argument. For the moment, it guarantees that the functions $b_{i,w}(z)$ are in fact defined for the z-values in question.

In order to study the Jacobian determinant $\det(\partial b_{i,w}/\partial z_j)$, notice first that

$$\begin{aligned}\frac{\partial}{\partial z_j} b_{i,w} &= \frac{\partial^2}{\partial z_j \partial \overline{w}_i}\left[\log \frac{K_{\Omega_0}(z,w)}{K_{\Omega_0}(w,w)}\right] \\ &= \frac{\partial^2}{\partial z_j \partial \overline{w}_i}[\log(K_{\Omega_0}(z,w)],\end{aligned}$$

because the expression $K_{\Omega_0}(w,w)$ has no z-dependence. Thus the relevant quantities can be calculated by substituting the asymptotic expansion for $K_{\Omega_0}(z,w)$ into the formula given. The following version of this substitution, and the subsequent calculations, is motivated by the somewhat simpler calculation when Ω_0 is the unit ball.

In order to calculate the boundary behavior of $\det[\partial b_{i,w}/\partial z_j]$ for a general strongly pseudoconvex domain Ω_0, and thus to complete the proof of Lemma 4.5.5, we shall use some standard notation as follows.

- $\psi : \mathbb{C}^n \to \mathbb{R}$ is a C^∞ function such that $\Omega_0 = \{z \in \mathbb{C}^n : \psi(z) > 0\}$ and $\nabla\psi$ is nonzero at every point of $\partial\Omega_0$,
- $X(z,w)$ represents the "Levi polynomial" of ψ, first,

$$\begin{aligned}X(z,w) := \psi(w) &+ \sum_{j=1}^{n} \frac{\partial \psi}{\partial w_j}\bigg|_w (z_j - w_j) \\ &+ \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \psi}{\partial w_j \partial w_k}\bigg|_w (z_j - w_j)(z_k - w_k),\end{aligned}$$

  and
- $\delta(w) := \mathrm{dis}(w, \partial\Omega_0)$.

Let $p_0 \in \partial\Omega_0$. For the moment, we restrict ourselves to the situation that $z, w \in \Omega$ satisfy:

$$|w - p_0| < \epsilon$$

and

$$|z - w| < \frac{3}{2}\delta(w).$$

Note that this implies $|z - p_0| \le 3\epsilon$. Choose ϵ sufficiently small so that, by a complex affine linear change of the coordinates in $\mathbb{C}^n$,

$$\begin{aligned}p_0 = (0, \ldots, 0); \qquad & \frac{\partial \psi}{\partial x_1}\bigg|_{p_0} = 1, \\ \frac{\partial \psi}{\partial y_1}\bigg|_{p_0} = \frac{\partial \psi}{\partial y_i}\bigg|_{p_0} = & \frac{\partial \psi}{\partial x_i}\bigg|_{p_0} = 0, \qquad i \ge 2,\end{aligned}$$

and

$$\frac{\partial^2 \psi}{\partial w_i \partial \overline{w}_j}\bigg|_{w=p_0} = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases},$$

where $i = 1, \ldots, n$. (first, $\psi(w) = \mathrm{Re}\, w_1 - |w_1|^2 - \ldots - |w_n|^2 +$ higher order terms.)

A term which has its absolute value not exceeding $C\delta^r$ for some constant C, as $\delta \to 0$, will be written $\lesssim \delta^r$. A term which is uniformly comparable in absolute value to δ^r (i.e., which has absolute value $\le C\delta^r$ and $\ge C^{-1}\delta^r$ for some positive constant C) as $\delta \to 0$ will be written $\sim \delta^r$. And, if the limit (as $\delta \to 0$) of the term divided by δ is 1, then the term will be written $\cong \delta$.

With this notation and $\delta = \delta(w)$:

1. $\psi(w) = (\cong\delta) = (\cong\mathrm{Re}\, w_1)$;
2. $\frac{\partial\psi}{\partial w_1} = \frac{1}{2} + (\lesssim\delta)$;
3. $\frac{\partial\psi}{\partial w_i} = (\lesssim\delta),\ i \ge 2$.

Therefore, for such w and z in Ω_0 with $|z - w| < \frac{3}{2}\delta(w)$, we see that

$$\begin{aligned} |X(z,w)| &= |(\cong\delta) + \frac{1}{2}(z_1 - w_1) + (\lesssim\delta^2)| \\ &\ge |(\cong\delta)| - \frac{3}{4}\delta - |(\lesssim\delta^2)| \\ &\ge \frac{1}{4}|(\cong\delta)| - |(\lesssim\delta^2)|. \end{aligned}$$

In particular, $X(z,w) = (\sim\delta)$ (the bound above is obvious).

The determinant $\det(\partial^2 b_{i,w}/\partial z_j \partial \overline{w}_i)$ becomes, upon substitution of the expansion

$$X^{-(n+1)}(z,w)[\varphi(z,w) + X^{(n+1)}(z,w) \cdot \widetilde{\varphi}(z,w) \log X(z,w)]$$

for $K_{\Omega_0}(z,w)$,

$$\begin{aligned} &(-1)^n(n+1)^n \det\Bigg[\frac{\partial^2}{\partial z_j \partial \overline{w}_i}(\log X(z,w)) \\ &\quad -(n+1)^{-1}\frac{\partial^2}{\partial z_j \partial \overline{w}_i}\log(\varphi + X^{n+1}(z,w)\widetilde{\varphi}\log X(z,w))\Bigg]_{i,j=1}^n . \end{aligned}$$

Now

$$\frac{\partial^2}{\partial z_j \partial \overline{w}_i}\log X(z,w) = X^{-1} \cdot \frac{\partial^2 X}{\partial z_j \partial \overline{w}_i} - \frac{\partial X}{\partial z_j}\frac{\partial X}{\partial \overline{w}_i} \cdot X^{-2}.$$

Thus, up to a nonvanishing absolute constant factor, the determinant to be evaluated is

$$\begin{aligned} &X^{-2n}\det\Bigg[X \cdot \frac{\partial^2 X}{\partial z_j \partial \overline{w}_i} - \frac{\partial X}{\partial z_j}\frac{\partial X}{\partial \overline{w}_i} \\ &\quad - (n+1)^{-1}X^2 \cdot \frac{\partial^2}{\partial z_j \partial \overline{w}_i}\log(\varphi + X^{n+1}\widetilde{\varphi}\log X)\Bigg]_{i,j=1}^n . \end{aligned} \tag{4.5.2}$$

The terms in the determinant can be easily checked to have the following order-of-magnitude behavior:

$$X^2 \frac{\partial^2}{\partial z_j \partial \overline{w}_i} \log(\varphi + X^{n+1}\widetilde{\varphi} \log X) = (\lesssim \delta^2)$$

[since $\varphi(p_0, p_0) \neq 0$]. Also,

$$\begin{aligned}
X \frac{\partial^2 X}{\partial z_j \partial \overline{w}_i} &= (\lesssim \delta^2), \qquad i \neq j \\
X \frac{\partial^2 X}{\partial z_i \partial \overline{w}_i} &= -X + (\lesssim \delta^2) = (\sim \delta), \qquad i = 1, \ldots, n \\
\frac{\partial X}{\partial z_1} \frac{\partial X}{\partial \overline{w}_1} &= (\sim 1) \\
\frac{\partial X}{\partial z_1} \frac{\partial X}{\partial \overline{w}_i} &= (\lesssim \delta), \qquad i \neq 1 \\
\frac{\partial X}{\partial z_j} \frac{\partial X}{\partial \overline{w}_1} &= (\lesssim \delta), \qquad j \neq 1 \\
\frac{\partial X}{\partial z_j} \frac{\partial X}{\partial \overline{w}_i} &= (\lesssim \delta^2), \qquad i \neq 1, j \neq 1.
\end{aligned}$$

Thus the entire expression (4.5.2) becomes

$$(\sim \delta)^{-2n} \det \begin{bmatrix}
(\sim 1) & (\lesssim \delta) & \cdots & \cdots & \cdots & (\lesssim \delta) \\
(\lesssim \delta) & (\sim \delta) & & & & \\
\vdots & & (\sim \delta) & & (\lesssim \delta^2) & \\
\vdots & & & \ddots & & \\
\vdots & & (\lesssim \delta^2) & & \ddots & \\
(\lesssim \delta) & & & & & (\sim \delta)
\end{bmatrix}$$

[The diagonal entries are of size $(\sim \delta)$ except the $(1,1)$-entry; the off-diagonal entries except the first row and the first column are of size $(\lesssim \delta^2)$.] Thus, the determinant of the Jacobian of the Bergman representative coordinate map at w p_0 is of $(\sim \delta(w)^{-(n+1)})$.

It is time to establish Lemma 4.5.5. By compactness of $\partial \Omega_0$, one can choose finitely many boundary points and associated ϵ-balls around them and corresponding ws from each ball to end up with an ϵ-neighborhood of the boundary $\partial \Omega_0$ for which the Jacobian determinant of the Bergman representative coordinate map is bounded away from zero. $\square$

Proof of Proposition 4.4.5. Now we give (at long last) the proof of Proposition 4.4.5. The basic idea is to exploit the fact that, in Bergman representative coordinates, an automorphism is given by a linear map. Thus estimation of its derivatives can be accomplished by estimating (1) its differential and (2) the relationship between representative coordinates and Euclidean coordinates.

Now the proof proceeds by contradiction. If the conclusion is false, then there are

(i) a sequence of domains Ω_ν converging in the C^∞ topology to a limit domain Ω_0;
(ii) a sequence $\{\alpha_\nu : \Omega_\nu \to \Omega_\nu\}$ of automorphisms;

and

(iii) a sequence of points $\{p_\nu \in \Omega_\nu\}$ and a Euclidean differential operator

$$\mathcal{D} = \left(\frac{\partial}{\partial z_1}\right)^{j_1} \left(\frac{\partial}{\partial z_2}\right)^{j_2} \cdots \left(\frac{\partial}{\partial z_n}\right)^{j_n}, \quad j_1, \ldots, j_k > 0,$$

with

$$\lim_{\nu \to \infty} |\mathcal{D}\alpha_\nu(p_\nu)| = +\infty.$$

Passing to a subsequence, we may assume that the sequences $\{p_\nu\}$ and $\{\alpha_\nu(p_\nu)\}$ converge to points $p_0, q_0 \in \mathrm{cl}(\Omega_0)$, respectively. We also may assume that both $\{\alpha_\nu\}$ and $\{\alpha_\nu^{-1}\}$, respectively, converge uniformly on compact subsets of Ω_0 to an automorphism α_0 of Ω_0 and its inverse α_0^{-1}, respectively (the possibility of establishing this last assertion was treated in Section 4.1 as well as in [Greene/Krantz 1981]). Now repeat the reasoning used in the proof of Lemma 4.5.4 to show that $p_0 \in \partial\Omega_0$. The same reasoning implies (because the inverse sequence $\{\alpha_\nu^{-1}\}$ converges to α_0^{-1}) that q_0 is also in $\partial\Omega_0$.

Select, by Lemma 4.5.5, a point $w_0 \in \Omega_0$ with these properties:

(A) $K_{\Omega_0}(p_0, w_0) \neq 0$;
(B) If $d_0(z) =$ the Jacobian determinant $\det(\partial b_{j,w_0}/\partial z_k)\big|_z$, $j, k = 1, \ldots, n$, then

$$\liminf_{z \to p_0} |d_0(z)| > 0.$$

[Here b_{j,w_0} are the Bergman representative coordinate functions that we introduced earlier.]

Because $K_{\Omega_0}(\cdot, w_0)$ extends to be a C^∞ function on the set

$$\left\{ z \in \mathrm{cl}(\Omega_0) \colon \mathrm{dis}(z, w) < \frac{3}{2}\mathrm{dis}(w_0, \partial\Omega_0) \right\},$$

property (A) implies that the Bergman representative coordinate functions b_{j,w_0} have C^∞ extensions to a neighborhood of p_0 in $\mathrm{cl}(\Omega_0)$. Property (B) is thus equivalent to the assertion that $d_0(p_0) \neq 0$. In particular, there is a number $\epsilon > 0$ such that the functions b_{j,w_0}, $j = 1, \ldots, n$, form a C^∞ coordinate system (holomorphic in Ω_0) on

$$\mathrm{cl}(\Omega_0) \cap \{z \in \mathbb{C}^n : \mathrm{dis}(z, p_0) \leq \epsilon\}.$$

[Notice that we are *not* claiming that the functions b_{j,w_0} are holomorphic across $\partial\Omega_0$; rather, these functions extend to be C^∞ across $\partial\Omega_0$ in the sense that their real and imaginary parts are C^∞ as real functions. In general they will only be holomorphic on Ω_0 itself.]

By Lemma 4.5.5, the Bergman representative coordinate functions b^ν_{j,w_0}, for Ω_ν, $j = 1, \ldots, n$, and $\nu = 1, 2, \ldots, \infty$, on

$$\mathrm{cl}(\Omega_\nu) \cap \{z \in \mathbb{C}^n : \mathrm{dis}(z, p_0) \leq \epsilon\}$$

converge in the C^∞ sense to the b_{j,w_0} on

$$\mathrm{cl}(\Omega_0) \cap \{z \in \mathbb{C}^n : \mathrm{dis}(z, p_0) \leq \epsilon\}.$$

In particular, for all ν sufficiently large, the functions b^ν_{j,w_0}, $j = 1, \ldots, n$ form a C^∞ coordinate system on

$$\mathrm{cl}(\Omega_0) \cap \{z \in \mathbb{C}^n : \mathrm{dis}(z, p_0) \leq \epsilon\}.$$

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain, $\alpha : \Omega \to \Omega$ be an automorphism with Euclidean components $(\alpha_1, \ldots, \alpha_n)$, and $\mathcal{J}_\alpha(z)$ denote the Jacobian determinant of α at z. Recall the following transformation formulas:

$$K_\Omega(z, w) = \overline{\mathcal{J}_\alpha(w)} \cdot \mathcal{J}_\alpha(z) \cdot K_\Omega(\alpha(z), \alpha(w)), \tag{1}$$

$$b_{j,w}(z) = \sum_{\ell=1}^{n} \overline{\left(\frac{\partial \alpha_\ell}{\partial w_j}\right)}\Bigg|_w b_{\ell,\alpha(w)}(\alpha(z)), \tag{2}$$

$$\left(\frac{\partial b_{j,w}}{\partial z_k}\right)\Bigg|_z = \sum_{\ell,m=1}^{n} \overline{\left(\frac{\partial \alpha_\ell}{\partial w_j}\right)}\Bigg|_w \cdot \left(\frac{\partial \alpha_m}{\partial z_k}\right)\Bigg|_z \cdot \left(\frac{\partial b_{\ell,\alpha(w)}}{\partial z_m}\right)\Bigg|_{\alpha(z)}, \tag{3}$$

$$\det\left(\frac{\partial b_{j,w}}{\partial z_k}\right)\Bigg|_z = \overline{\mathcal{J}_\alpha(w)} \cdot \mathcal{J}_\alpha(z) \cdot \det\left(\frac{\partial b_{\ell,\alpha(w)}}{\partial z_m}\right)\Bigg|_{\alpha(z)}. \tag{4}$$

Formula (1) is the standard transformation formula for the Bergman kernel; formulas (2) and (3) follow from (1) by differentiation; and formula (4) can by derived from (2) by using a little algebra.

The next observation is that $\det(\partial b_{j,\alpha_0(w_0)}/\partial z_k)\big|_{w_0} \neq 0$. To prove this assertion, notice that, by Lemma 4.5.5, the determinant equals

$$\lim_{\nu \to \infty} \det(\partial b^\nu_{j,\alpha_\nu(w_0)}/\partial z_k)\big|_{\alpha_\nu(p_\nu)};$$

this expression in turn equals, by formula (4),

$$\lim_{\nu \to \infty} \big(\mathcal{J}_{\alpha_\nu}(w_0)\big)^{-1} \cdot \big(\overline{\mathcal{J}_{\alpha_\nu}(p_\nu)}\big)^{-1} \cdot \det\big(\partial b^\nu_{j,w_0}/\partial z_k\big)\big|_{p_\nu}.$$

Since, by Lemma 4.5.4, the expression $|\mathcal{J}_{\alpha_\nu}|$ is bounded above on $\mathrm{cl}(\Omega_\nu)$ (uniformly in ν) and since

$$\lim_{\nu\to\infty} \det\left(\frac{\partial b^\nu_{j,w_0}}{\partial z_k}\right)\Bigg|_{p_\nu} = \det\left(\frac{\partial b_{j,w_0}}{\partial z_k}\right)\Bigg|_{p_0} \neq 0,$$

it follows that indeed $\det\bigl(\partial b_{j,\alpha_0(w_0)}/\partial z_k\bigr)\big|_{q_0} \neq 0$.

From the nonvanishing of this last determinant, it follows that the functions $b_{j,\alpha(w_0)}$ form a C^∞ coordinate system in some neighborhood in $\mathrm{cl}(\Omega_0)$ of q_0. In particular, there is a positive number η such that these functions form a C^∞ coordinate system on $\mathrm{cl}(\Omega_0) \cap \{z \in \mathbb{C}^n : \mathrm{dis}(z, q_0) \leq \eta\}$. Lemma 4.5.5 then implies that, for all sufficiently large ν, the functions $b^\nu_{j,\alpha(w_0)}$ form a C^∞ coordinate system on $\mathrm{cl}(\Omega_0) \cap \{z \in \mathbb{C}^n : \mathrm{dis}(z, \nu_0) \leq \eta\}$; moreover, this coordinate system converges in the C^∞ topology to the coordinate system $b_{j,\alpha(w_0)}$ on $\mathrm{cl}(\Omega_0) \cap \{z \in \mathbb{C}^n : \mathrm{dis}(z, q_0) \leq \eta\}$.

For any ν sufficiently large, $\mathrm{dis}(p_\nu, p_0) \leq \epsilon$ and $\mathrm{dis}(\alpha(p_\nu), q_0) \leq \eta$. Thus, for all sufficiently large ν, the mapping α_ν in a neighborhood of p_ν is completely determined—in w_0-Bergman coordinates (of Ω_ν) going to $\alpha_\nu(w_0)$-Bergman coordinates (of Ω_ν)—by formula (3). This mapping is linear with bounded differential. But, since both w_0-Bergman coordinates (of Ω_ν) and $\alpha_\nu(w_0)$-Bergman coordinates of Ω_ν are converging in the C^∞ topology to C^∞ coordinate systems (independent of ν), it follows by the chain rule that the Euclidean derivatives of each fixed order α_ν at $\alpha_\nu(p_\nu)$ are bounded above uniformly in ν as $\nu \to \infty$. This contradiction completes the proof of Proposition 4.4.5. $\square$

5

Lie Groups Realized as Automorphism Groups

5.1 Introduction

If Ω is a bounded domain in a complex Euclidean space, then the group Aut (Ω) of its holomorphic automorphisms is a finite-dimensional Lie group, as already discussed (Theorems 1.3.11, 1.3.12). It is natural to ask:

> **Question.** *Which Lie groups occur as the automorphism group of a bounded domain?*

Quite satisfactory answers are known. Bounded domains with noncompact automorphism group are in a sense unusual (cf. Corollary 3.4.4). Therefore it is natural to focus upon the compact Lie groups in asking which groups appear. In fact, every compact Lie group occurs as the automorphism group of a bounded domain in some complex Euclidean space, indeed a strictly pseudoconvex domain with real analytic boundary. This fact was proved independently and by different methods in [Bedford/Dadok 1987] and [Saerens/Zame 1987]. These proofs are the subject of this chapter.

In more detail:

Theorem 5.1.1 (Bedford–Dadok, Saerens–Zame). *Let G be a compact Lie group. Then there exist a positive integer N and a bounded strongly pseudoconvex domain Ω in $\mathbb{C}^N$ with a smooth (C^∞) boundary such that* Aut (Ω) *is Lie isomorphic to G.*

The semicontinuity theorem of Greene–Krantz (Theorem 4.4.3) makes it possible to choose the boundary of the domain in the theorem to be real analytic, as already stated. This will be discussed after the proof of the C^∞ result as stated.

5.2 General Philosophy

Before introducing the proofs, let us discuss the general philosophy underlying this theorem. Let G be a compact Lie group. It is a basic fact of Lie group

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theory that G can be Lie isomorphically embedded into a unitary group $U(n)$, for some $n > 0$. (This fact is an aspect of the famous Peter–Weyl theorem. cf. [Chevalley 1946].) Therefore it is automatic to construct a bounded strongly pseudoconvex domain whose automorphism group contains a subgroup that is isomorphic to the given group G: the unit ball B^n suffices, since $U(n)$ is a subgroup of its automorphism group. On the other hand, it is a general principle that perturbation of the boundary of the domain in the smooth category will lose some of the automorphisms. [This was discussed earlier, in Chapter 4, in connection with the semicontinuity theorem (Theorem 4.4.3).] Hence the key issue here is how to perturb the ball—or some other domain with G contained in its automorphism group—so that G is kept while the other unwanted automorphisms are eliminated.

We first present the proof by Saerens and Zame and then the proof by Bedford and Dadok. The techniques are so different that both proofs are worth considering carefully.

5.3 The Saerens/Zame Proof

5.3.1 Unitary Representation

Start with the injective Lie group homomorphism $\iota : G \to U(n)$ of G into some unitary group $U(n)$ already mentioned. In order for such a *faithful unitary representation* to exist, n of course needs to be sufficiently large.

5.3.2 G-action by Left Multiplication

Consider the group $GL(n, \mathbb{C})$ of nonsingular $n \times n$ matrices with complex entries. Let G act on $GL(n, \mathbb{C}) \times \mathbb{C}^m$ as follows.

$$\begin{aligned} G \times (GL(n, \mathbb{C}) \times \mathbb{C}^m) &\longrightarrow GL(n, \mathbb{C}) \times \mathbb{C}^m \\ (g, (z, w)) &\mapsto g(z, w) := (g \cdot z, w), \end{aligned}$$

where:

- the action of g on $(z, w) \in GL(n, \mathbb{C}) \times \mathbb{C}^m$ is only on the first component $z \in GL(n, \mathbb{C})$ by left multiplication.
- the positive integer m will be determined later, and the role of $\mathbb{C}^m$ will also be clarified at the same time.

5.3.3 Averaging a Plurisubharmonic Exhaustion

Now consider the following real-valued function $\varphi : GL(n, \mathbb{C}) \times \mathbb{C}^m \to \mathbb{R}$ defined by

$$\varphi(z, w) = |\det z|^{-2} + \sum_{i,j=1}^{n} |z_{ij}|^2 + \sum_{k=1}^{m} |w_k|^2.$$

This function is a smooth (in fact real analytic), strictly plurisubharmonic (psh for shorthand) exhaustion function for $GL(n, \mathbb{C}) \times \mathbb{C}^m$, which is an open connected subset of $\mathbb{C}^{n^2+m}$.

Take a bi-invariant measure ν of total mass 1 on the compact Lie group G (the Haar measure), and consider the averaged function

$$\varphi^G(z, w) = \int_G \varphi(g \cdot z, w) \, d\nu(g).$$

This new function is also a real analytic, strictly psh exhaustion function for $GL(n, \mathbb{C}) \times \mathbb{C}^m$ and is obviously G-invariant.

5.3.4 A G-Invariant Strongly Pseudoconvex Domain

Now take a regular value $T \in \mathbb{R}$, that is, a real number T such that $d\varphi^G$ is nowhere singular on $(\varphi^G)^{-1}(T)$. [Such T are dense in $\mathbb{R}$, by the Morse-Sard theorem (Theorem 5.3.2); see Section 5.3.7 for more details on this matter.] One can take T to be sufficiently large that $(\varphi^G)^{-1}(-\infty, T)$ contains the set $U(n) \times \{0\}$. Denote by D^G the connected component of $(\varphi^G)^{-1}(-\infty, T)$ that contains the set $U(n) \times \{0\}$. By its construction, D^G is a G-invariant, bounded domain in $\mathbb{C}^{n^2+m}$ with a C^∞ smooth boundary. It has in fact real analytic boundary, by construction.

5.3.5 Preparation for Perturbation of the Boundary

Since $d\varphi^G$ is nonsingular at each point of ∂D^G, there exists an open neighborhood W of ∂D^G on which $d\varphi^G$ is nonsingular. Choose $r > 0$ such that $(\varphi^G)^{-1}(-r + T, T + r) \subset W$; such an $r > 0$ exists because ∂D^G is compact. Replacing W by $(\varphi^G)^{-1}(-r + T, T + r)$, we may assume that W itself is a G-invariant open neighborhood of ∂D^G, consisting of only regular points of φ^G.

Now consider the quotient by the G-action. By construction, the G-action is a fixed-point-free, properly discontinuous action. Therefore the quotient spaces W/G and $\partial D^G/G$ are smooth manifolds.

5.3.6 Scalar Invariants

Finding a suitable perturbation of the boundary of D^G uses an idea from the theory of curvature invariants in the sense of Tanaka–Chern–Moser. Here is a brief summary.

This concerns the local CR-invariants of the real hypersurfaces that will play an important role in the perturbation step. Consider a smooth real-valued function $\phi : \mathbb{C}^{n+1} \to \mathbb{R}$ that defines a smooth hypersurface $M = \{\phi = 0\}$ passing through the origin 0. In case M is strongly pseudoconvex, the function ϕ can be written, after a suitable change of coordinate system, say $(z_1, \ldots, z_n, \zeta)$ with $\zeta = u + iv$ about 0, in what is called the Chern–Moser normal form (see pp. 241–243 of [Burns/Shnider/Wells 1978] for further details and precise terminology). In this "normal form,"

$$\phi(z_1, \ldots, z_n, \zeta) = v - \sum_{\alpha=1}^{n} |z_\alpha|^2 - \sum_{p,q \geq 2} N_{p,q}$$

where each $N_{p,q}$ is a polynomial in the multi-variables $z, \bar{z}$ of type (p, q), first p of zs and q of $\overline{z}$s, with coefficients that are formal power series in the variable u as follows.

$$N_{p,q} = \sum N_{a_1 \cdots a_p; \bar{b}_1 \cdots \bar{b}_q}(u)\ z_1^{a_1} \cdots z_p^{a_p} \bar{z}_1^{b_1} \cdots \bar{z}_q^{b_q}$$

and

$$N_{a_1 \cdots a_p; \bar{b}_1 \cdots \bar{b}_q}(u) = \sum_{j=0}^{\infty} N^{(j)}_{a_1 \cdots a_p; \bar{b}_1 \cdots \bar{b}_q} u^j .$$

The origin 0 in M is called *spherical* (or *umbilical* in [Burns/Shnider/Wells 1978]; for the original introduction and developments, see [Chern/Moser 1974]) if the coordinates can be chosen so that $N^{(0)}_{a_1 a_2 \bar{b}_1 \bar{b}_2} = 0$ for any $a_1 a_2 \bar{b}_1 \bar{b}_2$. Otherwise, 0 is called *nonspherical.* This notion is independent of the choice of the normal form and is in fact preserved by biholomorphic transformations.

At a nonspherical point, further normalization, called the *restricted normal form*, is available (see Lemma 3.1 of [Burns/Shnider/Wells 1978]). In [Burns/Shnider/Wells 1978], "curvature invariants" for $j \geq 0, p \geq q \geq 2, p \geq 3$ are given by

$$K^j_{p,q} := \sum |N^{(j)}_{a_1 \cdots a_p; \bar{b}_1 \cdots \bar{b}_q}|^2$$

at the origin. (The curvature invariants make sense only at nonspherical points.) These are local CR invariants, meaning that the CR equivalences preserve the value of these terms.

5.3.7 Jets and Multi-Jets

The proof also involves the concept of *jets.* Again a brief summary.

Jets

Let X, Y be smooth manifolds and let $f, g : X \to Y$ be smooth maps with $f(x) = y = g(x)$ for some $x \in X$ and $y \in Y$. Then f and g are said to have first-order contact at x if every first-order partial derivative of f coincides with the corresponding derivative of g at x in some local coordinates around x and y in X and Y respectively. Notice that this concept does not depend upon the choices for local coordinate systems for X at x and for Y at y.

Likewise, f and g are said to have k-th order contact if they have the same partial derivatives at p of order up to and including k. Again, for every k, this concept does not depend upon the choices for local coordinate systems for X at x and for Y at y. For each k, it is obvious that this defines an equivalence relation; denote it by $\cong_k$, for the germs of smooth mappings. For a smooth map $f : X \to Y$ satisfying $f(x) = y$, denote by $j^k f|_{x,y}$ the equivalence class of the germ of f at x with respect to the relation $\cong_k$.

Denote by $J^k(X,Y)_{x,y}$ the collection of all the equivalence classes just defined. This is not in general a vector space as it does not have any obvious addition or scalar multiplication. However, in case Y is a Euclidean space, it is a vector space in an obvious way. In particular, $J^1(X,\mathbb{R})_{x,y}$ is naturally isomorphic to the cotangent space of X at x.

It is customary to call $J^k(X,Y)_{x,y}$ the space of *k-th order jets* (or simply *k-th jets*) of maps from X to Y at (x,y) and to consider the space

$$J^k(X,Y) = \bigcup_{(x,y)\in X\times Y} J^k(X,Y)_{x,y} \quad \text{(disjoint union)}.$$

This union is usually called the *jet bundle* for smooth maps from X to Y. Notice that the space of k-th jets and the k-jet bundle are finite-dimensional smooth manifolds for each $k = 1, 2, 3, \ldots$.

Likewise, it makes sense to consider the map

$$j^k f : X \to J^k(X,Y) : x \mapsto j^k f|_{x,f(x)}$$

which is usually called the *k-jet* of the smooth map $f : X \to Y$. It is a smooth map with respect to the obvious smooth structure on $J^k(X,Y)$.

Multi-Jets

Now we shall introduce the concept of "multi-jets" (although, for our exposition here we only need double-jets).

First, we define

$$X^{(s)} := \left\{ (x_1, \ldots, x_s) \in \prod^s X \mid x_j \neq x_k \text{ if } j \neq k \right\}$$

and let $\alpha : J^k(X,Y) \to X$ be the projection defined by $\alpha(\sigma) = x$ if and only if $\sigma = j^k f|_{x,y}$ for some $y \in Y$ and some germ of a smooth $f : X \to Y$ with $f(x) = y$. Then let $\alpha^s := \prod^s \alpha : \prod_{\ell=1}^s J^k(X,Y) \to \prod^s X$ be the product map. Then one can consider the *space of s-fold k-th jets* defined by

$$J^k_{(s)}(X,Y) := (\alpha^s)^{-1}(X^{(s)}).$$

This is what is called in [Saerens/Zame 1987] a *multi-jet*. One can easily generalize this formalism to define the concept of the s-fold multi-jet bundle

$J^k_{(s)}(X,Y)$ and the map $j^k_{(s)}f : X^{(s)} \to J^k_{(s)}(X,Y)$, where the last is nothing but

$$j^k_{(s)}f(x_1,\dots,x_s) = (j^k f(x_1),\dots,j^k f(x_s))$$

for every $(x_1,\dots,x_s) \in X^{(s)}$.

Transversality

The *transversality* concept in differential topology is also needed for the proof. The idea of transversality grew out of the idea of regular value, already used in Section 5.3.4. For completeness and motivation, we discuss this first. Let $f : M \to N$ be a smooth map from a smooth manifold M to another smooth manifold N. Then one would like to know when the pre-image $f^{-1}(y)$ is necessarily a smooth submanifold of M for $y \in N$. A satisfactory answer comes of course from the implicit function theorem:

A point $y \in N$ is called a *regular value* of the smooth map $f : M \to N$ if, for any $x \in f^{-1}(y)$, the differential $df_x : T_xM \to T_yN$ is surjective. The implicit function theorem then implies:

Theorem 5.3.1. *Let M and N be smooth manifolds and let $f : M \to N$ be a smooth mapping. If $y \in N$ is a regular value for f, then the pre-image $f^{-1}(y)$ is an embedded submanifold of M.*

One notices that, due to the logic involving the empty set, any point $y \in N \setminus f(M)$ becomes a regular value. Of course in such a case $f^{-1}(y)$ coincides with the empty set, and that is surely a submanifold. (The dimension of empty submanifold is usually understood to be -1.) One might like to disregard such a "pathological" case, but in fact there is no particular reason to do so; in fact it will play an important role in many cases, including our current discussion.

Do regular values exist? The following familiar theorem guarantees their abundance.

Theorem 5.3.2 (Sard's Theorem; cf. e.g., [Munkres 1966]). *The set of regular values for a smooth map $f : M \to N$ is dense in N.*

In fact, if we denote the set of regular values by R, then $N \setminus R$ is of measure zero. Note that the concept "measure zero" in differential topology does not have to involve any specific choice of a measure. A set is *measure zero* if and only if it has measure zero in every local coordinate system in the sense that its intersection with each coordinate domain has measure zero in $\mathbb{R}^n$ when mapped to $\mathbb{R}^n$ by the local coordinate map.

The following notion of transversality grew out of the concept of regular values.

Definition 5.3.3. Let M, N be smooth manifolds and let Z be a submanifold of N. Let $f : M \to N$ be a smooth mapping. Then we say that f is *transversal* to Z, if the equality

$$df_x(T_xM) + T_{f(x)}Z = T_{f(x)}N$$

for any $x \in f^{-1}(Z)$. It is customary to denote transversality by $f \pitchfork Z$.

The following is a well-known result in differential topology (cf. e.g., [Hirsch 1976]).

Theorem 5.3.4 (Transversality). *Let M, N be smooth manifolds and Z a submanifold of N. Let $f : M \to N$ be a smooth mapping. If f is transversal to Z, then $f^{-1}(Z)$ is an embedded submanifold of M.*

The reader must have noticed, by the logic involving the empty set, that f is transversal to Z whenever $f(M) \cap Z = \emptyset$. On the other hand, if it happens to be the case that $\dim N > \dim M + \dim Z$, then f can be transversal to Z if and only if $f(M) \cap Z = \emptyset$. Again, this seemingly somewhat pathological logic is going to play an important role in what follows.

Now what about the generalization of Sard's theorem (Theorem 5.3.2)? The Saerens/Zame proof uses the following standard theorems on this subject (cf. e.g., [Golubitsky/Guillemin 1973]):

Theorem 5.3.5. *Let X and Y be smooth manifolds.*

(1) [Thom transversality theorem] *Let W be a submanifold of $J^k(X, Y)$ and let $T_W := \{f \in C^\infty(X, Y) \mid j^k f \pitchfork W\}$. Then T_W is a dense G_δ-subset of $C^\infty(X, Y)$ in the C^∞ topology.*

(2) [Multi-jet transversality theorem] *Let W be a submanifold of $J^k_{(s)}(X, Y)$ and let $T_W := \{f \in C^\infty(X, Y) \mid j^k_{(s)} f \pitchfork W\}$. Then T_W is a dense G_δ-subset of $C^\infty(X, Y)$ in the C^∞ topology.*

5.3.8 Application of Transversality to ∂D^G

We now return to the actual proof of Theorem 5.1.1 at the point where we had a G-invariant domain D^G with a smooth strongly pseudoconvex boundary, and the regular G-invariant neighborhood W of ∂D^G (end of Subsection 5.3.5).

Consider Ψ the set of all smooth, strictly psh, G-invariant, proper functions defined on $GL(n, \mathbb{C}) \times \mathbb{C}^m$ that are nonsingular at every point of W, with W as in Subsection 5.3.5. This set is nonempty as we constructed such a function φ^G by an averaging method. However, unlike what is claimed in [Saerens/Zame 1987] by Saerens and Zame, it is actually *not* true that Ψ is an open subset of $C^\infty(GL(n, \mathbb{C}) \times \mathbb{C}^m, \mathbb{R})$, since G-invariance is not an open condition.

Fortunately, this incorrect claim is not essential for the rest of the arguments. Here is a way to fix the situation. Consider the subset

$$\mathcal{D} := \{h \in C^\infty(GL(n, \mathbb{C}) \times \mathbb{C}^m, \mathbb{R}) \mid h(g \cdot x) = h(x), \\ \forall x \in GL(n, \mathbb{C}) \times \mathbb{C}^m \text{ and } \forall g \in G\}.$$

This is a closed linear subspace of the Fréchet (i.e., complete, semi-normed) space $C^\infty(GL(n,\mathbb{C}) \times \mathbb{C}^m, \mathbb{R})$. Consider now the set

$$\Psi_G = \{\phi \in C^\infty(GL(n,\mathbb{C})/G \times \mathbb{C}^m, \mathbb{R}) \mid \phi \circ \pi \in \Psi\}.$$

Here, $\pi : GL(n,\mathbb{C}) \times \mathbb{C}^m \to GL(n,\mathbb{C})/G \times \mathbb{C}^m$ is the standard quotient map. Since G is compact, the map π is proper. It then follows by the chain rule that the map $\pi^* : C^\infty(GL(n,\mathbb{C}) \times \mathbb{C}^m, \mathbb{R}) \to \mathcal{D}$ defined by $\pi^*(\psi) := \psi \circ \pi$ is a continuous mapping. Since the function space Ψ is an open subset of $\mathcal{D}$ in the inherited topology from $C^\infty(GL(n,\mathbb{C}) \times \mathbb{C}^m, \mathbb{R})$, and since $\Psi_G = [\pi^*]^{-1}(\Psi)$, we see immediately that $\Psi_{/G}$ is an open subset of $C^\infty(GL(n,\mathbb{C}) \times \mathbb{C}^m, \mathbb{R})$. This is what we need for the rest of the argument.

Let the correspondence $\phi \mapsto \phi_G : \Psi \to \Psi_G$ be defined by $\phi_{/G}(G\cdot x) = \phi(x)$. This gives rise to the natural map

$$\pi_k^* : J^k\Psi \to J^k\Psi_G,$$

defined by $\pi_k^*\big(j^k\phi_G\big|_{Gx}\big) = j^k\phi|_x$ for every $x \in GL(n,\mathbb{C}) \times \mathbb{C}^m$.

5.3.9 Elimination of Spherical Jets by Perturbation

Recall the definition of spherical (boundary) point in Section 5.3.6. The concept of spherical point depends only upon the jet of order at most 4. Therefore it makes sense to define the concept of *spherical jets* (of normalized defining functions) following the obvious method, instead of the concept of spherical point associated with the (normalized) defining function. Denote by S^k the set of spherical jets in $J^k\Psi$ and let $\Sigma^k := \pi_k^*(S^k)$. Furthermore, for p, q with $p > q \geq 3$ and $p+q \leq k$, the scalar curvature invariant functions $\widetilde{K}^0_{p,q}$ are also defined on $J^k\Psi_G \setminus \Sigma^k$, analogously to the curvature functions for Ψ. Also, let

$$S^k_{p,q} = \{\psi \in J^k\Psi \mid K^0_{p,q}(\psi) = 0\}$$

and

$$\Sigma^k_{p,q} = \{\psi \in J^k\Psi_G \mid K^0_{p,q}(\psi) = 0\}.$$

Lemma 5.3.6. *There exists $\ell > 0$ such that, for every $m \geq \ell$, the following estimates hold:*

$$\text{codim } (\Sigma^4 \text{ in } J^4\Psi_G) \geq 2(n^2 + m)$$

and

$$\text{codim } (\Sigma^4_{p,q} \text{ in } J^4\Psi_G) \geq 2(n^2 + m)$$

whenever the positive integers p, q satisfy the conditions $p > q \geq 3$ and $p + q \leq m$.

Notice that $2n^2+2m = \dim_{\mathbb{R}} W$. The proof of this lemma uses only general facts on the jets and the curvature invariants introduced in [Burns/Shnider/Wells 1978]. The proof we sketch here is reorganized by B.-L. Min in his thesis ([Min, B.-L. 2009]; see also [Min, B.-L. 2009a]). We refer to this last paper for further details.

A sketch of the proof of Lemma 5.3.6. The proof is a direct computation. In [Burns/Shnider/Wells 1978], the codimension of the space S^4 of spherical jet in $J^4\Psi$ was computed to be $t^2(t-1)^2/4-(t-1)^2$ where $t = n^2+m = \dim_{\mathbb{C}} W$. On the other hand, $\dim_{\mathbb{R}} J^4\Psi = \dim_{\mathbb{R}} W + 1 + \dim_{\mathbb{R}} A^4_{2n^2+2m}$ where A^k_r is the vector space of polynomials of degree $\leq k$ in r variables without constant terms.

Note that $\dim_{\mathbb{R}} G \leq n^2$ as $G \in U(n)$. Consequently, $\dim_{\mathbb{R}} J^4\Psi_{/G} \geq n^2+2m+1+\dim A^4_{n^2+2m}$, and this eventually gives rise to

$$\text{Codim } (\Sigma^4 \text{ in } J^4\Psi_G) \geq \frac{1}{4}m^4 - \text{ lower order terms in } m.$$

As n is fixed, and m can be chosen sufficiently large, one can see (due to the remarks in the first paragraph of this proof) that the assertion of the lemma follows. $\square$

On the other hand, let $m \geq \ell$ be an integer as in the preceding lemma, and let

$$\Sigma = S^4 \cup \left(\bigcup_{\substack{p>q\geq 3 \\ p+q\leq m}} \Sigma^4_{p,q} \right).$$

Now apply the transversality theorem (Theorem 5.3.5) on jets and multijets introduced above. Recall the special neighborhood W for the boundary of the domain D^G defined earlier. For such a W, there exists a dense $\mathcal{G}_\delta$-subset of $\Psi_{/G}$ such that ψ in the G_δ-subset has the following two properties:

(1) If a map $j^4\psi : W/G \to J^4\Psi_G$ is transversal to Σ^4 and, at the same time, to $\Sigma^4_{p,q}$, then $j^4\psi(y) \notin \Sigma^4 \cup \Sigma^4_{p,q}$ for any $y \in W/G$.
(2) If

$$J^4{\Psi_G}^{\times} := \{\text{nonspherical jets in } J^4\Psi_G\}$$

and

$$J^4\Psi^{\times} := \{\text{nonspherical jets in } J^4\Psi\},$$

then there exists a set Q of $4(n^2+m)+1$ distinct curvature functions $\widetilde{K}_1, \ldots, \widetilde{K}_Q$, where $\widetilde{K}_\ell = \widetilde{K}_{p_\ell,q_\ell}$ for p_ℓ and q_ℓ satisfying $p_\ell > q_\ell \geq 3$ and $p_\ell + q_\ell \leq m$, such that the map

$$\widetilde{K} := (\widetilde{K}_1, \ldots, \widetilde{K}_Q) : J^4{\Psi_G}^{\times} \to \mathbb{R}^Q$$

has maximal rank. Let $\varDelta$ denote the diagonal of $\mathbb{R}^Q \times \mathbb{R}^Q$. Then the inverse image $(\widetilde{K}, \widetilde{K})^{-1}(\varDelta)$ is a submanifold of $J^4\varPsi_G{}^\times \times J^4\varPsi_G{}^\times$. The function ψ has its double jet $j^4_{(2)}\psi : (W/G)^{(2)} \to J^4_{(2)}\varPsi_{/G}$, transversal to $(\widetilde{K}, \widetilde{K})^{-1}(\varDelta)$.

Property (1) holds on a dense G_δ by the codimension estimates in the previous lemma. Property (2) holds on a dense G_δ by the multijet transversality theorem, Theorem 5.3.5 (2). Thus properties (1) and (2) hold simultaneously on a dense G_δ-set.

5.3.10 Construction of $\varOmega$

It may be useful to summarize what has been done up to this point. We started with the embedding of the given compact Lie group G into the unitary group $U(n)$ of some sufficiently large n. Then we considered the real analytic strictly psh function

$$\varphi(z, w) = |\det z|^{-2} + \sum |z_{jk}|^2 + \sum |w_\ell|^2$$

defined on $GL(n, \mathbb{C}) \times \mathbb{C}^m$. Then, exploiting the compactness of the given Lie group G, we have used the averaging process

$$\varphi^G(z, w) := \int_G \varphi(g \cdot z, w)\ d\nu(g)$$

so that the new function φ^G is invariant under the G-action and is strictly psh and real analytic. Then we choose a regular value T so that $D^G := (\varphi^G)^{-1}(-\infty, T)$ is defined to be a G-invariant, bounded, strongly pseudoconvex domain with a real analytic boundary. Furthermore, we observed that there exists a special G-invariant open neighborhood W of ∂D^G such that $d\varphi^G$ is nonsingular at every point of W.

Then, using jets and transversality theorems, we were able to perturb φ^G as follows.

Construct first $\phi : GL(n, \mathbb{C})/G \times \mathbb{C}^m \to \mathbb{R}$ by $\phi(G \cdot x) = \varphi^G(x)$. Then perturb ϕ to obtain $\psi : GL(n, \mathbb{C})/G \times \mathbb{C}^m$ so that $\widetilde{\psi} := \psi \circ \pi$ is still arbitrarily close to ϕ on compact subsets (and hence in particular on W). Notice that here one needs to take m sufficiently large. Of course $\widetilde{\psi}$ is still strictly psh and smooth of class $\mathcal{C}^\infty$, and $d\widetilde{\psi}$ is nonsingular at any point of W. Furthermore, if we now let

$$\varOmega = \widetilde{\psi}^{-1}(-\infty, T),$$

then $\varOmega$ is a bounded strongly pseudoconvex domain in $\mathbb{C}^{n^2+m}$ that has the following properties:

(i) $G \subset \text{Aut}\ (\varOmega)$.
(ii) $\partial\varOmega$ has no point at which the jet of ψ is spherical.
(iii) If $x, y \in \partial\varOmega$ such that $x \notin G \cdot y$, then $K(j^4\psi(x)) \neq K(j^4\psi(y))$.

Now to continue the proof, we wish to show that $G = \text{Aut}\ (\Omega)$. Let $h \in \text{Aut}\ (\Omega)$. Since the scalar curvature invariant function is a CR invariant, and since h extends to a diffeomorphism of $\text{cl}(\Omega)$ by Fefferman's extension theorem, $h(x) = y$ implies that $x \in G \cdot y$.

Thus $h(x) = g_x \cdot x$ for some $g_x \in G$ that is *a priori* depending on x. But recall that the elements x and $h(x)$ are in $GL(n, \mathbb{C}) \times \mathbb{C}^m$. Hence we may write $x = (z, w)$ and $h(x) = h(z, w) = (h_1(z, w), h_2(z, w))$. Now $g_x \cdot x = h(x)$ means

$$g_{(z,w)} = h_1(z, w)z^{-1} \qquad \text{and} \qquad h_2(z, w) = w.$$

Therefore the map $g : \partial\Omega \to G$, $g(z, w) = g_{(z,w)}$, defines a CR-function. However, $U(n)$ inside $GL(n, \mathbb{C})$ is totally real. Therefore the differential of this map has to vanish identically. This means that $g = g_{(z,w)}$ is independent of $x = (z, w) \in \partial\Omega$ and hence depends only on h. first, for every $h \in \text{Aut}\ (\Omega)$ there exists $g \in G$ such that $h(z, w) = (g \cdot z, w)$ for any $(z, w) \in \Omega$. Hence $G = \text{Aut}\ (\Omega)$ as desired. This completes the construction and the proof of Theorem 5.1.1.

5.4 The Bedford/Dadok Proof

An alternative approach to the realization of a given compact Lie group as the automorphism group of a bounded domain was given by E. Bedford and J. Dadok ([Bedford/Dadok 1987]). Their essential idea was to realize the given group as the isometry group of a perturbation of the unit ball in some real Euclidean space $\mathbb{R}^n$ and then pass to the complex setting by considering a suitable modification of the "tube domain" in $\mathbb{C}^n$ over the domain in $\mathbb{R}^n$. Their paper also considers the question of realizing a given compact Lie group as the automorphism group of a compact-closure (and strongly pseudoconvex) domain in a Stein manifold, rather than in a complex Euclidean space: the point here is that this realization is possible in rather lower dimensions than if one requires a domain in $\mathbb{C}^n$. We shall outline the approaches in the two cases, the $\mathbb{C}^n$ case first. Complete details are given in [Bedford/Dadok 1987] for both.

5.4.1 Structure of the Proof

Suppose that G is a compact Lie group and that (following the notation of [Bedford/Dadok 1987]) ω is a bounded domain in some Euclidean space $\mathbb{R}^n$ with the following properties:

(a) there is an injective homomorphism of G into $O(n)$, the image of which we again denote by G, such that ω is invariant under G;
(b) if $g : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation with $g(\omega) = \omega$, then $g \in G$.

(We shall see later that, for suitable n, such an ω can be obtained as a C^∞ small perturbation of the unit ball in $\mathbb{R}^n$.) Now let

$$\Omega = (\omega + i\mathbb{R}^n) \setminus V \subset \mathbb{C}^n$$

where $V = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \colon z_1^2 + \cdots + z_n^2 = \frac{1}{2}\}$. The role of removing V from the "tube domain" $\omega + i\mathbb{R}^n$ will become apparent momentarily. Note that each $g \in G$ takes Ω to itself if G is taken to act on $\mathbb{C}^n$ by complex linear extension of its action on $\omega \subset \mathbb{R}^n$: this is clear since g takes $\omega + i\mathbb{R}^n$ to itself and g takes V to itself—because g on Ω maps the set $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_1^2 + \cdots + x_n^2 = \frac{1}{2}\}$ to itself, since $g \in O(n)$.[1]

The domain Ω is of course unbounded, but it is biholomorphic to a bounded open set (since it is contained in a proper cone). This immediately implies, using the fact that bounded holomorphic functions extend across deleted subvarieties, that any automorphism of Ω extends to be an automorphism of the tube domain $\omega + i\mathbb{R}^n$. Now automorphisms of tube domains are completely understood. In particular it is shown in [Yang 1982] that every automorphism of $\omega + i\mathbb{R}^n$ has the form $z \mapsto Az + b + ic$ for some $b, c \in \mathbb{R}^n$ and some $A \in GL(n, \mathbb{R})$. Here $Az + b$ must map ω to itself, so from property (b) of ω above, $b = 0$ and $A \in G \subset O(n)$. Now, for $z \mapsto Az + ic$ to map V to itself, it must be that $c = 0$: this is so because A maps V to itself but $V \neq V + ic$ if $c \neq 0$. Hence the original automorphism $z \mapsto Az + b + ic$ is in fact an element of G.

5.4.2 How to Obtain a Bounded Domain

The domain Ω does not as such answer the question of realizing G as the automorphism group of a bounded domain with a smooth boundary, since Ω is neither bounded nor smooth (because of the removal of V, which has real codimension 2). However one can modify Ω as follows: the domain Ω is pseudoconvex so it admits a C^∞ strictly plurisubharmonic exhaustion function $\varphi : \Omega \to \mathbb{R}$. By averaging with respect to the action of the compact group G on Ω, one can obtain such a φ that is G-invariant, so that its c-sublevel sets $\Omega_{\varphi,c} := \{z \in \Omega \colon \varphi(z) < c\}$ are C^∞, bounded and G-invariant, for generic choice of c (by Sard's theorem (Theorem 5.3.2)), first for c regular values of φ. For each fixed c, there is an arbitrarily small (in the C^∞ sense) perturbation, to be denoted $\widehat{\Omega}_{\varphi,c}$, which guarantees that $\widehat{\Omega}_{\varphi,c}$ is still contained in Ω, G-invariant and strongly pseudoconvex, and has the further property that $\operatorname{Aut}(\widehat{\Omega}_{\varphi,c})$ preserves the function $\sum_{j=1}^n z_j^2$. This follows from the arguments discussed earlier (Section 5.4.1; see also Sections 5.3.3, 5.3.4, and 5.3.8.) about introducing orbit-stabilizing perturbations. Since G itself preserves $\sum_{j=1}^n z_j^2$,

[1] The inclusion relation $g(V) \subset V$ follows by the "persistence of identities" upon passing from a totally real maximal dimension submanifold to a whole connected open set in $\mathbb{C}^n$.

the possibility of carrying this perturbation process in a G-equivariant way follows easily. By choosing the perturbations sufficiently small (for each c_j), the property can be retained that for some fixed increasing sequence $c_j \to +\infty$, the $\widehat{\Omega}_{\varphi,c_j}$ are increasing (i.e., $\widehat{\Omega}_{\varphi,c_j} \subset \widehat{\Omega}_{\varphi,c_{j+1}}$ and $\bigcup_{j=1}^{+\infty} \widehat{\Omega}_{\varphi,c_j} = \Omega$.

With these choices made, it follows that, for j sufficiently large, $\mathrm{Aut}\,(\widehat{\Omega}_{\varphi,c_j})$ must be exactly G.

To see this, it suffices to show that if $c_j \to +\infty$, c_j a regular (i.e., noncritical) value for φ, and $\alpha_j \in \mathrm{Aut}\,(\widehat{\Omega}_{\varphi,c_j})$, then there is a subsequence α_{j_k} of the α_js which converges uniformly on compact subsets of Ω to an automorphism of Ω, first to an element of G. For, if this is known, then $\mathrm{Aut}\,(\widehat{\Omega}_{\varphi,c_j})$ restricted to some fixed (nonempty) $\widehat{\Omega}_{\varphi,c}$ lies, when j is large enough, in a small neighborhood of $G\big|_{\Omega_{\varphi,c}}$ and hence, by the results of Chapter 4, in fact $= G$ (since it contains G).

To check the indicated convergence result for a subsequence of the α_j, note first that some subsequence α_{j_k} of the α_js converges uniformly on compact subsets of Ω to some holomorphic function $\alpha_0 : \Omega \to \Omega \cup \partial\Omega$. This follows from standard normal families arguments since Ω is biholomorphic to a bounded domain. Note that we need not worry about possible "divergence to infinity" for this reason: Re $(\sum z_j^2)$ is preserved by $\mathrm{Aut}\,(\widehat{\Omega}_{\varphi,c})$ by construction. And, the real parts of the z_js are bounded on $\widehat{\Omega}_{\varphi,c}$. It follows that the imaginary parts of the coordinates of $\varphi_j(0,\ldots,0)$ are bounded for $\varphi_j \in \mathrm{Aut}\,(\widehat{\Omega}_{\varphi,c_j})$, the bound being uniform in j. The limit α_0 is in $\mathrm{Aut}\,(\Omega) = G$, provided it does not "degenerate," i.e., provided that $\alpha_0(\Omega) \subset \Omega$, for which it suffices to show that $\alpha_0(\Omega) \not\subset \partial\Omega$.

Now $\alpha_0(\Omega)$ cannot contain points of $\partial\omega + i\mathbb{R}^n$ that are not in V since such points are strongly pseudoconvex, by the standard argument about strongly pseudoconvex boundary points of domains not biholomorphic to the ball (cf. [Rosay 1979]) and the "scaling version" of Rosay's argument presented in Chapter 9 (see Theorem 9.2.1). On the other hand, it cannot be that $\alpha_0(\Omega) \subset V$ since this would give a retraction of $\Omega \cup V$ onto V, which is impossible for homological reasons: $\Omega \cup V$ is contractible, but $V \cap (\omega + i\mathbb{R}^n)$ is homologically nontrivial in dimension n since $\{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 = \frac{1}{2}\}$ is not homologically trivial in V.

5.4.3 Construction of ω

Turning now to the construction of a suitable ω as a C^∞ small perturbation of the unit ball in some $\mathbb{R}^n$, we note first a general idea of metric perturbations and group actions: Suppose that (M, g_0) is a Riemannian manifold with metric g_0 invariant under a faithful action on M of a compact Lie group G. (By *faithful*, we mean here that only the identity in G acts as the identity map of M.) Thus, in effect, G can be thought of as a subgroup of the isometry group Isom (M, g_0) of M with respect to the metric g_0. Now, in general, it is not necessarily the case that there is a metric g on M that is C^∞ close to g_0

such that Isom $(M, g) = G$. For example, if a metric of the k-dimensional sphere S^k is invariant under the standard $SO(k+1)$ action on S^k, then it is necessarily a multiple of the standard S^k metric, and hence its isometry group is $O(k+1)$, not just $SO(k+1)$. However, what is true is that there is always a metric g on M, C^∞ close to g_0, such that the metric g is invariant under the G-action and Isom (M, g) has the same orbits as the G-action. Such an orbit-stabilizing perturbation of g_0 is obtained by making G-invariant alterations of the g_0-metric in tubular neighborhoods of sufficiently many G-orbits of maximal dimension. Then the detailed argument is similar to but easier than the corresponding ideas in the Saerens–Zame argument already presented, so we omit the details at this time. In summary, one can stabilize a given G-orbit by making a high-order derivative of the metric g in normal directions to the orbit larger than for other (remote) orbits: this will stabilize a neighborhood of the orbit. This process can be successively adjusted to stabilize smaller neighborhoods and the limit orbit itself. Then a dense set of other orbits can be stabilized, by the Baire category theorem. Hence all orbits can be stabilized.

Thus the problem of finding a suitable ω as above can be solved if it can be converted to an orbit stabilization situation. As pointed out in [Bedford/ Dadok 1987], this can be arranged by choosing first a *diagonal embedding.* Suppose that the group G has a faithful representation as a subgroup of $O(n)$ for some n. Then G has an action on $\mathbb{R}^{n^2} \cong \mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n$ (n summands) by letting G act on each summand by its $O(n)$ representation. The G can be considered as a subgroup of $O(n^2)$, and this faithful representation of G has the following property: *If a subgroup H of $O(n^2)$ has the same orbits as G, i.e., $Hx = Gx$ for all $x \in \mathbb{R}^{n^2}$, then $H = G$.*

The role of the diagonal embedding process can be made more vivid by constructing a concrete example. Consider the action of $SO(3)$ on $S^2 \subset \mathbb{R}^3$, $S^2 = \{(x, y, z) \colon x^2 + y^2 + z^2 = 1\}$ as usual, and the action of $O(3)$ on S^2. The 2-sphere is itself an orbit for both actions, and, moreover, any metric invariant under $SO(3)$ has to be a multiple of the standard metric on S^2 and hence must be invariant under $O(3)$. No process of *orbit stabilization*—indeed no process whatever—can produce a metric on S^2 which is $SO(3)$ invariant but not $O(3)$ invariant: it cannot be arranged that Isom $(S^2, g) = SO(3)$ exactly with $SO(3)$ acting in the standard way as indicated. In fact it cannot be arranged that Isom $(S^2, g) = SO(3)$, acting any way at all. The reason is that a faithful $SO(3)$ action must *a priori* have an orbit of dimension $=$ $\dim SO(3) -$ maximum isotropy dimension $= 3 - 1 = 2$. Thus every faithful $SO(3)$ action on S^2 must make S^2 homogeneous so that an invariant metric must have constant Gauss curvature; and then S^2 with that metric must be isometric to S^2 with a multiple of its standard metric. But such a metric has isometry group $O(3)$, not just $SO(3)$.

All this difficulty of distinguishing $SO(3)$ from $O(3)$ by orbits can be repaired, as it were, by considering the *diagonal action.* first we let $SO(3)$ act on $\mathbb{R}^9$ as follows. Consider $A \in SO(3)$, a 3×3 orthogonal matrix. Then associate to A a *diagonal-associate* $\widehat{A} \in O(9)$, first the 9×9 matrix with three 3×3 diagonal blocks being A, and all other matrix elements 0:

$$\widehat{A}_{ij} = \begin{cases} A_{ij} & \text{if } 1 \leq i, j \leq 3, \\ A_{i-3,j-3} & \text{if } 4 \leq i, j \leq 6, \\ A_{i-6,j-6} & \text{if } 7 \leq i, j \leq 9, \\ 0 & \text{otherwise.} \end{cases}$$

The transformation $\widehat{A}$, constructed from $A \in SO(3)$, gives an orthogonal action on $\mathbb{R}^9$.

The crucial point that makes this construction of interest is this: if H is a subgroup of $O(9)$ the action of which on S^8 (or, equivalently on $\mathbb{R}^9$) has each H-orbit contained in some orbit of the *diagonal action* (action by $\{\widehat{A}\colon A \in SO(3)\}$), then each element $h \in H$ has the form $\widehat{A}$ for some $A \in SO(3)$. This will be checked momentarily. Note that this means that if a Riemannian metric g on S^8 is invariant under the action of $\{\widehat{A}\colon A \in SO(3)\}$ and also has the *orbit stabilization* property that Isom (g) has the same orbits as the orbits of $\{\widehat{A}\colon A \in SO(3)\}$, then Isom $(g) = \{\widehat{A}\colon A \in SO(3)\} \cong SO(3)$. Since such orbit stabilization can always be induced by a small perturbation of S^8, by making $\{A\}$-invariant perturbations normal to enough $\{\widehat{A}\colon A \in SO(3)\}$-orbits, one finds then a metric on S^8 with its isometry group isomorphic to $SO(3)$. The $O(3)$ versus $SO(3)$ difficulty for the actions on S^2 is eliminated by moving up to S^8. [Here we use implicitly the *rigidity* of small perturbations of S^8: for such, isometries of the metric are always realized as the restriction of a rigid motion of $\mathbb{R}^9$, hence, changing the origin if need be, by $O(9)$ elements. See the end of Subsection 5.4.4 for details of this idea.]

It remains to see why a subgroup H of $O(9)$ which has orbits contained in $\{\widehat{A}\}$-orbits must itself consist of elements of $\widehat{A}$ form. For this consider a 9×9 matrix $h \in H \subset O(9)$. We write images as column vectors here, so the first column of the matrix h is the image under h of $e_1 = (1, 0, \ldots, 0)$, this image written in column form. This image is of course in the H-orbit of $e_1 = (1, 0, \ldots, 0)$ and hence by hypothesis is in the $\{\widehat{A}\}$ orbit of $e_1 = (1, 0, \ldots, 0)$: it equals $\widehat{A}e_1$ for some $A \in SO(3)$. In particular, this column has its bottom six entries $= 0$. Similarly, the fourth column of the h-matrix has its top three and bottom three entries $= 0$. The seventh column has its top six entries $= 0$.

Now we wish to see that the top three entries of column 1 of the h matrix $=$ the middle three entries of column 4 $=$ the bottom three entries of column 7 (same order, top to bottom, in the three cases). For this, we consider the h-image of $e_1 + e_4 + e_7$ where $e_i =$ the vector with 1 in the i-th position, all other components $= 0$. This h-image is (written as a column) the sum of the first, fourth and seventh columns. And, noting the forms of these columns already shown, this is the top three entries of the first column followed by the middle three of the fourth column followed by the bottom three of the seventh column. On the other hand $h(e_1 + e_4 + e_7)$ belongs to the H-orbit of $e_1 + e_4 + e_7$, and hence by hypothesis to the $\{\widehat{A}\}$-orbit of $e_1 + e_4 + e_7$. So $h(e_1 + e_4 + e_7) = \widehat{A}(e_1 + e_4 + e_7)$ for some $A \in SO(3)$. But $\widehat{A}(e_1 + e_4 + e_7)$ (as a column vector) consists of its top three entries repeated in order two additional times. This shows that the h-matrix has the correct form to be an $\widehat{A}$-matrix as far as the first, fourth, and seventh columns are concerned.

Similar reasoning applied to e_2, e_5 and e_8 together with $e_2 + e_5 + e_8$ and e_3, e_6 and e_9 together with $e_3 + e_6 + e_9$ completes the proof that the H-matrix has repeated block-diagonal form. The block, call it B, must belong to $O(3)$, since $h \in O(9)$. To see that $B \in SO(3)$, consider $h(e_1 + e_5 + e_9)$. This (column) vector is, from top to bottom, first column of B, second column of B, third column of B. Therefore, in order for the element $\widehat{B}(e_1 + e_5 + e_9)$ to coincide with the element $\widehat{A}(e_1 + e_5 + e_9)$, for some $A \in SO(3)$, it must be that $B = A$. So $h = \widehat{B}$ for some $B \in SO(3)$.

Note that the map of $SO(3)$ onto the orbit of $e_1 + e_5 + e_9$ is injective: $\widehat{A_1}(e_1 + e_5 + e_9) = \widehat{A_2}(e_1 + e_5 + e_9)$ implies that $A_1 = A_2$. It follows from general considerations that this is true generically: $A \mapsto \widehat{A}v$ is injective for generic vectors $v \in \mathbb{R}^9$, i.e., the set of v for which this is true is dense and open in $\mathbb{R}^9$.

Thus one is indeed in the situation where orbit stabilization suffices. The orbit stabilization process is in fact simpler in this case than for a general Riemannian action. And one sees that there is a G-invariant C^∞-small perturbation of the unit sphere which lies in the unit sphere except for a set of small measure and which stabilizes G-orbits in the sense that the (abstract) isometry group for the perturbation ω has the same orbits as G acting on the perturbation ω. It follows then that any affine mapping of $\mathbb{R}^{n^2}$ that preserves this perturbed domain ω is in fact in $O(n^2)$ and hence in G: the reason is that, because of the coincidence of the perturbation ω with the unit sphere everywhere but on a set of small measure, such an affine mapping must carry some open subset of the unit sphere to itself and hence be in $O(n^2)$. Further details can be found in [Bedford/Dadok 1987].

5.4.4 Isometry Group of a Riemannian Manifold

Note that, with ω so chosen, G is in fact the full isometry group of $\partial\omega$, the boundary of ω. This follows from the fact that $\partial\omega$, being C^∞ close to the unit sphere, is thus rigid in the sense that all its intrinsic (abstract) isometries extend to be isometries of $\mathbb{R}^{n^2}$. This rigidity follows from E. Cartan's "type number" local rigidity theorem: the unit sphere has maximal type number and hence so does every hypersurface C^∞ close enough to it. (Refer to [Hermann 1968] for these matters. See also [Spivak 1975], Volume 5, Chapter 12, p. 244 ff and the discussion on *type numbers* and rigidity.) From another only slightly different viewpoint, $\partial\omega$, being C^∞ close to the unit sphere, has positive sectional curvature and thus is rigid, again by E. Cartan's result. Thus any isometry of $\partial\omega$ extends to an isometry of $\mathbb{R}^{n^2}$ so that $G =$ the isometry group of $\partial\omega$ considered as an abstract Riemannian manifold. Thus one obtains: if G is a compact Lie group, then there is a compact Riemannian manifold (M, g) such that Isom $(M, g) \cong G$.

Curiously, the natural question in geometry that this result answers was never considered successfully in the context of pure Riemannian geometry itself, prior to its arising in the present context of complex analysis in [Bedford/Dadok 1987] and [Saerens/Zame 1987].

5.4.5 Stein Domains

The second major line of thought in [Bedford/Dadok 1987] concerns realization of compact Lie groups as automorphism groups of bounded domains (i.e., domains with compact closure) in Stein manifolds which are not necessarily biholomorphic to bounded domains in $\mathbb{C}^n$. This more general class of domains yields a possible realization in lower dimensions. In effect, one can go from complex dimension n^2 for the Euclidean space case if $G \subset O(n)$ to dimension equal to that of G itself, clearly much lower when n is large.

Theorem 5.4.1 (Bedford–Dadok). *If G is a connected compact Lie group the dimension of whose center is not* 1*, then there is a strongly pseudoconvex domain Ω with the real analytic boundary contained in the complexification $G_{\mathbb{C}}$ of G and with $G \subset \Omega$ such that* $\mathrm{Aut}\,(\Omega) \cong G$ *and* $\mathrm{Aut}\,(\Omega)$ *consists exactly of the action of G on itself by translation extended holomorphically to Ω.*

If the dimension of the center of G is 1, then a similar domain Ω exists in $G_{\mathbb{C}} \times \mathbb{C}$.

This result is established by using the decomposition of G into the product of its center and simply connected simple factors, up to a finite quotient. The essential point is then to use the result of H. Cartan showing that, under quite general circumstances, the automorphism group of a product is the product of the automorphism groups of the factors. (This will be discussed in more detail later.)

5.4.6 Decomposition of G into $T \times G_s$

The product decomposition result is a standard part of Lie group theory (cf. [Helgason 1962]): *Every connected compact Lie group G has the form $(T^k \times G_1 \times \ldots \times G_\ell)/H$ where T^k is a k-dimensional torus ($k = 0$ is allowed), the G_is are simply connected compact simple groups, and H is a finite subgroup.* While the result is usually considered only in a Lie-group-theoretic context, it actually has an illuminating differential-geometric interpretation (and, indeed, proof).

This arises as follows: any left-invariant metric on the compact Lie group can be averaged with respect to the Haar measure on right translations of G. This produces a bi-invariant metric $\langle\,,\rangle$ on G. For this bi-invariant metric, the covariant derivative $D_X Y$, where X and Y are left-invariant vector fields, is $\frac{1}{2}[X,Y]$. And, again for left-invariant vector fields, the Riemann curvature tensor $R(X,Y,Z,W)$ is $-\frac{1}{4}\langle[X,Y],[Z,W]\rangle$ (cf. [Milnor 1963]; note that the sign convention for R in that reference is opposite to ours). This curvature tensor is parallel. Moreover, the Riemann sectional curvatures attached to it are all nonnegative, as follows immediately from the formula: the sectional curvature of the 2-plane spanned by an orthonormal pair X, Y is $-R(X,Y,X,Y) = \langle[X,Y],[X,Y]\rangle \geq 0$.

Let $\mathcal{I}$ be the set of all left-invariant vector fields X such that $[X,Y]=0$ for all left-invariant vector fields Y and set $\mathcal{I}_p = \{X(p) \colon X \in \mathcal{I}\}, p \in G$. If $X \in \mathcal{I}$, then X is globally parallel, since $D_Y X = \frac{1}{2}[Y,X] = 0$ for every (left-invariant) Y so $DX \equiv 0$. Thus the family of subspaces $\mathcal{I}_p \subset T_pG$, $\forall p \in G$, is a parallel family (i.e., invariant under parallel translation). The parallel nature of the family $\mathcal{I}_p$ can be interpreted in terms of the curvature tensor R: $\mathcal{I}$ is exactly the set of all left-invariant vector fields X such that $R(X,Y,Z,W)=0$ for all left-invariant vector fields Y, Z, W. So the parallel nature of R implies that of the family $\mathcal{I}_p$.

The de Rham decomposition theorem (cf. [Kobayashi/Nomizu 1963], Theorem 6.2, p. 192, Vol. I) now implies that the universal cover $\widehat{G}$ of G splits as a product $T \times G_s$ where the tangent space of the torus T at each point is the lift of $\mathcal{I}$ at the image of the point under the covering projection. And thus, for the pullback to $\widehat{G}$ of the metric of G, the torus T is flat. Moreover G_s is necessarily compact. (In the notation G_s, "s" stands for semi-simple, for reasons that will appear later.)

The group G_s is compact because, if G_s were noncompact, then there would be a geodesic ray $\gamma : [0,+\infty) \to G_x$ emanating from a pre-image of the identity. [Recall that a *ray* is a curve γ on $[0,+\infty)$ with $\mathrm{dis}(\gamma(0),\gamma(t)) = t$ for all $t \geq 0$.] But if v is the tangent vector $\gamma'(0)$ and V the associated left-invariant vector field on G, then there is a left-invariant vector field on G with $[V,W] \neq 0$. This would mean that $-R(V,W,V,W)$ would be a positive constant along the ray, implying the existence of a conjugate point to the initial point of the ray, a contradiction. Alternatively, one could show that G_s is compact by noting that it is complete and has positive Ricci curvature bounded away from 0: this follows by noting that, at a pre-image of the identity, there is at least one 2-plane of positive sectional curvature containing a given vector $v \neq 0$, associated, as above, to W such that $[V,W] \neq 0$. So the Ricci curvature of v is positive. Since curvature is parallel, the Ricci curvature is positive and bounded away from 0 everywhere. The compactness of G_s of course implies that any covering-space quotient of it is finite-to-one.

5.4.7 Decomposition of G_s

There is potentially a further decomposition of G_s that arises as follows. Since the metric is bi-invariant, its Lie derivative as a tensor with respect to a left-invariant vector field Y must be 0. This gives

$$0 = Y\langle X, Z\rangle - \langle [Y, X], Z\rangle - \langle X, [Y, Z]\rangle$$

for X, Y, Z left-invariant vector fields, using the usual Leibniz property to compute the $\mathcal{L}_Y$ Lie derivative of $\langle\, ,\rangle$ as a tensor. But $\langle X, Z\rangle$ is constant so that $Y\langle X, Z\rangle = 0$. It follows that $\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle$. This same formula holds if we consider the lifts of left-invariant vector fields on G to vector fields on $\widehat{G}$. Let $\mathcal{L}$ = the Lie algebra of such lifts. Then the relationship $\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle$ implies that the orthogonal complement of an ideal in $\mathcal{L}$ is again an ideal, as one sees immediately. From this viewpoint, the space of vector fields in $\mathcal{L}$ tangent to G_s is exactly the orthogonal complement of the ideal in $\mathcal{L}$ consisting of vector fields tangent to T. Now the fact that orthogonal complements of ideals are again ideals implies that the tangent ideal of G_s can be successively decomposed into, finally, an orthogonal direct sum of simple ideals. Since G_s is simply connected, this implies a corresponding product decomposition of G_s into a product: the ideal decomposition is parallel by bi-invariance, so the de Rham decomposition theorem again applies. Thus, in outline, one arrives at the Lie group decomposition result as stated. Of course, the argument just discussed can be considered exclusively in Lie group terms: the appeal to the de Rham decomposition theorem is used just to give a differential geometric perspective.

The irreducibility of the ideals arising in this final decomposition implies that the positive Ricci curvature on each irreducible factor is in fact constant: the bi-invariant metrics are Einstein. Thus the Ricci curvature tensor itself can be thought of as being the original bi-invariant metric up to a constant factor. The $R(X, Y, Z, W) = -\langle [X, Y], [Z, W]\rangle$ formula shows that this Ricci curvature is in fact, again up to a constant, equal to the traditional "Killing form" $K(X, Y) = -tr\,(ad(X)ad(Y))$, where $ad(X)$ is the map on the tangent space determined by Lie bracketing with X. Thus the original metric and the Killing form metric are themselves Einstein metrics. The uniqueness (up to constant factors) of bi-invariant metrics on the simple factors can of course be seen directly from the irreducibility of the tangent ideals.

The decomposition of $\widehat{G}$ into $T \times G_s$, and the associated information about G itself, can also be viewed in the context of the Toponogov splitting theorem for complete manifolds of nonnegative sectional curvature, at least as far as the $T \times G_s$ decomposition is concerned. (The further decomposition of G_s into simple factors does not fit into this picture, however.) The reader is invited to consult [Cheeger/Ebin 1975] or [Petersen 2006] for further details of this perspective on decomposition.

5.4.8 Torus Group Case

We now begin constructing domains in the complexification of a compact connected Lie group G with automorphism group $= G$.

As already noted, the product decomposition of a compact connected Lie group offers a natural approach to finding domains with automorphism group equal to the given compact Lie group. If such domains can be found for each factor in the product then, under quite general and rather easily arranged circumstances, the product of these domains will serve for the whole (product) group. We now turn to this situation in more detail.

The first case to consider is that of a k-dimensional torus $T = \{(\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k : |\alpha_i| = 1, \forall i\}$. Recall the classical concept of a Reinhardt domain: an open and connected set $\Omega \subset \mathbb{C}^k$ such that Ω is invariant under the mappings $(z_1, \ldots, z_k) \mapsto (\alpha_1 z_1, \ldots, \alpha_k z_k)$ where each α_i has modulus 1. The torus T acts on such a domain, by definition.

A Reinhardt domain, say Ω, is completely specified by its "log profile"

$$\mathrm{Log}\ (\Omega) := \{(\log |z_1|, \ldots, \log |z_k|) \in \mathbb{R} \cup \{-\infty\} : (z_1, \ldots, z_k) \in \Omega\}.$$

We allow $-\infty$ values to accommodate the possibility that U contains points with some or all coordinates $= 0$. We write Log $(z_1, \ldots, z_k)$ for the k-tuple $(\log |z_1|, \ldots, \log |z_k|)$, including the possible $-\infty$ values.

Note that $\mathrm{Log}^{-1}(1, \ldots, 1) = T \subset \mathbb{C}^k$. Thus $\mathrm{Log}^{-1}(V)$, where V is some neighborhood of $(1, \ldots, 1)$ in $\mathbb{R}^k$, is a tubular neighborhood of the real n-dimensional submanifold T of $\mathbb{C}^k$. Note also that T is a totally real submanifold of $\mathbb{C}^k$ in the sense that the tangent space of T and the J-image of this tangent space intersect in the 0-vector only. (Here J is the standard almost complex structure on $\mathbb{R}^{2k} = \mathbb{C}^k$.) That T is totally real is clear at the point $(1, 1, \ldots, 1) \in \mathbb{C}^k$, since the tangent space in $\mathbb{R}^{2k}$ coordinates $(x_1, y_1, \ldots, x_k, y_k)$, $x_j + iy_j = z_j$, is the set of vectors of the form $(0, b_1, 0, b_2, \ldots, 0, b_k)$, each $b_j \in \mathbb{R}$. The same holds at other points of T since these arise from $(1, 1, \ldots, 1)$ by a complex linear map which preserves T. Thus we can identify, for each (sufficiently small) neighborhood V in $\mathbb{R}^k$ of $(1, 1, \ldots, 1)$, the set $\mathrm{Log}^{-1}(V)$ with a tubular neighborhood of T in its own complexification: $T_{\mathbb{C}}$ is characterized in a neighborhood of T by being a complex k-dimensional manifold containing T as a totally real submanifold.

Suppose now that Ω is a Reinhardt domain and Log (Ω) is a bounded convex domain in $\mathbb{R}^k$. Then, by [Bedford 1980], the automorphisms of Ω must have the form:

$$(z_1, \ldots, z_k) \mapsto (c_1 z^{m_1}, \ldots, c_k z^{m_k}),$$

where we are using multi-index notation

$$z^{m_j} = z_1^{m_j^1} \cdots z_k^{m_j^k},$$

and where it is required that the matrix $(m_j^\ell) \in GL(k, \mathbb{Z})$. A mapping of this form maps Ω to Ω if and only if the affine mapping $z \mapsto Mz + \log |c|$ is an affine mapping of Log (Ω) to itself. Here $M =$ the matrix (m_j^ℓ), $z = (z_1, \ldots, z_k) \in \mathbb{C}^k$ and $c = (c_1, \ldots, c_k) \in \mathbb{C}^k$.

Now, if $k \geq 2$, then, generically, domains in $\mathbb{R}^k$ have no nontrivial affine self-mappings. In particular, there are domains V in $\mathbb{R}^k$ that are small perturbations of a (small) ball around the origin in $\mathbb{R}^k$. For such V, as before, the domain $\mathrm{Log}^{-1}(V)$ is a tubular neighborhood of T in its complexification $T_{\mathbb{C}}$, where as earlier we identify T with a totally real submanifold of $\mathbb{C}^k$. And, for such V (which have no nonidentity affine self-mapping), the automorphism group of $\mathrm{Log}^{-1}(V)$ is exactly T.

In case $k = 1$, any connected bounded open neighborhood of 0 in $\mathbb{R}^k = \mathbb{R}^1$ has an affine self-mapping that is not the identity, first reflection at its mid-point, the neighborhood being of course an open interval. Thus, in this case, for any V, $\mathrm{Log}^{-1}(V)$ has an automorphism other than those in T. (One such automorphism which is associated to the affine "inversion" indicated is the automorphism $z \mapsto R_1R_2/z$ of $\{z\colon R_1 < |z| < R_2\}$, $0 < R_1 < R_2 < +\infty$ to itself.) So special consideration and indeed an extra dimension (as stated in the theorem) is needed in this case. Indeed no Riemann surface has automorphism group isomorphic to $\{z \in \mathbb{C}\colon |z| = 1\}$ (cf. Chapter 2): the extra dimension is definitely required.

The reader can find an explicit construction dealing with this special case in [Bedford/Dadok 1987].

In summary form: set $\Omega = \{(z, w) \in \omega \times \mathbb{C}\colon r_1(z) < |w| < r_2(z)\}$, where ω is a smoothly bounded, triply-connected domain in $\mathbb{C}$ with $\mathrm{Aut}\,(\omega)$ being the identity alone, and r_1, r_2 are continuous functions on the closure of ω, smooth on ω itself, with $0 < r_1 < r_2$ on the closure of ω. Then, if $r_1(z)r_2(z)$ is not the modulus of a holomorphic function on ω, then $\mathrm{Aut}\,(\Omega)$ is isomorphic to $\{\alpha \in \mathbb{C}\colon |\alpha| = 1\} = T$. The proof can be found in [Bedford/Dadok 1987]. Note that it is not hard to see that there are, for example, perturbations of the unit ball in $\mathbb{C}^2$ for which the automorphisms group is exactly the set of maps (isomorphic to T)

$$(z_1, z_2) \mapsto (\alpha z_1, \alpha z_2), \quad \alpha \in \mathbb{C}, \quad |\alpha| = 1.$$

The point of the more intricate construction of Bedford/Dadok is that the above Ω lies in $T_{\mathbb{C}} \times \mathbb{C}$.

5.4.9 The Case of Simple Lie Groups

The next stage in the application of the product decomposition to finding domains in $G_{\mathbb{C}}$ with specified automorphism group is to consider the case $G =$ a compact simple group. In this case the usual representation of G acting on its own Lie algebra is faithful up to a finite kernel. In more detail, if v is a vector in the tangent space of G at the identity and $\gamma(t)$ is the corresponding one-parameter subgroup, then we define $Ad\ g$, $g \in G$ acting on v, by

$$(Ad\ g)(v) = \frac{d}{dt} g^{-1}\gamma(t)g\Big|_{t=0},$$

this being again a tangent vector to G at the identity. This gives a representation

$G \to$ linear endomorphism of the tangent space of G at the identity.

The simplicity of G implies that the kernel of this representation is finite. Indeed, to check this one needs only check that the kernel contains no 1-parameter subgroup, since the kernel is a closed subgroup of G. This follows from the simplicity of G and the associated nondegeneracy of the Killing form.

Thus, up to a finite quotient, G can be considered to be a matrix group. The image of the Ad representation is in fact a subgroup of the orthogonal group of linear transformations of the tangent space at the identity, orthogonal relative to the bi-invariant metric (which is the Killing form, as already discussed).

This gives an explicit way to construct a neighborhood basis of G inside $G_{\mathbb{C}}$; first, if ω is a neighborhood of zero in the tangent space of G at the identity, then we can set $\Omega_\omega = G \cdot \exp(i\omega)$ (ignoring the quotienting, which is easily handled by "lifting"), where exp is the usual exponentiation of matrices. Of course one can handle this matter "intrinsically": since exp in the 1-parameter subgroup sense is defined on ω, and since G is totally real in $G_{\mathbb{C}}$ and exp is real analytic, there is a unique way to define exp holomorphically on a sufficiently small neighborhood of the identity in $G_{\mathbb{C}}$. In particular, if w is sufficiently small, then $\exp(i\omega)$ is defined in this way, simply from holomorphic function theory.

Note that such a tubular neighborhood is G-invariant (for left multiplication action of G), and that this G-action is holomorphic on this G-invariant neighborhood of G in $G_{\mathbb{C}}$. The final step in completing the construction is to show that, for some suitable choice of ω, these G-induced automorphisms are the only automorphisms of the tubular neighborhood.

To begin with, we restrict the neighborhood ω of 0 in the Lie algebra of G (which we identify as usual with the tangent space of G at the identity) to be a perturbation of a small ball around 0 in the Lie algebra in the bi-invariant metric. As far back as Grauert's proof of the existence of real analytic embedding of real analytic manifolds [Grauert 1958], it was noted that for such ω, the associated tubular neighborhood Ω_ω is C^∞ strongly pseudoconvex. This is a general phenomenon, not involving the fact that G is a Lie group: every compact real analytic manifold has a neighborhood basis of smooth strongly pseudoconvex domain inside its own complexification (again [Grauert 1958]). In particular, such tubular neighborhoods are Stein manifolds, by Grauert's solution of the Levi problem since they have no compact positive-dimensional subvarieties. Each of these Stein tubular neighborhoods has compact closure in a slightly larger tubular neighborhood which is also a Stein manifold. Then it follows that a given such tubular neighborhood has a defined, positive definite Bergman metric in the manifold sense. This Bergman metric is constructed from the Bergman kernel obtained from the space of L^2 holomorphic $(k, 0)$ forms, $k =$ the complex dimension of the complexification, as discussed in Section 3.2. This follows easily from embedding in complex Euclidean space the slightly larger Stein manifold in which the given tubular neighborhood has compact closure. The given tubular neighborhood thus inherits holomorphic L^2 forms from the ambient Euclidean space; these restrictions/pullbacks to the tubular neighborhood in the submanifold (of $\mathbb{C}^N$) are automatically L^2, and there are enough of them to guarantee a positive definite Bergman metric. This argument is a straightforward generalization of the argument showing that a bounded domain in $\mathbb{C}^N$ has a defined and positive definite Bergman metric.

Returning to the specific situation of an Ω_ω in $G_{\mathbb{C}}$ with ω so chosen as above, note that $\mathrm{Aut}\,(\Omega_\omega)$ contains G in the sense that (left) multiplication by elements of G acts as biholomorphic maps on Ω_ω. *A priori*, it could be that $\mathrm{Aut}\,(\Omega_\omega)$ is larger than G, or even that the connected component of the identity in $\mathrm{Aut}\,(\Omega_\omega)$ was larger than G. [Note that $\mathrm{Aut}\,(\Omega_\omega)$ is a Lie group here and indeed a Lie group with the isotropy of points of Ω_ω compact, since $\mathrm{Aut}\,(\Omega_\omega)$ is a closed subgroup of the isometry group of the Bergman metric of Ω_ω.]

Now the homology group $H_d(\Omega_\omega, \mathbb{Z})$ is isomorphic to $H_d(G, \mathbb{Z})$, since ω is convex; thus Ω_ω has a strong deformation retract onto $G \subset \Omega_\omega$ by linearly contracting ω to 0 in the Lie algebra. Since $H_d(G, \mathbb{Z}) = \mathbb{Z}$, $d = \dim_{\mathbb{R}} G$, it follows by topological considerations that there is an orbit of $\mathrm{Aut}\,(\Omega_\omega)$ in Ω_ω with dimension at most d ([Bedford 1983a]). Since $\mathrm{Aut}\,(\Omega_\omega)$ contains G in the sense mentioned, such an $\mathrm{Aut}\,(\Omega_\omega)$-orbit of dimension at most d must in fact be a finite union of G-orbits (of dimension exactly d). And any one of these must be stable under the identity component $\mathrm{Aut}^0(\Omega_\omega)$ of $\mathrm{Aut}\,(\Omega_\omega)$, by continuity.

Let Gx_0 (following the notation of [Bedford/Dadok 1987]) be such an $\mathrm{Aut}^0(\Omega_\omega)$-stable orbit. Then $\mathrm{Aut}^0(\Omega_\omega)$ acts as isometries on Gx_0, when Gx_0 is equipped with the restriction of the Bergman metric of Ω_ω. Identifying Gx_0 with G (since left multiplication by "elements of G" is a simply transitive action on Gx_0), one obtains that $\mathrm{Aut}^0(\Omega_\omega)$ is in effect a subgroup of the identity component of the isometry group of G with the left-invariant metric obtained by restricting the Bergman metric to Gx_0 (identified with G). Note that this need not be the bi-invariant metric of G itself (if $x_0 \notin G \subset G_{\mathbb{C}}$), but it is left invariant. The form of such isometries was determined in [Ochiai/Takahashi 1976]: for each $f \in \mathrm{Aut}^0(\Omega_\omega)$, there are elements $a, b \in G$ such that $f(g \cdot x_0) = agb \cdot x_0$, where $\cdot$ denotes the G-action operation.

Since G is transitive on Gx_0, an "extra" automorphism in $\mathrm{Aut}^0(\Omega_\omega)$, that is one that is not in G, can be obtained as an automorphism φ fixing x_0 followed by one in G. Such an automorphism φ fixing x_0, and stabilizing the orbit Gx_0 at x_0, acts on the tangent space $T_{x_0}(Gx_0)$ of Gx_0 at x_0. The Cauchy–Riemann equations then determine an action on $J(T_{x_0}(Gx_0))$. Thus, since φ is a Bergman metric isometry, this determines the action of φ on geodesics with tangent vectors in $J(T_{x_0}(Gx_0))$.

The domain ω determines the domain Ω_ω as far as its transversal-to-G nature is concerned. So, in this situation, it is natural to suppose that a suitable choice of ω will rule out the possibility of any nontrivial such action of $d\varphi$ on the $J(T_{x_0}(Gx_0))$. And then, again by Cauchy–Riemann equations, the action $d\varphi$ along G would also be necessarily trivial. Then no "extra" automorphisms in $\mathrm{Aut}^0(\Omega_\omega)$ would exist.

This intuitive expectation is in fact correct. In [Bedford/Dadok 1987], it is shown that for this it suffices to choose ω so that (i) $\omega = -\omega$ and (ii) the only $\sigma \in$ automorphisms of the Lie algebra of G with $\sigma(\omega) = \omega$ is the identity. (Note here that multiplication by -1 is not an automorphism of the Lie algebra so the conditions are consistent.) Of course, ω continues to be chosen so that $G \cdot \exp(i\omega)$ is strongly pseudoconvex and smoothly bounded in $G_{\mathbb{C}}$. For the sufficiency of this genericity condition, the reader is referred to [Bedford/Dadok 1987].

5.4.10 Connected Lie Group Case with Product Decomposition

Once the situation is in hand for the torus factor and the simple group factors in the product decomposition $G = T \times G_1 \times \cdots \times G_\ell / H$, H finite, the group G as a whole is treated as follows. A domain in the complexification $G_{\mathbb{C}}$ of G written as $G \cdot \exp(i\omega)$, some ω, can be obtained in particular with $\omega = \omega^0 \times \cdots \times \omega^\ell$ in obvious notation. By a result of H. Cartan

$$\mathrm{Aut}\,(\Omega_\omega) = T \times \mathrm{Aut}\,(\Omega_1) \times \cdots \times \mathrm{Aut}\,(\Omega_\ell),$$

(where $T = \mathrm{Aut}\,(\Omega_0)$ and $\Omega_j = \Omega_{\omega^j}$) provided that the ws are chosen so that no permutation-of-factors automorphisms arise: this choice of ws is always possible. A lifting argument disposes of the H-quotienting (see [Bedford/Dadok 1987] for details), and one obtains a pseudoconvex product domain in $G_{\mathbb{C}}$ with automorphism group G.

We replace this domain with a bounded strongly pseudoconvex domain with smooth boundary by considering sub-level sets of a C^∞ strictly plurisubharmonic exhaustion function φ, first $\{z \colon \varphi(z) < \lambda\}$, λ a noncritical value of φ. The normal families method of [Greene/Krantz 1985b] can be applied to obtain a bounded, strongly pseudoconvex domain with smooth boundary which is clearly G-invariant and has no "extra" automorphisms so that its automorphism group is G. By using a real analytic φ, one can in fact make this final domain have real analytic boundary.

5.4.11 Some Remarks

If one is not restricted to bounded strongly pseudoconvex domains, for instance if one is interested in constructing complex manifolds with prescribed automorphism group, there is more recent work, even when the given Lie group is noncompact. See for instance [Winkelmann 2004], [Kan, S.-J. 2007].

On the other hand, the following question was posed by Greene and Krantz some years ago:

Question ([Greene/Krantz 1982a]). Let Ω be a bounded, strongly pseudoconvex domain in $\mathbb{C}^n$ with C^∞ boundary, whose automorphism group is compact. Let H be a closed subgroup of the automorphism group. Then, for any open neighborhood $\mathcal{U}$ of Ω in the C^∞ topology, does there exist $\Omega' \in \mathcal{U}$ such that $\mathrm{Aut}\,(\Omega)$ is Lie-group-isomorphic to H?

A significant partial answer is reported recently: see [Min, B.-L. 2009]. the result is as follows.

Theorem 5.4.2 ([Min, B.-L. 2009]). *Let Ω be a bounded, strongly pseudoconvex domain in $\mathbb{C}^N$ with C^∞ boundary, with its automorphism group G compact. If $N > 5\dim_{\mathbb{R}} G + 4$, then, for any closed subgroup H of G and any open neighborhood $\mathcal{U}$ (in the C^∞ topology on domains) of Ω, there exists $\Omega' \in \mathcal{U}$ such that* $\mathrm{Aut}\,(\Omega')$ *is Lie-group-isomorphic to H.*

Whether the codimension condition $N > 5\dim_{\mathbb{R}} G+4$ is sharp is not known at this writing. Of course some restriction on the dimension is clearly required; see for example the discussion on $O(3)$ and $SO(3)$ actions in Section 5.4.3.

6

The Significance of Large Isotropy Groups

Two facts about isotropy groups have played a central role up to now in our study of automorphism groups of bounded domains: First, that assigning to each element of the isotropy its differential at the fixed point gives an injective isomorphism onto a subgroup of the linear group of invertible linear maps of the tangent space at the point to itself (Corollary 1.3.3). Second, that this injective isomorphism is a homeomorphism onto a compact subgroup so that in particular the isotropy group itself is compact (Corollary 1.3.7). As noted already in the Preface, these facts about the automorphisms of bounded domains represent a fundamental difference between bounded domains and $\mathbb{C}^n$ itself, for instance. This difference is at bottom derived from the even more basic difference—that maps into a bounded domain form a normal family while maps into $\mathbb{C}^n$ do not.

The compactness of isotropy groups and the related properness of the action of the automorphism group (Theorem 1.3.12) are closely associated to the question of the existence of automorphism-invariant metrics, which have played such an important role in our developments. Indeed, the properness of the automorphism group action is equivalent logically to the existence of an invariant Riemannian (or Hermitian) metric (cf. the discussion at the end of Section 1.1 and the discussion immediately after the proof of Theorem 1.3.12).

Given this central role of isotropy subgroups, it is natural to ask about them in more detail as groups and, in particular, what conclusions one can draw if the isotropy at a point is large in some sense. Of course the most typical case is that in which the isotropy group consists of the identity alone: the whole automorphism group of a bounded strongly pseudoconvex domain consists "generically" of the identity alone (Theorem 4.1.4). But, in Riemannian geometry, there is a long history of considering manifolds with metrics for which the isometry group is large in one sense or another, and it is natural to consider the analogous sort of problem in the complex case. In Riemannian geometry, the classical conditions of homogeneity—the isometry group is transitive—and of "free mobility"—the isometry group is transitive and the isotropy is the full orthogonal group—come to mind. One can also consider

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manifolds which are homogeneous and for which the isotropy is transitive on directions but not necessarily the full orthogonal group. (These exist—see [Besse 1987]). Thus, the general situation of considering the consequences of a "large" group of isometries is a familiar one.

The subject of this chapter is the consequences of assuming specifically that the isotropy is large at one single point in a sense that will be made precise later. The Riemannian analogue of the particular condition involved also has a history in Riemannian geometry: the Riemannian manifolds satisfying the corresponding Riemannian condition are the "weak models" in the sense of [Greene/Wu 1977]. But here we operate in the complex situation, though we soon reduce it to the metric one.

The Riemann surface version of this general question of large isotropy at one point has a definitive and straightforward answer that has been known for a long time (cf. [Aumann/Carathéodory 1934]).

Theorem (Bruné, Aumann, Carathéodory et al.). If Ω is a bounded domain in $\mathbb{C}$ and if, for some $p \in \Omega$, I_p is infinite, then Ω is biholomorphic to the unit disc.

This actually holds for hyperbolic Riemann surfaces (that is, those Riemann surfaces covered holomorphically by the unit disc), not just for bounded domains, as the following arguments will show.

We shall discuss the proof of this result in a moment. But first we note that the natural generalization of this question to higher dimensions arises not directly from the theorem as stated. but from a peripheral observation about the theorem's hypothesis. first, if the isotropy at a point of a bounded domain in $\mathbb{C}$, or more generally of a Riemann surface that admits an automorphism-invariant metric, is infinite, then in fact the isotropy at that point is as large as possible: expressed in terms of the differentials of the elements of the group at the point, it is the full "circle group" $\{z \in \mathbb{C}\colon |z| = 1\}$, which we shall hereafter denote by T.

Indeed, the simplest way to prove the theorem of Bruné et al. is to establish this preliminary fact first. It is this more or less immediate consequence of the infinity of the isotropy that leads to a natural generalization to higher dimensions: [Greene/Krantz 1985a] in the noncompact case, [Oeljeklaus E. 1970] in the compact case; cf. [Bland/Duchamp/Kalka 1987] also. It turns out that the natural hypothesis for higher dimensions can be weaker than having the largest possible isotropy $U(n)$, $n =$ dimension of the manifold. It suffices for complete information to suppose that the isotropy at one point should be transitive on real directions at that point: given any two nonzero vectors v and w at the point, there should be an isotropy element the differential of which takes v to a multiple of w. These higher-dimensional results are the main subject of this chapter and will be covered in detail in later sections.

But let us first explore the situation of complex dimension 1 more explicitly from a viewpoint that will lead to the higher-dimensional situation most naturally. If Ω is a bounded domain, or more generally a hyperbolic Riemann surface, and if p is a point of Ω, then the differential map $\iota\colon f \mapsto df$, $f \in I_p$, is an injective continuous isomorphism onto a necessarily compact subgroup, as already discussed (Corollary 1.3.7 for bounded domains; hyperbolic Riemann surfaces, from Proposition 2.4.4). In the case of complex dimension 1, it is an isomorphism onto a subgroup of $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ as a multiplicative group, if we interpret the differential in complex terms. And since the only compact subgroups of $\mathbb{C}^\times$ are subgroups of the *circle group* T, the image of ι must lie in T. [In the bounded domain case, this is of course apparent from Cauchy estimates directly: if df were not in T, then either df or df^{-1} would have absolute value larger than 1, and the differentials of powers of f or f^{-1} would then go to infinity in absolute value, contradicting Cauchy estimates.]

Now it is easy to see that any infinite subgroup of T is dense in T: given any (small) $\epsilon > 0$, there would necessarily be two distinct elements with angular separation less than ϵ, i.e., two elements of the form e^{θ_1} and e^{θ_2} with $|\theta_1 - \theta_2| < \epsilon$. But then the elements arising as powers of $e^{i\theta_1} \cdot e^{-i\theta_2}$ would be ϵ-dense in T. Since this is so for all positive ϵ, the group would be dense in T.

Now, in our case, the subgroup of T is not only infinite and hence dense, it is also compact and hence closed. Thus it is all of T, as we claimed above.

Now suppose that M is a Riemann surface with a point p having the following property:

(†) There is a smooth complete Hermitian metric h on M such that, for each pair of nonzero vectors v and w tangent to M at p, there is an element of I_p which takes v to a multiple of w and which is an isometry for the Hermitian metric.

Of course, if M has an automorphism-invariant metric, and in particular if M is a hyperbolic Riemann surface (or a bounded domain in particular), then this property (†) is equivalent to I_p acting transitively on real directions in the sense defined above, which is in turn equivalent to $\iota(I_p)$ being the whole circle group, with ι the injection onto differentials as above. But the property we are now assuming can hold even when M does not have an invariant metric, nor even a metric invariant for all of I_p, e.g., if M is the Riemann sphere.

Now consider the arc length parameter geodesics emanating from p. These are all isometry-equivalent, in the obvious sense, according to the hypothesis assumed in the previous paragraph. Hence either all are minimizing to infinity or all have a cut point (first point beyond which they are not minimizing) at the same distance from p. In the first case, the geodesic exponential map is a diffeomorphism onto M. In the second case, the Riemann surface M must be compact, and it is not hard to see that in fact it must be homeomorphic to a sphere. [This point will be treated in detail later on.]

In the case of a bounded domain, first the situation of the theorem of Bruné et al., the compact case is of course not relevant. We shall return to the general hyperbolic case in a moment. Restricting our attention to the bounded domain case, M must be diffeomorphic to a disc and, since it is hyperbolic in the Riemann surface sense, it must be biholomorphic to the disc.

In the more general hyperbolic Riemann surface case of the Theorem of Brune et al., the compact case is again irrelevant: one does not even need to prove that the Riemann surface would be a sphere and hence not hyperbolic, but rather it suffices to note that a compact hyperbolic Riemann surface has a finite automorphism group (Proposition 2.5.2) so that its isotropy cannot be transitive on real directions at any point.

However, it is possible for a nonhyperbolic Riemann surface to have property (†) above: as noted, the Riemann sphere has this property. It is easy to see from the discussion at the end of Section 2.2 that a torus has finite isotropy, though it has infinite automorphism group. So the Riemann sphere is in fact the only compact example with property (†) that is not hyperbolic. The only noncompact, nonhyperbolic Riemann surface with property (†) is of course $\mathbb{C}$ itself since, by the argument given, it would have to be diffeomorphic to the plane, but not biholomorphic to the unit disc.

These lines of development suggest that there might be generalizations to higher dimensions concerning manifolds with isotropy that is transitive on real directions, or more precisely, for which some compact subgroup of the isotropy is transitive on real directions. The compact subgroup condition is used to guarantee via averaging a suitably invariant metric h as above.

These expectations are in fact fulfilled, as this chapter continues.

6.1 Complex Manifolds with Large Isotropy at One Point

In this section, precise formulations will be presented of the general idea just discussed, that a compact subisotropy transitive on real directions at one point controls a complex manifold almost completely in both the compact and noncompact cases.

In the noncompact case, only $\mathbb{C}^n$ and the ball B^n can have this property, and in the compact case only $\mathbb{CP}^n$. While it was relatively easy to check these conclusions for Riemann surfaces, as in the previous paragraphs, the results are much more subtle in higher complex dimensions, and the proof involves an extended sequence of steps in both the noncompact and compact cases. We discuss the noncompact case first.

Theorem 6.1.1. *Let M be a noncompact complex manifold of complex dimension n. Let $p \in M$. Assume that there is a compact subgroup H of the isotropy group I_p of p with the following property: for any two real tangent vectors η, ξ at p there is an element $h \in H$ such that $dh|_p(\eta) = \lambda\xi$ for some real number λ.*

Then M is either biholomorphic to the unit ball in $\mathbb{C}^n$ or biholomorphic to $\mathbb{C}^n$.

The hypothesis of compactness of the subgroup H is essential. An example is provided by $\mathbb{C}^2 \setminus \{(1,0)\}$, $p = (0,0)$. Here the group I_p acts transitively on real directions at p, though no compact subgroup of I_p does—so there is no inconsistency with Theorem 6.1.1. To see that I_p does act with this transitivity, let v, w be nonzero tangent vectors at $p = (0,0)$. Choose a complex nonsingular linear transformation $A : \mathbb{C}^2 \to \mathbb{C}^2$ with $Av = w$. Of course A may not induce a map of $\mathbb{C}^2 \setminus \{(1,0)\}$ into itself since $A((1,0))$ may not equal $(1,0)$. However a biholomorphic map $\widehat{A}$ from $\mathbb{C}^2 \setminus \{(1,0)\}$ to itself fixing $(0,0)$ and having $d\widehat{A}\big|_{(0,0)} = A$ can be obtained by composing $A : \mathbb{C}^2 \to \mathbb{C}^2$ with suitable "shears" of the forms $(z,w) \mapsto (z, w + f(z))$ and $(z,w) \mapsto (z + g(w), w)$, where $f, g : \mathbb{C} \to \mathbb{C}$ are holomorphic functions. In particular, we can require $f(0) = g(0) = 0$ and $df|_0 = dg|_0 = 0$ and still have compositions of such "first-order constant" shears acting transitively on $\mathbb{C}^2 \setminus \{(0,0)\}$. Then a composition $F : \mathbb{C}^2 \to \mathbb{C}^2$ of such shears has $F((0,0)) = (0,0)$, $dF|_{(0,0)} =$ identity, and $F(A((1,0))) = (1,0)$. The map $\widehat{A}$ defined by $(z,w) \mapsto F(A((z,w)))$ then takes $\mathbb{C}^2 \setminus \{(1,0)\}$ biholomorphically to itself, takes $(0,0)$ to $(0,0)$, and has differential at $(0,0)$ taking v to w.

We return now to the direct consideration of Theorem 6.1.1. Since the elements of H are holomorphic, one can think of the action of H as "complex" and the differentials at p of elements of H as being not just real linear maps on the real tangent space at p, but as complex linear maps on the complex (holomorphic) tangent space at p. To make this more explicit, we write as usual $(z_1, \ldots, z_n)$ for the complex coordinates (local holomorphic coordinates around p in the manifold case). As usual, the holomorphic tangent space is defined at p as the span over $\mathbb{C}$ of the vectors $\partial/\partial z_j$; these vectors belong to the complexification of the real tangent space at p. This space is independent of the choice of holomorphic local coordinates. Also, as usual, we set $z_j = x_j + iy_j$, $j = 1, \ldots, n$, and x_j, y_j real. As a real mapping, the differential of an element $h \in H$ has matrix representation given by the real $2n \times 2n$ Jacobian matrix with elements $\partial u_j/\partial x_k, \partial u_j/\partial y_k, \partial v_j/\partial x_k, \partial v_j/\partial y_k$, $j, k = 1, \ldots, n$, where u_j, v_j are the real component functions of $h(x_1, y_1, \ldots, x_n, y_n)$. Associated to this situation is the complex $n \times n$ Jacobian matrix $(\partial h_j/\partial z_k)$, where $h = (h_1, \ldots, h_n)$ is the complex component expression of $h(z_1, \ldots, z_n)$. We think of this matrix as giving a $\mathbb{C}$-linear map of the holomorphic tangent space at p to itself. Of course the real Jacobian and the complex Jacobian are each determined by the other, via the Cauchy–Riemann equations.

Theorem 6.1.1 as stated deals with the situation wherein the real differentials at p of elements of H act transitively on the real vectors at p up to scalar multiples. It is natural to ask whether the same result holds if one assumes not this "real transitivity" up to scalar multiples, but rather transitivity of the action of the complex differentials up to scalar multiples. Precisely, one could take as hypothesis that, for each pair of nonzero vectors $v, w \in \mathbb{C}^n$, there should be an element $h \in H$ such that the complex Jacobian of h takes v to λw for some $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. This complex transitivity up to scalars is clearly implied by real transitivity up to scalars.

When $n = 1$, this complex condition is satisfied automatically. But real-up-to-scalars transitivity is not. Thus the complex condition is strictly weaker in case $n = 1$. But as pointed out in [Bland/Duchamp/Kalka 1987], when $n \geq 2$, the complex condition, seemingly weaker, in fact implies the real condition. The argument for this is topological, as follows.

Complex transitivity up to complex scalars is equivalent to transitivity of the complex projective space $\mathbb{CP}^{n-1}$ associated to the holomorphic tangent space at p. Now we can assume that the real differentials at p of elements of H are isometric relative to some Hermitian metric (at p) because H is compact, so we can average an arbitrary Hermitian metric to obtain this result. Then we can show in effect that transitivity on $\mathbb{CP}^{n-1}$ implies transitivity on the unit sphere S^{2n-1} relative to the Hermitian metric. Again, since H is compact and hence has closed orbits, transitivity on this S^{2n-1} is equivalent to some orbit having an interior point. For then every point of the orbit is interior so the orbit is both open and closed, and is hence all of S^{2n-1}. So suppose, for proof by contradiction, that no orbit of H acting on S^{2n-1} is open. Transitivity on $\mathbb{CP}^{n-1}$ then shows that, for each point $v \in S^{2n-1}$, there is a unit vector w tangent to S^{2n-1} at v (i.e., w is perpendicular to v) such that w has zero projection on $\mathbb{CP}^{n-1}$ under the usual projection of the tangent space at v on the tangent space to $\mathbb{CP}^{n-1}$ at the equivalence class of v in $\mathbb{CP}^{n-1}$. Since the kernel on tangent vectors of this projection is codimension 1, simple connectivity of S^{2n-1}, when $n \geq 2$, implies that w can be chosen as a continuous function of v. This gives a continuous section of the Hopf fibration $S^{2n-1} \to \mathbb{CP}^{n-1}$. This is a contradiction to the nontriviality of the Hopf fibration. Thus, again as pointed out in [Bland/Duchamp/Kalka 1987], Theorem 6.1.1 has the following consequence.

Corollary 6.1.2. *If M is a noncompact complex manifold of dimension $n \geq 2$ and there is a point $p \in M$ and a compact subgroup H of I_p which has differentials acting on the complex holomorphic tangent space at p transitively up to complex scalar multiples, then M is biholomorphic to the unit ball in $\mathbb{C}^n$, or to $\mathbb{C}^n$ itself.*

In the case of a compact, complex manifold M which is directionally complex transitive at a point p in the sense we have been discussing, there is again a natural "model" manifold, first, $\mathbb{CP}^n$. And it is actually true that this is the only case that occurs. But the argument here, used in the original proof in [Oeljeklaus E. 1970], is somewhat different. The case $n = 1$ follows from uniformization as discussed in Chapter 2. When $n \geq 2$, one again deduces, as discussed, real transitivity. Moreover, the infinitesimal generators of H acting on real differentials, considered as vector fields of S^{2n-1}, must be of dimension at least $2n - 1$, by the transitivity. It follows that the space of these vector fields together with the J-image of each of these vector fields must be of dimension $2n$ when evaluated at points near p but not equal to p. (At p, all vector fields vanish.) Note that these holomorphic vector fields that arise as $J(\tau)$, τ an infinitesimal generator of H, are themselves holomorphic and have integral curves defined for all times $t \in (-\infty, +\infty)$. It follows that some orbit of I_p, albeit not of a compact subgroup of I_p, is an open neighborhood of p

with p deleted. The example that makes this idea clearer is rotation around 0 of the Riemann sphere $\mathbb{C} \cup \{\infty\}$, with J of its infinitesimal generator being the generator of (exponential) dilation $z \to e^t z$, $t \in (-\infty, +\infty)$, as already discussed (see Section 1.6).

The compact case (Theorem 6.1.3 below) now follows from [Oeljeklaus E. 1970]. This result was obtained later in the incisive paper of [Bland/Duchamp/Kalka 1987] by using Theorem 6.1.1 as already shown by [Greene/Krantz 1985a] and considering the cut locus of the exponential map for an H-invariant metric. Note that, in the noncompact case, the reduction to the case where $M \setminus \{p\}$ is homogeneous does not apply: the unit ball with the origin removed is not homogeneous since automorphisms extend to be automorphisms of the whole unit ball taking the origin 0 to 0. Hence Aut $(B^n \setminus \{0\})$ consists only of unitary rotations about the origin and is therefore not transitive on $B^n \setminus \{0\}$.

To state explicitly the result for the compact case:

Theorem 6.1.3 ([Oeljeklaus E. 1970], later [Bland/Duchamp/Kalka 1987]). *If M is a compact complex manifold of dimension n and if there are a point $p \in M$ and a compact subgroup H of I_p such that H acts transitively on the holomorphic tangent spaces of M at p up to (complex) scalar multipliers, then M is biholomorphic to $\mathbb{CP}^n$.*

We turn now to the proof of Theorem 6.1.1. The proof via the methods of [Bland/Duchamp/Kalka 1987] for the theorem in the compact case just stated will be summarized in Section 6.7.

6.2 Proof of Theorem 6.1.1: An Invariant Metric

For simplicity of exposition, we first present the proof in the case that the manifold M is biholomorphic to a domain in $\mathbb{C}^n$, i.e., wherein M admits global coordinates. Later we shall describe the modifications needed to make the proof apply to noncompact complex manifolds in general.

Let Ω be a domain in Theorem 6.1.1 so that there is a compact subgroup H of I_x for some $x \in \Omega$ which has the transitivity property stated in the theorem. The first step of the argument is to construct a special metric invariant under H.

Step 1. *Construction of an Invariant Metric.* There is a Hermitian metric h_0 on Ω that is invariant under the action of elements of H.

Proof. Let $g = (g_{ij})$ be the Euclidean metric on Ω. Set

$$h_0 = \int_{\alpha \in H} \alpha^* g \, d\alpha,$$

where integration is with respect to the bi-invariant Haar measure on H. Then h_0 is an H-invariant Hermitian metric on M. □

Step 2. *Construction of an Invariant Exhaustion Function.* There is an exhaustion function for Ω that is invariant under the action of H.

Proof. Select a C^∞ function $F_1 : \Omega \to \mathbb{R}$ such that, for each $t \in \mathbb{R}$, the set $F_1^{-1}((-\infty, t])$ has compact closure in Ω. The existence of such a C^∞ "exhaustion function" on any noncompact manifold is a standard result of differential topology.

Then, for each $q \in \Omega$, set

$$F(q) = \int_{\alpha \in H} F_1(\alpha(q))\, d\alpha.$$

This function F has the required properties. □

Step 3. *Construction of a Complete Invariant Metric.* There is a complete metric on Ω which is invariant under the action of H.

Proof. Let h_0 be as in Step 1 and F be as in Step 2. For any C^∞ function $\chi : \mathbb{R} \to \mathbb{R}$, we set

$$h_\chi = \left(e^{\chi \circ F}\right) \cdot h_0.$$

This metric is C^∞ Hermitian and is also, by construction, invariant under the action of H. If χ is selected so that it increases sufficiently rapidly as $t \to \infty$, in particular so that

$$\mathrm{dis}_{h_\chi}\left(F^{-1}((-\infty, t]), F^{-1}([t+1, +\infty))\right) \geq 1,$$

then the metric h_χ will be complete. □

In what follows, $h = h_\chi$ will always denote a complete H-invariant metric as in Step 3.

Step 4. *Global Regularity of the h-Exponential Map.* Let p be the fixed point for H of Ω. The (geodesic) exponential map

$$\exp_p : T\Omega_p \to \Omega,$$

for the invariant metric h, is a global diffeomorphism of the tangent space $T\Omega_p \cong \mathbb{R}^{2n}$ onto Ω.

Proof. Since Ω is complete under the metric h, but noncompact, there is at least one half-infinite geodesic emanating from p, call it $C_0 : [0, +\infty) \to \Omega$, such that $C_0(0) = p$ and such that $\mathrm{dis}(p, C_0(t)) = t$ for all $t \geq 0$. [One calls such a curve a *geodesic ray* emanating from p.] This assertion is proved by a standard limiting process (see, for instance, [Gromoll/Meyer 1969]).

Since the metric is H-invariant, we know that the map $t \mapsto \alpha(C_0(t))$ is, for each $\alpha \in H$, a geodesic ray. Because H acts transitively on real tangent directions at p (here is one place that we use our hypothesis), it follows that *every* arc length parameter geodesic $C : [0, \infty) \to \Omega$ emanating from p is in fact a geodesic ray: every arc length parameter geodesic from p is the image of C_0 under some α. One need only take $\alpha \in H$ such that

$$\left(D\alpha\big|_p\right)(C_0'(0)) = C'(0).$$

It is a standard result of Riemannian geometry (p. 100, vol. II, [Kobayashi/ Nomizu 1963]) that, if every arc length parameter geodesic from a point of a Riemannian manifold is a ray, then the exponential map is a global diffeomorphism. □

6.3 Proof of Theorem 6.1.1: Biholomorphisms of Metric Balls

Throughout this section, we continue to fix $p \in \Omega$ as in the statement of the theorem, and choose an invariant metric h as in Step 3 of the last section.

The principal result of the present section is that any metric ball centered at p is biholomorphic to the unit ball in $\mathbb{C}^n$. It follows that Ω is the increasing union of domains biholomorphic to the ball. It does *not* then follow automatically that Ω is biholomorphic to the ball—even in the case that Ω is bounded ([Fornæss 1976]). But, since we have the point p fixed at the center of each ball, we shall in fact be able to extract a normal family of mappings that converges to a biholomorphism of Ω to the unit ball B^n in $\mathbb{C}^n$ (or to a biholomorphism with all of $\mathbb{C}^n$).

Our proof proceeds by the continuity method. We set

$$R_0 = \{r > 0 : B(p,r) \text{ is biholomorphic to } B^n\}.$$

The argument has four steps:

Step 1. *Nonemptiness of R_0:* The set R_0 contains at least one element.

Proof. We construct equivariant coordinates by averaging over the group action. In detail, choose $r > 0$ so small that the metric ball $B(p,r)$ has a holomorphic coordinate system. Denote that system by $w_1, \ldots, w_n$, and suppose that $w_j(p) = 0$, $j = 1, \ldots, n$. Let

$$S = \{q : \alpha(q) \in B(p,r) \text{ for all } \alpha \in H\}.$$

In what follows, if $\alpha \in H$ then let A_α denote the Jacobian matrix of α at p in the $w_1, \ldots, w_n$ coordinates. We will let A_α^{-1} act on the vector $(w_1(\alpha(q)), \ldots, w_n(\alpha(q)))$ by matrix multiplication.

Define a map F of S into $\mathbb{C}^n$ by

$$F : q \longmapsto \int_{\alpha \in H} A_\alpha^{-1}(w_1(\alpha(q)), \ldots, w_n(\alpha(q)))\, d\alpha.$$

The map F is defined on a neighborhood U of p because $\bigcap_{\alpha \in H} \alpha(B(p,r))$ contains a neighborhood of p by the compactness of H. In addition, F is holomorphic on U and has nonsingular Jacobian at p; in fact, the Jacobian of F at p is the identity matrix in $w_1, \ldots, w_n$ coordinates.

Thus there is an $r_0 > 0$ such that F is a holomorphic coordinate system on $B(p, 2r_0)$ (that is to say, the component functions $F_1, \ldots, F_n$ form a coordinate frame, with nondegenerate Jacobian matrix). Furthermore, the action of H on $B(p, 2r_0)$ is linear when expressed in the F-coordinates. By assigning to $\mathbb{C}^n$ a suitable Hermitian metric—that is, a complex linear change of coordinates—the action of H can be taken to be unitary in the F-coordinates.

We conclude that the F-image of $B(p, r_0)$ must be a standard Euclidean ball

$$\left\{(z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_j |z_j|^2 < \rho\right\}$$

for some $\rho > 0$ because $F(S(p, r_0))$ is a $(2n-1)$-dimensional orbit of a subgroup of the unitary group and is thus a sphere. Hence $B(p, r_0)$ is biholomorphic to the unit ball B_n of $\mathbb{C}^n$. $\square$

Step 2. *Connectivity of* R_0*:* If $r_1 \in R_0$ and if $0 < r_2 < r_1$, then $r_2 \in R_0$.

Proof. By hypothesis, there is a biholomorphic map

$$f : B(p, r_1) \to B^n.$$

Without loss of generality, we may suppose that $f(p) = 0$. Then the map

$$H \ni \alpha \to f \circ \alpha \circ f^{-1}$$

takes H isomorphically to a subgroup H_0 of the unitary group $U(n)$ (the unitary group being the isotropy group of the origin in the domain the ball).

If $0 < r_2 < r_1$, then the sphere $S(p, r_2)$ (in the h-metric) is the orbit under H of each point $q \in S(p, r_2)$ (see also what was said on this matter in the proof of Step 1). Thus $f(S(p, r_2))$ is the orbit under H_0 of $f(q)$. In particular, $f(S(p, r_2))$ is a sphere in B^n with center at $0 \in \mathbb{C}^n$. That is to say,

$$f(S(p, r_2)) = \{z \in \mathbb{C}^n : \|z\| = s\}$$

for some $s < 1$. Since $f(p) = (0, \ldots, 0)$, connectivity considerations show that

$$f(B(p, r_2)) = \{z \in \mathbb{C}^n : \|z\| < s\}.$$

Hence the mapping

$$q \mapsto \frac{1}{s} f(q)$$

sends $B(p, r_2)$ biholomorphically onto the unit ball B^n. $\square$

Step 3. *Openness of R_0:* If $r_1 \in R_0$ then there is an $\epsilon > 0$ such that $[r_1, r_1 + \epsilon) \subseteq R_0$.

Proof. Let $r_1 \in R_0$. Suppose for the moment the following:

> **Lemma A.** *The sphere $S(p, r_1)$ is a strongly pseudoconvex hypersurface.*

This lemma will be proved in Section 6.5. Assume it for now. Then there is an $\epsilon > 0$ and a C^∞ strictly increasing convex function $\chi : (r_1 - \epsilon, r_1 + \epsilon) \to \mathbb{R}$ such that $\chi \circ (\mathrm{dis}(p, \cdot))$ is C^∞ plurisubharmonic on $\{q \in \Omega : r_1 - \epsilon < \mathrm{dis}(p, q) < r_1 + \epsilon\}$. This is a standard fact about strongly pseudoconvex domains: any defining function can be made strictly plurisubharmonic in some neighborhood of the boundary by such a composition with a suitable convex increasing function.

Then, for each $r \in [r_1, r_1 + \epsilon)$, the sphere $S(p, r)$ is C^∞ strongly pseudoconvex, since it has a strictly plurisubharmonic defining function: the level sets of $\chi \circ \mathrm{dis}(p, \cdot)$ are of course the same as those of $\mathrm{dis}(p, \cdot)$ itself, first the spheres around p. In other words, the ball $B(p, r)$ is a strongly pseudoconvex domain. Therefore the $\overline{\partial}$ problem can be solved on $B(p, r)$ by the methods of [Folland/Kohn 1972]. We shall solve such a problem in a moment.

Let $F : B(p, r_1) \to B^n$ be a biholomorphic mapping onto the unit ball in $\mathbb{C}^n$ and $F_1, \dots F_n$ the component functions of F. The action of H expressed in $F_1, \dots, F_n$ coordinates is then linear and, in fact, unitary. If $r > r_1$, then there is a diffeomorphism D_r of the closure of $B(p, r)$ onto the closure of $B(p, r_1)$ that commutes with elements of H; that is, $D_r \circ \alpha = \alpha \circ D_r$ for all $\alpha \in H$. Such a diffeomorphism D_r can be obtained by deformation along the radial geodesics from p, for instance. Also, the D_r can be chosen so that, when r is close to r_1, the corresponding D_r is close to the identity on the closure of $B(p, r)$ in the C^∞ topology.

The functions $F_i \circ D_r$ need not be holomorphic on $B(p, r)$—after all, D_r itself is not holomorphic. However D_r was selected to be C^∞ close to the identity. Thus $\overline{\partial}(F_i \circ D_r)$ is C^∞ small. Let $g_i : B(p, r) \to \mathbb{C}$ be the Kohn solution of the equation $\overline{\partial} g_i = \overline{\partial}(F_i \circ D_r)$; that is, g_i is orthogonal in the L^2 sense to holomorphic functions on $B(p, r)$ relative to the Hermitian metric h. By standard estimates for the canonical solution of the $\overline{\partial}$ problem (see [Greiner/Stein 1977]), each function g_i will be small in the uniform topology and hence have small derivatives.

Now, for r near to r_1, the g_i will be C^1 close to 0 on the closure of $B(p, r)$ and the functions

$$F_1 \circ D_r - g_1, \dots, F_n \circ D_r - g_n \tag{6.1}$$

will form a holomorphic coordinate system on $B(p, r)$. [See [Greene/Krantz 1981], as well as our discussion in Sections 3.5–3.6, for another example of this type of argument.]

The action of H on $B(p, r)$ in the coordinates (6.1) is again linear. In fact, it is the same linear action as that of H on $B(p, r_1)$ in $F_1, \dots, F_n$ coordinates.

To see this, let $\alpha \in H$ and let A_α be the matrix that represents the action of α on the coordinates F_i. By this we mean

$$A_\alpha(F_1, \ldots, F_n) = (F_1 \circ \alpha, \ldots, F_n \circ \alpha).$$

Also note that

$$\begin{aligned}(F_1 \circ D_r \circ \alpha, \ldots, F_n \circ D_r \circ \alpha) &= ((F_1 \circ \alpha)D_r, \ldots, (F_n \circ \alpha)D_r) \\ &= A_\alpha(F_1 \circ D_r, \ldots, F_n \circ D_r).\end{aligned}$$

Taking $\overline{\partial}$ of both sides of this last identity yields

$$A_\alpha(\overline{\partial}(F_1 \circ D_r), \ldots, \overline{\partial}(F_n \circ D_r)) = \overline{\partial}(F_1 \circ D_r \circ \alpha, \ldots, F_\alpha \circ D_r \circ \alpha).$$

Thus

$$\begin{aligned}\overline{\partial}(A_\alpha(g_1, \ldots, g_n)) &= A_\alpha(\overline{\partial}g_1, \ldots, \overline{\partial}g_n) \\ &= A_\alpha(\overline{\partial}(F_1 \circ D_r), \ldots, \overline{\partial}(F_n \circ D_r)) \\ &= (\overline{\partial}(F_1 \circ D_r \circ \alpha), \ldots, \overline{\partial}(F_n \circ D_r \circ \alpha)).\end{aligned}$$

Since α is holomorphic and is also isometric for the metric h, it also holds that

$$\overline{\partial}(g_1 \circ \alpha, \ldots, g_n \circ \alpha) = (\overline{\partial}(F_1 \circ D_r \circ \alpha), \ldots, \overline{\partial}(F_n \circ D_r \circ \alpha))$$

and that the $g_j \circ \alpha$ are h-orthogonal to the space of holomorphic functions. By the uniqueness of the Kohn solution of the $\overline{\partial}$ problem, we see that $(g_1 \circ \alpha, \ldots, g_n \circ \alpha) = A_\alpha(g_1, \ldots, g_n)$. The linearity of the action of H, when expressed in the coordinate system (6.1), then follows.

Since H acts linearly on $B(p, r)$ (in suitable coordinates) for all r sufficiently close to r_1, it follows by the same argument as in Step 1 that $B(p, r)$ is in fact biholomorphic to the unit ball. $\square$

Step 4. *Closedness of* R_0*:* If $r \in R_0$ for all $0 < r < r_1$ then $r_1 \in R_0$.

Proof. As already noted, there are no general results about increasing unions of balls that make our result automatic (cf. [Fornæss 1976]).

We first consider the relationship between two metric balls $B(p, r_1)$ and $B(p, r_2)$ in Ω when $0 < r_1 < r_2$. Let $f_{r_j} : B(p, r_j) \to B^n$, for $j = 1, 2$, be biholomorphic mappings of these two balls into the unit ball of $\mathbb{C}^n$. By an argument that was already used in the proof of Step 2, the map f_{r_2} sends $B(p, r_1)$ biholomorphically onto a ball $B(0, s) \subseteq \mathbb{C}^n$, some $0 < s < 1$. Then the map $[1/s]f_{r_2}$ and the map f_{r_1} are both biholomorphisms of $B(p, r_1)$ onto B^n which take p to 0. Therefore there is an $\omega \in U(n)$ such that

$$\frac{1}{s} \cdot \omega^{-1} \circ f_{r_2}\Big|_{B(p,r_1)} = f_{r_1}.$$

It follows from the repeated application of this argument that if $r_1 < r_2 < \cdots$ and if $\lim_{r_k \to \infty} r_k = r_\infty$, then there is a sequence of biholomorphic mappings F_j of $B(p, r_j)$ onto balls about 0 in $\mathbb{C}^n$ (not necessarily unit balls) with the property that

$$F_{j+1}\big|_{B(p,r_j)} = F_j \ , \quad j = 2, 3, \ldots .$$

Taken together, these maps determine a biholomorphic mapping of $B(p, r_\infty) = \bigcup_j B(p, r_j)$ onto an open ball around $(0, \ldots, 0)$ or onto $\mathbb{C}^n$. In particular, it follows that $r_\infty \in R_0$ provided that $\bigcup_j F_j(B(p, r_j))$ is not equal to $\mathbb{C}^n$.

To prove that the union is not equal to $\mathbb{C}^n$ requires estimation of the radial derivatives of the map determined by the F_j. This estimate is a natural byproduct of the proof of Lemma A and related ideas. This matter will be treated in Section 6.5. $\square$

6.4 Proof of Theorem 6.1.1, Assuming Lemma A

Again let Ω be a domain in $\mathbb{C}^n$ and let p be a point in Ω such that there is a compact subgroup H of the isotropy group I_p that acts transitively on real directions as in the statement of the theorem. Let h be the complete invariant metric as constructed in the previous section. By the argument given at the end of that section, each h-metric ball $B(p, r)$, $r > 0$, is biholomorphic to the unit ball $B^n \subseteq \mathbb{C}^n$. From the proof of this fact, these biholomorphisms are closely related to each other (when r varies). In particular, a biholomorphism $F : B(p, r) \to B^n \subseteq \mathbb{C}^n$ with $F(p) = 0$ necessarily has the property that, for each positive $0 < r_1 < r$, the restriction $F\big|_{B(p,r_1)}$ is a biholomorphic map onto a ball in $\mathbb{C}^n$ centered at 0.

Now select a biholomorphic map F_1 of $B(p, 1)$ onto the unit ball $B^n \subseteq \mathbb{C}^n$. With F_1 so fixed, choose next a map F_2 of $B(p, 2)$ into $\mathbb{C}^n$ such that

(i) $F_2(p) = 0$;
(ii) $dF_2\big|_p = dF_1\big|_p$;
(iii) $F_2 : B(p, 2) \to B^n(0, r_2) \subseteq \mathbb{C}^n$ for some $r_2 > 0$.

Such an F_2 can be obtained from a biholomorphic map $G_2 : B(p, 2) \to B^n$ with $G_2(p) = 0$ simply by applying a unitary rotation and a dilation.

With F_2 so chosen, write $F_2(B(p, 1)) = B^n(0, r) \subseteq \mathbb{C}^n$. Then $F_2 \circ F_1^{-1} \equiv G$ maps $B^n(0, 1) = B^n \subseteq \mathbb{C}^n$ biholomorphically onto $B^n(0, r) \subseteq \mathbb{C}^n$. Moreover, $G(0) = 0$ and $dG\big|_0 = \mathrm{id}$. It follows now from elementary analysis (or from Schwarz's lemma) that $r = 1$ and $F_2\big|_{B(p,1)} = F_1$.

The construction that we have just presented can be iterated: Choose $F_3 : B(p, 3) \to \mathbb{C}^n$ such that (i)–(iii) hold with the index 2 replaced by the index 3 and the index 1 replaced by the index 2. Then it follows, just as above, that $F_3\big|_{B(p,2)} = F_2$. Continuing in this fashion, we obtain a sequence of biholomorphic maps onto balls $F_k : B(p, k) \to \mathbb{C}^n$ such that $F_k\big|_{B(p,k_1)} = F_{k_1}$

whenever $0 < k_1 < k$. These maps together yield a biholomorphism of Ω onto $\bigcup_k F_k(B(p,k))$. Since each $F_k(B(p,k))$ is a ball centered at zero, and since the maps F_k agree on the intersections of their domains, it follows that $\bigcup_k F_k(B(p,k))$ is either a ball centered at 0 or is $\mathbb{C}^n$. $\square$

6.5 Statement and Discussion of Lemma A

In order to complete the proof of Theorem 6.1.1, it remains to establish Lemma A. Recall what this lemma said.

Lemma A. *For any $r_1 \in R_0$, $S(p,r_1) \equiv \partial(B(p,r_1))$ is a strongly pseudoconvex hypersurface.*

Now we discuss the proof. The proof repeats ideas that we have already seen earlier, and so it will be only summarized now. (The reader may consult [Greene/Krantz 1985a] for all the details.)

Suppose that $F : B(p,r_0) \to \mathbb{C}^n$ is a holomorphic embedding with the property that, for each $0 < r < r_0$, $F(B(p,r))$ is a Euclidean ball centered at 0. Assume further that F commutes with the action of the compact group H in I_p that acts transitively on directions. Observe that, for each $\alpha \in H$, the map $F \circ \alpha \circ F^{-1}$ is a biholomorphic map of Euclidean balls that fixes the origin and is therefore linear.

Now we define a norm

$$\|F\|_1 \equiv \sup\{\|dF\big|_q(v)\| : v \in TM_q, \|v\|_h = 1, \mathrm{dis}(p,q) < r_0\}.$$

Here $\| \ \|_h$ is the length of a vector with respect to the Hermitian metric h.

Sublemma 6.5.1. *The quantity $\|F\|_1$ is finite.*

Sketch of the proof. The concept of the proof is as follows: F commutes with H, and H acts in the tangential direction to the spheres. And of course a Lie group (such as H) is smooth, so F automatically satisfies a smoothness *estimate* in tangential directions. The size of that estimate will, by scaling, depend in a natural way on the radius. If we let $s(r)$ denote the Euclidean distance from $0 \in \mathbb{C}^n$ to any point $F(q)$ where $\mathrm{dis}(p,q) = r$, then this quantity is independent of the choice of q in the sphere of radius r. And we have the estimate (with $s = s(r)$)

$$\|dF\big|_q(v)\| \le C_0 \cdot s \cdot \mathrm{dis}(p,q) \qquad (6.2)$$

for any vector v that is tangent to the h-metric sphere of radius r about p.

Now we use the complex structure. By the Cauchy–Riemann equations (or, more formally, by using the complex structure tensor J), we know that the normal vector $\nu = \nu_q$ at any point q of the sphere of radius r is just i times a (real) tangent vector (first, the tangent vector that is usually designated as "complex normal"). Thus, essentially for free, we obtain from (6.2) the estimate

$$\|dF\big|_q(\nu)\| \le C_0 \cdot s \cdot \mathrm{dis}(p,q). \tag{6.3}$$

Combining (6.2) and (6.3) yields the uniform bound on first derivatives of F that is required by the sublemma. $\square$

A similar argument allows us to estimate any number of derivatives of F on a metric ball in Ω. This yields the following.

Proposition 6.5.2. *If $F : B(p, r_0) \to B^n$ is a biholomorphic mapping that commutes with the action of H, then F extends to a C^∞ map on $B(p, r_0) \cup S(p, r_0)$ and $F^{-1} : B^n \to B(p, r_0)$ extends to a C^∞ map on the closed unit ball $cl(B^n) \subseteq \mathbb{C}^n$.*

Lemma A now follows immediately because $S(p, r_1)$ corresponds, under a C^2 diffeomorphism induced by a biholomorphic mapping, to the unit sphere in Euclidean space (which itself is strongly pseudoconvex).

6.6 Alternative Viewpoints and the Modification for the Theorem in the Manifold Case

The process of averaging with respect to the action of the compact group H was crucial to the proof just discussed (when M was a domain in $\mathbb{C}^n$) of Theorem 6.1.1. But, outside of this basic construction, the use of Lie group theory was minimized, at the price of using rather powerful results from complex analysis.

It is worth noting, however, that another viewpoint illuminates the picture in a different light. Let us return to the situation of the H-invariant complete Hermitian metric h and the associated spheres $S(p, r), r > 0$. These are smooth (real) hypersurfaces, since $\exp_p$ is a C^∞ diffeomorphism: $S(p, r) = \{\exp_p v \colon \|v\| = r,\ v \in T_pM\}$, by standard Riemannian geometry. [The metric h has infinite injectivity radius,[1] etc.; see [Kobayashi/Nomizu 1963] or [Petersen 2006].]

As such, these spheres inherit a CR-structure of hypersurface type from the complex structure of M. This would also be true in the general case of M being an open complex manifold, not just in the case of M being a domain in $\mathbb{C}^n$.

Now, by the hypothesis about the transitivity in directions of H, H acts on each $S(p, r)$, $r > 0$, transitively. And, since the elements of H are holomorphic maps of M, the induced actions on $S(p, r)$ are via CR-diffeomorphisms.

The reason for considering the situation is this: a homogeneous CR-manifold of hypersurface type which is homeomorphic to S^{2n-1}, $n \ge 3$ is

[1] The *injectivity radius* ι_p of a Riemannian manifold from a point, say p, is the supremum of positive radii within which no two distinct geodesic rays emanating from p meet away from p. Then the *injectivity radius* of the manifold is the infimum of all these ι_p, over all p. Hence, if $\iota = +\infty$, no two geodesics emanating from the same point meet except at their initial point.

CR-diffeomorphic to S^{2n-1} with its standard Cauchy-Riemann structure ([Lehmann/Feldmueller 1987]). Thus, for complex dimension $n \geq 3$, one has an alternative approach to Theorem 6.1.1: each h-metric ball $B(p,r)$ is an open set with smooth boundary and its boundary is CR-equivalent to a standard ball. In particular, it is then quite easy to show that each $B(p,r)$ is H-equivariantly biholomorphic to the unit ball in $\mathbb{C}^n$, obviating for example the use of fairly delicate $\overline{\partial}$ methods. In particular, the conclusion of Lemma A is immediate, since a CR-hypersurface that is intrinsically CR-diffeomorphic to a CR-strongly pseudoconvex hypersurface—the boundary of the ball in this case—is necessarily strongly pseudoconvex in the usual sense, of embedding strong pseudoconvexity.

The proof we gave was needed, however. For $n = 2$, the homogeneity of a CR-structure on $S^{2n-1} = S^3$ definitely does not imply CR-equivalence to the standard $S^3 \subset \mathbb{C}^4$ structure. This was discovered long ago, by E. Cartan. So the alternative approach just outlined does not apply.

The modifications of the proof given for M = a domain in $\mathbb{C}^n$ to make it apply to a general noncompact complex manifold are relatively straightforward. First note that the H. Cartan result that shows that elements of H with equal differentials at p are equal extends immediately to the manifold case, because H is compact. The compactness means that if $\varphi : U \to \mathbb{C}^n$, $p \in U$ is a local coordinate system, then there is a neighborhood V of p such that $g(V) \subset U$ for all $g \in H$. In this situation, the argument given for Lemma A applies immediately. The constructions for invariant metrics and exhaustions and the behavior of the equivariant exponential map remain as given. So does the proof of openness of R_0, except that one must make the additional observation that $B(p,r)$, the interior of $S(p,r)$, $r > r_0$ but close to r_0, is a Stein manifold so that $\overline{\partial}$ is solvable. This follows from the strong pseudoconvexity of the boundary together with the absence of nontrivial compact subvarieties. This latter point is disposed of as follows. By the maximum principle for the strictly plurisubharmonic function ψ, such a variety would have to lie in $B(p,r_0)$. But this is impossible since $B(p,r_0)$ is itself a Stein manifold, or easier, biholomorphic to an open subset of $\mathbb{C}^n$. The interested reader can consult [Greene/Krantz 1985a] for complete details of the argument in the manifold case.

6.7 The Compact Manifold Case

Now we shall give an outline of the proof of Theorem 6.1.3, that a compact complex manifold with directional homogeneity at one point must be a complex projective space. We shall discuss the proof given in [Bland/Duchamp/Kalka 1987] rather than the original proof given in [Oeljeklaus E. 1970], since the former is closer to the noncompact argument already given.

As in the noncompact case, one begins with an Hermitian metric h that is invariant under the action of the compact group H. Such a metric can be obtained by averaging, with respect to the action of H, an arbitrary Hermitian metric, as before. Also, as in the noncompact case, the map that sends $g \in H$ to $dg|_p$ (real differential) is an injective homomorphism of H into linear maps on T_pM to T_pM which are isometric for $h|_p$ and are "complex linear" on T_pM in the sense of commuting with the almost complex structure tensor $J : T_pM \to T_pM$. The injectivity of the homomorphism follows either from the argument in the style of the Cartan lemma as in Section 1.3 or more simply from the fact that isometries of h are determined by their differentials at a fixed point.

The existence of a compact group of isometries fixing a point p of a compact Riemannian manifold M and having differentials that are transitive on (real) tangent directions at that point already imposes severe restrictions on the manifold, independently of any complex structure considerations. In particular, the Riemannian exponential map $\exp_p : T_pM \to M$ has to behave as follows. There is an $r_0 > 0$ such that $\exp_p$ is a diffeomorphism of $\{v \in T_pM \colon \|v\| < r_0\}$ into M and $M = \exp_p(\{v \in T_pM \colon \|v\| \leq r_0\})$. Here every geodesic from p of length $\leq r_0$ is minimizing, but no geodesic from p of length $> r_0$ is minimizing. This follows immediately from transitivity of isometries of h on directions at p: the cut distance along a geodesic from p must be independent of which geodesic is chosen.[2]

In this situation, the "cut points" at distance r_0 from 0 in T_pM must be conjugate points, that is $d\exp_p|_v$ must be singular for each $v \in T_pM$ with $\|v\| = r_0$. This follows easily: if $d\exp_p|_v$ were nonsingular for one and hence every such v, one can see that M would have to be nonorientable (e.g., $\mathbb{RP}^{2k}$) whereas M, a complex manifold, must be orientable (cf. [Bland/Duchamp/Kalka 1987] for details). Thus the image under $\exp_p$ of $\{v \in T_pM \colon \|v\| = r_0\}$ must be of real codimension at least 2 in M.

Now, by Corollary 6.1.2, the complement in M of $\exp_p(\{v \in T_pM \colon \|v\| = r_0\})$, which is $\exp_p(\{v \in T_pM \colon \|v\| < r_0\})$, is biholomorphic to the unit ball in $\mathbb{C}^n$ or to $\mathbb{C}^n$ itself. In either case, this complement admits nonconstant holomorphic functions. So $\exp_p(\{v \in T_pM \colon \|v\| = r_0\})$ must have real codimension no more than 2, hence exactly 2. From now on, we denote this submanifold by C_p, following the notation of [Bland/Duchamp/Kalka 1987].

Also, the complement $M \setminus C_p$ cannot be biholomorphic to the unit ball. For then it would admit bounded nonconstant holomorphic functions. These would extend to all of M holomorphically, a contradiction, since M is compact and hence admits no bounded, nonconstant holomorphic functions.

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[2] In general, geodesics need not be shortest connections, although they are in short ranges. When one considers a geodesic ray emanating from, say, a point p, the first point at which the geodesic stops being the shortest connection (in Riemannian geometry, it is said in such a case that the geodesic stops "minimizing") is called a *cut point*. Considering all possible geodesic rays from p, one may consider the collection of cut points—this is the *cut locus* of p. See [Petersen 2006], [Kobayashi/Nomizu 1963] for detail.

Thus M is a compactification of $\mathbb{C}^n$. It is a general fact (cf. [Peternell/Schneider 1991]) that, for any compactification of $\mathbb{C}^n$ as a compact complex manifold M, $M \setminus \mathbb{C}^n$ is a complex subvariety of pure codimension 1, agreeing with the result already obtained that $\exp_p(\{v \in T_pM \colon \|v\| = r_0\})$ has real codimension 2.

Transitivity on directions shows that the space $C_p = M \setminus \mathbb{C}^n$ in our case is in fact a smooth, complex hypersurface. (This result is proved explicitly in [Bland/Duchamp/Kalka 1987] without appeal to the general result on compactification of $\mathbb{C}^n$.)

In complex dimension $n \leq 6$, every smooth compactification M of $\mathbb{C}^n$ is biholomorphic to $\mathbb{CP}^n$, with $M \setminus \mathbb{C}^n$ going to $\mathbb{C}^{n-1} \subset \mathbb{C}^n$ (cf. [Peternell/Schneider 1991]). Whether this is true for $n \geq 7$ seems to be unknown. So a separate argument in our directionally homogeneous case is required for $n \geq 7$.

Now $M \setminus C_p$ is biholomorphic to $\mathbb{C}^n$, via a biholomorphism to $\mathbb{C}^n$ via a biholomorphism which is equivariant with respect to the H-action on $M \setminus C_p$ and a compact, complex-linear group action on $\mathbb{C}^n$. This latter action can be taken to be a subgroup of $U(n)$. Thus the natural fibration of $\mathbb{C}^n \setminus \{(0, \ldots, 0)\}$ by complex lines through the origin can be transferred to $M \setminus C_p$ as a foliation of $M \setminus C_p \setminus \{p\}$ which is acted on by H in the sense that the H-image of such a leaf is another leaf. To complete the proof, one shows that each leaf meets C_p transversally in a single point: this gives a map of complex lines through the origin in $\mathbb{C}^n$ into C_p and identifies M into $\mathbb{C}^n$ compactified by $\mathbb{CP}^{n-1}$. The details of this follow the methods of [Brenton/Morrow 1978].

A subtlety arises in this approach from the fact that the group H need not be the full unitary group (here, we interpret H via identification with the linear group of differentials at p as before). These are compact groups of complex linear mappings of $\mathbb{C}^n$ which are transitive on the sphere but smaller than $U(n)$. These groups are, however, classified (cf. [Besse 1987]). They have the following property: For each real vector $v \neq 0$, let I_v be the set of H such that $g(v) = v$. Then let $F(v)$ be the set of w such that $g(w) = w$ for all $g \in I_v$. For the unitary group, $F(v)$ is the 2-dimensional subspace $\mathrm{Span}(v, Jv)$. For general compact groups H acting complex linearly and transitively on the unit sphere, $F(v)$ is either of dimension 2 or 4. This dimension is of course independent of the choice of v by the transitivity of H on directions. This follows from the classification mentioned. It can also be established directly; this is done in [Bland/Duchamp/Kalka 1987] by noting that if the dimension of $F(v)$ is k, then a group structure is induced on S^{k-1}, so $k = 2$ or 4 are the only possibilities.

In the case $k = 2$, [Brenton/Morrow 1978] applies directly. In the case of $k = 4$, a second application of [Brenton/Morrow 1978] is required: the 4-dimensional $F(v)$-subspace when compactified by the "points at infinity" of C_p becomes $\mathbb{CP}^2$ by [Brenton/Morrow 1978], and the structure of M as $\mathbb{CP}^n$ follows from that. The reader is referred to [Bland/Duchamp/Kalka 1987] for full details.

7

Some Other Invariant Metrics

Chapters 3 and 4 explored extensively the use of the Bergman metric in studying the function theory of bounded domains and in particular their automorphism groups. The Bergman metric also played a role in Chapter 5 in the Bedford/Dadok proof of the realizability of every compact Lie group as an automorphism group. In all these instances, the invariance of the Bergman metric made it possible to bring to bear powerful differential geometric methods that would not have been otherwise available. But there are other invariant metrics that also deserve attention for their utility in such investigations. The purpose of this chapter is to give a quick survey of some of them.

We denote throughout by Hol (M, N) the set of holomorphic mappings from a complex manifold M into another complex manifold N.

7.1 The Carathéodory Metric

7.1.1 The Carathéodory Metric and Distance

Let Ω be a domain in $\mathbb{C}^n$ and let D denote the unit open disc in $\mathbb{C}$ as before. Then what is known as the infinitesimal Carathéodory metric is defined as follows:

Definition 7.1.1. *For $p \in \Omega$ and $v \in \mathbb{C}^n$ the (infinitesimal) Carathéodory metric of v at p is defined to be*

$$\mathsf{C}_\Omega(p; v) := \sup\{|dh_p(v)| \colon h \in \text{Hol }(\Omega, D), h(p) = 0\},$$

where dh is the differential of h considered as a map from $\mathbb{C}^n$ to $\mathbb{C}$.

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The following lemma is immediate.

Lemma 7.1.2. *If Ω is a (not necessarily bounded) domain in $\mathbb{C}^n$, then*

(1) $\mathsf{C}_\Omega(p,v) \geq 0$,
(2) $\mathsf{C}_\Omega(p,\lambda v) = |\lambda|\mathsf{C}_\Omega(p,v)$,
(3) $\mathsf{C}_\Omega(p,v+w) \leq \mathsf{C}_\Omega(p,v) + \mathsf{C}_\Omega(p,w)$,

for any $p \in \Omega$ and any $v, w \in \mathbb{C}^n$.

Thus the infinitesimal Carathéodory metric C_Ω is a semi-norm on $\mathbb{C}^n$, for each point $p \in \Omega$. Note that the definition of C_Ω depends only upon the set $\mathrm{Hol}\,(\Omega, D)$ and is thus a holomorphic invariant for Ω. Moreover, there is a general *distance-nonincreasing property*:

Proposition 7.1.3. *Let $f : \Omega \to \Omega'$ be a holomorphic mapping from a domain $\Omega \subset \mathbb{C}^n$ into another domain $\Omega' \subset \mathbb{C}^m$. Then*

$$\mathsf{C}_{\Omega'}(f(p), df_p(v)) \leq \mathsf{C}_\Omega(p,v)$$

for every $p \in \Omega$ and every $v \in \mathbb{C}^n$. In particular, if f is a biholomorphism, so that necessarily $n = m$, then

$$\mathsf{C}_{\Omega'}(f(p), df_p(v)) = \mathsf{C}_\Omega(p,v)$$

for every $p \in \Omega$ and every $v \in \mathbb{C}^n$.

There is a related and even more easily defined idea of "Carathéodory distance" defined as follows: set

$$\delta^{\mathsf{C}}_\Omega(p,q) = \sup\{d_D(f(p), f(q)) \colon f \in \mathrm{Hol}\,(\Omega, D)\},$$

where d_D denotes the Poincaré distance on the unit open disc D. (Earlier this was called the Carathéodory metric on Ω, and the semi-norm on tangent vectors was referred to as the infinitesimal Carathéodory metric.) If $F : \Omega_1 \to \Omega_2$ and $f : \Omega_2 \to D$ are holomorphic, then $f \circ F$ is of course a holomorphic map of Ω_1 to D. It follows immediately that $\delta^{\mathsf{C}}_{\Omega_1}(p,q) \geq \delta^{\mathsf{C}}_{\Omega_2}(F(p), F(q))$: holomorphic maps are distance-nonincreasing relative to the Carathéodory distance.

One can also define the integrated distance from the Carathéodory metric on tangent vectors. first, the distance between two given points $p, q \in \Omega$ is defined by

$$d^{\mathsf{C}}_\Omega(p,q) = \inf \int_\gamma \mathsf{C}_\Omega(z, dz) := \inf \int_0^1 \mathsf{C}_\Omega(\gamma(t), d\gamma|_t(d/dt))\; dt$$

where the infimum is taken over all piecewise C^1 curves $\gamma : [0,1] \to \Omega$ with $\gamma(0) = p, \gamma(1) = q$. Then this pseudodistance (in the sense that it is not in general positive definite) also has the distance-decreasing property under the action of holomorphic mappings. Call it the *integrated Carathéodory distance* of Ω.

In general, these two distances do not coincide (see [Barth 1977] and also [Sibony 1972]); the integrated distance is generally larger. We shall not explore this matter any further since it plays no role in our later considerations.

7.1.2 C-Hyperbolicity

The Schwarz lemma in classical complex analysis implies immediately that

$$\mathsf{C}_D(p, v) = \frac{|v|}{1 - |p|^2}$$

for every $p \in D$ and every $v \in \mathbb{C}$: the Carathéodory metric coincides with the Poincaré metric on the unit disc, up to a constant multiple. This, together with the distance-decreasing property (the preceding proposition), in particular implies that the Carathéodory metric C_Ω is positive definite whenever Ω is a bounded domain in $\mathbb{C}^n$. The Carathéodory distance on the unit disc equals the Poincaré distance, essentially by definition.

Readers must have noticed already that the exposition on the Carathéodory distance does not have to be restricted to domains in the complex Euclidean space $\mathbb{C}^n$. The definition has an obvious analogue for a complex manifold. The same applies to the Carathéodory metric; We can define, for M a compact manifold, $p \in M$ and $v \in$ the holomorphic tangent space of M at p:

$$\mathsf{C}_M(p, v) := \sup\{|dh_p(v)| \colon h \in \text{Hol } (M, D), h(p) = 0\},$$

where dh is the differential of h considered as taking the holomorphic tangent $\mathbb{C}$-linearly to $\mathbb{C}$. However, little is known in generality on the question of which complex manifolds possess positive definite Carathéodory metric. In general, the construction of nonconstant bounded holomorphic functions lies outside the realm of the usual L^2 $\bar{\partial}$-theory, for example (see [Greene/Wu 1977] for a general discussion of these questions).

The classic papers by Carathéodory and by Reiffen (See [Wu H. 1993] for an extensive survey on various metrics including the Carathéodory metric, and references.) show further properties also: the Carathéodory metric is a continuous metric defined on every complex manifold. The distance-decreasing property is the most prominent of all, in that it produces useful applications. On the other hand, since the definition of this metric depends on the bounded holomorphic functions, it is identically zero on any compact complex manifold, or compact complex manifolds with subsets taken away, when the set (that was removed) happens to be necessarily a removable singularity set for bounded holomorphic functions.

S. Kobayashi presented the following extended definition for the Carathéodory metric on compact manifolds ([Kobayashi 1976]): first, call a complex manifold M *C-hyperbolic* if, for the universal covering $\pi : \widetilde{M} \to M$, $\mathsf{C}_{\widetilde{M}}$ is positive definite. In this case, Kobayashi defines the *C-metric* (i.e., the *extended Carathéodory metric*) on M by

$$\widehat{\mathsf{C}}_M(q,\xi) := \mathsf{C}_{\widetilde{M}}(\widetilde{q}, d\pi|_{\widetilde{q}}^{-1}(\xi)).$$

Notice that C_M can be identically zero while $\widehat{\mathsf{C}}_M$ is positive definite. The compact Riemann surfaces with genus at least 2 give standard examples. Note that, if $\widehat{\pi}\colon \widehat{M} \to M$ is a holomorphic covering such that $\widehat{M}$ has positive definite Carathéodory metric, then the universal cover $\widetilde{M}$ of M has positive definite Carathéodory metric. This follows because there is a holomorphic covering $\pi_{\widehat{M}} : \widetilde{M} \to \widehat{M}$ so that holomorphic maps of $\widehat{M}$ into D give rise to holomorphic maps of $\widetilde{M}$ into D by composition. Thus C-hyperbolicity of M could have been defined equivalently in terms of positive definiteness of the Carathéodory metric on some holomorphic covering of M, rather than the universal cover as such. This concept will be considered further in Section 7.5.4.

7.2 The Kobayashi Metric and Distance

It is natural to consider a construction that is in a sense dual to the familiar Carathéodory metric, looking at the infimum over maps from D into a domain or manifold, rather than the supremum over maps from the domain or manifold into D. S. Kobayashi's introduction of the Kobayashi (pseudo-) distance followed such an idea.

7.2.1 Kobayashi Distance

Specifically let Ω be a domain in $\mathbb{C}^n$ or more generally a complex manifold. Set

$$\delta^{\mathsf{K}}_\Omega(p,q) = \inf\{d_D(a,b)\colon h \in \mathrm{Hol}\ (D,\Omega), h(a)=p, h(b)=q\}.$$

The following is clear by considering compositions of holomorphic maps $h : D \to \Omega_1$ and $f : \Omega_1 \to \Omega_2$.

Proposition 7.2.1. *If $f : \Omega_1 \to \Omega_2$ is a holomorphic map from one domain into another, then*

$$\delta^{\mathsf{K}}_{\Omega_2}(f(z), f(w)) \le \delta^{\mathsf{K}}_{\Omega_1}(z,w)$$

for every $z, w \in \Omega_1$.

It is obvious that $\delta^{\mathsf{K}}_\Omega \ge 0$, but there is no apparent reason why it has to satisfy the triangle inequality; in fact the triangle inequality is known to fail in some cases.[1]

[1]If Ω is a bounded Reinhardt domain in $\mathbb{C}^n$ containing the origin 0, then the set of unit tangent vectors in $T_0\Omega$ (identified with $\mathbb{C}^n$ here) is similar (i.e., homothetic) to Ω. Thus one only needs to take a nonconvex Reinhardt domain to see the failure of the triangle inequality: e.g., $\{(z,w) \in \mathbb{C}^2\colon |z|^2 + \sqrt{|w|} < 1\}$.

But the concept of Kobayashi distance can be refined to obtain a pseudometric space structure (i.e., the triangle inequality will hold, but the distance of distinct points may be 0). Here is how it can be done: let $p, q \in \Omega$. Then take a set $\{p_0, \ldots, p_N\}$ of finitely many points in Ω such that $p = p_0$ and $q = p_N$. Call such a finite set a *partition.* Then define

$$d_\Omega^{\mathsf{K}}(p,q) = \inf \sum_{j=1}^{N} \delta_\Omega^{\mathsf{K}}(p_{j-1}, p_j),$$

where the infimum is taken over all possible partitions with $p = p_0, \ldots, p_N = q$. This d_Ω^{K} is the *Kobayashi (pseudo-)distance* of Ω. For this, the triangle inequality is clear, since such a chain from p to q together with a chain from q to r gives a chain from p to r: this is analogous to the proof of the triangle inequality for Riemannian distance and associated concepts of "length spaces" in metric space theory.

Proposition 7.2.2 (Kobayashi). *The Kobayashi pseudodistance d^{K} satisfies:*

(i) *d_D^{K} coincides with the Poincaré distance for the unit open disc D.*
(ii) *d^{K} satisfies the triangle inequality.*
(iii) *d^{K} is distance-nonincreasing, i.e., $d_{\Omega_2}^{\mathsf{K}}(f(z), f(w)) \le d_{\Omega_1}^{\mathsf{K}}(z,w)$ for any $z, w \in \Omega_1$ whenever f is a holomorphic mapping from Ω_1 into Ω_2. In particular, biholomorphic mappings preserve the Kobayashi distance.*

Corollary 7.2.3. *The Kobayashi distance is positive definite on every bounded domain.*

The corollary follows from the distance-nonincreasing property since if p, q are in a bounded domain and $p \ne q$, then there is a coordinate function z_ℓ such that $z_\ell(p) \ne z_\ell(q)$, when the z_ℓ-image of Ω is contained in some disc around 0. As Kobayashi himself explained in seminar lectures many times, this distance was developed in his mind as a geometric principle for the "little Picard theorem." We refer to [Kobayashi 1970] (and [Kobayashi 1998]) for detailed discussions. Also, note that $d_{B^1}^{\mathsf{k}}$ is positive definite (here, B^1 is of course the same as the unit open disc D) as one sees easily from the geometry of the Poincaré metric: the minimum over chains from p to q is realized by a single map of the unit disc and $d_{B^1}^{\mathsf{K}} = \delta_{B^1}^{\mathsf{K}}$. Also $d_{\mathbb{C}}^{\mathsf{k}} = 0$, since $\delta_{\mathbb{C}}^{\mathsf{K}} = 0$. This implies Liouville's theorem as follows. Let $f : \mathbb{C} \to B^1$ be holomorphic. Then, for any $z \in \mathbb{C}$, by the distance-decreasing property,

$$d_{B^1}^{\mathsf{K}}(f(z), f(0)) \le d_{\mathbb{C}}^{\mathsf{K}}(z, 0) = 0.$$

Since $d_{B^1}^{\mathsf{K}}$ is positive definite, this implies that $f(z) = f(0)$ for every z.

A complex manifold (or a domain in $\mathbb{C}^n$) is called *Kobayashi hyperbolic* if its Kobayashi distance is positive definite. It is called *complete Kobayashi hyperbolic* if the manifold equipped with the Kobayashi distance (positive definite) is Cauchy complete. Kobayashi hyperbolicity is a reasonable generalization of the boundedness of domains for many purposes. In particular the following normal families result holds.

Theorem 7.2.4. *If M, N are complex manifolds and if N is complete Kobayashi hyperbolic, then the set of holomorphic mappings from M into N is a normal family; i.e., every sequence of holomorphic mappings from M into N either has a subsequence that converges uniformly on compact subsets or has a compactly divergent subsequence.*

The concept of compact divergence here is as usual: a sequence of maps $f_j : M \to N$ is called compactly divergent if, for every compact subset K of M and K' of N, respectively, there exists $j_0 > 0$ such that $f_j(K) \subset N \setminus K'$ whenever $j > j_0$. For the history and credits associated with this theorem as well as its proof, see pages 73–74 of [Kobayashi 1970].

7.2.2 Royden's Infinitesimal Kobayashi Metric

The infinitesimal version of the Kobayashi distance was developed by H. L. Royden ([Royden 1971]) shortly after Kobayashi's definition of the distance function d^{K}.

Definition 7.2.5. Let Ω be a domain in $\mathbb{C}^n$, or more generally a complex manifold. Identify $T_0 D$ with $\mathbb{C}$. Then the *infinitesimal Kobayashi metric* (the *Kobayashi–Royden pseudometric*) is a real-valued function K_Ω from the holomorphic tangent bundle $T\Omega$ of Ω defined to be

$$\mathsf{K}_\Omega(p, v) = \inf\{|\lambda| \colon f \in \text{Hol }(D, \Omega), f(0) = p, df|_0(\lambda) = v\}.$$

It is not clear from the definition alone whether this would present a function that gives arc length of piecewise C^1 curves. For that, Royden established the following.

Proposition 7.2.6 ([Royden 1971]). $\mathsf{K}_\Omega \colon T\Omega \to \mathbb{R}$ *is upper semicontinuous.*

Hence one can define arc length as follows: let $\Omega \subseteq \mathbb{C}^n$ be open and $\gamma : [0, 1] \to \Omega$ a piecewise C^1 curve. The *Kobayashi–Royden length* of γ is defined to be

$$L^{\mathsf{K}}(\gamma) = \int_0^1 \mathsf{K}_\Omega(\gamma(t), \gamma'(t))dt.$$

Royden also proved the following.

Proposition 7.2.7 ([Royden 1971]). *Denote by $\Gamma_\Omega(p, q)$ the set of all piecewise C^1 curves in Ω joining p and q. Then*

$$d^{\mathsf{K}}_\Omega(p, q) = \inf_{\gamma \in \Gamma_\Omega(p,q)} L^{\mathsf{K}}(\gamma).$$

Thus in effect the infinitesimal metric has the same relationship to the Kobayashi distance function as the Riemannian metric has to the Riemannian distance function on a Riemannian manifold. The difference here is that the Kobayashi distance has an independent definition, whereas the Riemannian distance function is definable only in terms of lengths of curves.

7.2.3 The Automorphism Group of a Kobayashi Hyperbolic Manifold

The Lie-group-theoretic properties of $\mathrm{Aut}\,(\Omega)$, Ω a bounded domain in $\mathbb{C}^n$, were deduced in Section 1.3 using normal families arguments. Specifically, these arguments showed that $\mathrm{Aut}\,(\Omega)$ was a Lie group and that the action map of $\mathrm{Aut}\,(\Omega)$ on Ω was proper (Theorem 1.3.12). From this, it was concluded that a C^∞ Hermitian metric on Ω existed that was invariant under $\mathrm{Aut}\,(\Omega)$; i.e., the elements of $\mathrm{Aut}\,(\Omega)$ acted as isometries of the metric, following [Palais 1961].

Theorem 7.2.4 suggests that similar lines of reasoning and similar conclusions should apply to complex manifolds that are complete hyperbolic, since normal families work in that case essentially as they do for bounded domains: this is effectively what Theorem 7.2.4 says. But as it happens, the completeness is not actually needed for the Lie-group-theoretic conclusions, even though it is needed for Theorem 7.2.4 as stated. Specifically, the following theorem and its corollaries give the full Lie group picture even in the absence of the completeness hypothesis.

Theorem 7.2.8. *If M is a connected Kobayashi hyperbolic complex manifold, then* $\mathrm{Aut}\,(M)$ *is a Lie group and, for each $p \in M$, the isotropy subgroup at p is compact.*

Proof. Let G be the group of isometries of the metric space M with its Kobayashi metric. Give G the compact-open topology as usual. Then the fact that a sequence of holomorphic mappings that converges uniformly on compact subsets of M has a holomorphic limit implies that $\mathrm{Aut}\,(M)$ is a closed subgroup of G.

Now M with its Kobayashi metric is a connected locally compact metric space since, in fact, the Kobayashi metric topology is the same as the manifold topology ([Barth 1972]). Thus a theorem of van Dantzig and van der Waerden from the classical period of metric space topology applies. first, in [v. Dantzig/v. d. Waerden 1928] it is shown that: *If X is a connected locally compact metric space, then the group of isometries of X is locally compact in the compact-open topology and its isotropy subgroup at $p \in X$ is compact for each $p \in X$. And, if X is compact, then the whole isometry group is compact.* Applying this result to the group G together with the Bochner–Montgomery theorem (Theorem 1.3.11) yields that G is a Lie group. This, together with the fact that $\mathrm{Aut}\,(M)$ is a closed subgroup of G, yields that $\mathrm{Aut}\,(M)$ is a

Lie group, by the well-known result that a closed subgroup of a Lie group is a Lie group. (See, for instance, [Warner 1971], p. 110, 3.42 Theorem.) □

For such an M, as in the theorem, the action of Aut (M) is actually proper. This matter will be discussed in detail shortly.

Corollary 7.2.9. *Let M be a connected hyperbolic manifold and let $p \in M$. Denote by $\mathcal{E}$ the set of $\mathbb{R}$-linear endomorphisms of the real tangent space of M at p that commute with the almost complex structure J. Define a map D from the isotropy subgroup I_p of* Aut (M) *at p into $\mathcal{E}$ by*

$$D(\alpha) = \textit{the real differential of } \alpha \textit{ at } p.$$

Then D is a continuous, injective homomorphism.

Proof. The map D is a homomorphism by the chain rule. The map is continuous by Cauchy estimates. To see that D is injective, note that, for each $\epsilon > 0$, $\alpha(B(p, \epsilon)) = B(p, \epsilon)$ where $B(p, \epsilon)$ is the ball of radius ϵ around p in the Kobayashi metric. Let U_ϵ be the connected component of $B(p, \epsilon)$ that contains p. (Actually, $B(p, \epsilon)$ is connected so $U_\epsilon = B(p, \epsilon)$, but this is not needed for the argument.) Then $\alpha(U_\epsilon) = U_\epsilon$. Moreover, for $\epsilon > 0$ sufficiently small, U_ϵ is biholomorphic to a bounded domain in $\mathbb{C}^n$ because M is a complex manifold U and the Kobayashi metric topology is the same as the manifold topology. The injectivity of D now follows from the corresponding injectivity for bounded domains (Corollary 1.3.3) applied to the I_p-invariant open set U_ϵ. □

Theorem 7.2.10. *Let M be a connected hyperbolic complex manifold. Then the action map $A :$* Aut $(M) \times M \to M \times M$ *defined by $A(\alpha, p) = (\alpha(p), p)$ is a proper mapping.*

Theorem 7.2.10 should be regarded as an extension of the compact isotropy assertion of Theorem 7.2.8: I_p is the pre-image $A^{-1}\{(p, p)\}$ of the one-point set $\{(p, p)\}$ in $M \times M$, so that properness of A implies in particular that I_p is compact for each $p \in M$. But the arguments used to prove Theorem 7.2.8 need to be strengthened to prove Theorem 7.2.10, though the essential points are mostly implicit in [Kobayashi 1970], p. 70 and [Kobayashi/Nomizu 1963], pp. 46 ff.

Proof of Theorem 7.2.10. The essential preliminary observation is the standard theorem of metric space topology that a connected, locally compact metric space is necessarily separable. Let S be a countable dense subset, $S = \{s_1, s_2, \ldots\}$. If $\{\varphi_j\}$ is a sequence of isometries of the space such that, for each i, $\{\varphi_j(s_i) : j = 1, 2, 3, \ldots\}$ converges, then it is clear that $\{\varphi_j\}$ converges to an isometry of the space. Thus, by the usual normal families diagonal process, a sequence $\{\varphi_j\}$ of isometries will have a subsequence $\{\varphi_{j_k}\}$ which converges (uniformly on compact sets) to an isometry φ_0 provided that, for each i, there is a compact set K_i such that $\varphi_j(s_i) \in K_i$ for all $j = 1, 2, \ldots$.

A connectedness argument shows that this happens provided that there is at least one point x in the space X such that $\{\varphi_j(x)\}$ is a convergent sequence ([Kobayashi/Nomizu 1963], Lemma 3, p. 47).

This last result is closely related to properness of the action, but does not quite imply properness of the action directly. To deduce properness of the action, one needs to know the following:

(∗) *Suppose X is a locally compact, connected metric space. Suppose also that $\{\varphi_j\}$ is a sequence of isometries of X and that $\{p_j\}$ is a convergent sequence in X with limit p_0. If $\{\varphi_j(p_j)\}$ is convergent to, say, q_0 in X, then there are subsequences $\{\varphi_{j_k}\}$ and $\{p_{j_k}\}$ such that $\{\varphi_{j_k}\}$ converges uniformly on compact subsets of X and (hence) $\lim \varphi_{j_k}(p_{j_k}) = q_0$.*

If, as in the theorem, $A : \text{Isom}\,(X) \times X \to X \times X$ is the action map defined by $A(\varphi, x) = (\varphi(x), x)$, then the statement $(*)$ is exactly the assertion that, for each compact set K in $X \times X$, $A^{-1}(K)$ is sequentially compact in $\text{Isom}\,(X) \times X$. This is the same as properness of A since $\text{Isom}\,(X)$ is second countable (in the usual compact-open topology); see [Kobayashi/Nomizu 1963], p. 46, so that sequential compactness implies compactness. □

Proof of statement $(*)$. Choose $\epsilon > 0$ such that the closed balls $\text{cl}(B(p_0, \epsilon))$ and $\text{cl}(B(q_0, \epsilon))$ are compact. Choose j_0 so large that, for all $j \geq j_0$, $d_X(p_j, p_0) < \epsilon/10$ and $d_X(\varphi_j(p_j), q_0) < \epsilon/10$. Then

$$\text{cl}(B(p_0, \epsilon/10)) \subset \text{cl}(B(p_j, \epsilon/3)) \subset \text{cl}(B(p_0, \epsilon))$$

by the triangle inequality. Similarly,

$$\text{cl}(B(q_0, \epsilon/10)) \subset \text{cl}(B(\varphi_j(p_j), \epsilon/3)) \subset \text{cl}(B(q_0, \epsilon)).$$

Now $\text{cl}(B(\varphi_j(p_j), \epsilon/3)) = \varphi_j(\text{cl}(B(p_j, \epsilon/3)))$ since φ_j is an isometry. From this, $\varphi_j(\text{cl}(B(p_0, \epsilon/10))) \subset \text{cl}(B(q_0, \epsilon))$ for all $j \geq j_0$. In particular, since $\varphi_j(p_0)$ belongs to the compact set $\text{cl}(B(q_0, \epsilon))$ for all $j \geq j_0$, there is a subsequence $\{\varphi_{j_k}\}$ such that $\{\varphi_{j_k}(p_0)\}$ converges. By Lemma 3, p. 47 of [Kobayashi/Nomizu 1963], there is a subsequence of $\{\varphi_{j_k}\}$ which converges uniformly on compact subsets of X to an isometry $\varphi_0 : X \to X$. Uniform convergence gives $\varphi_0(p_0) = q_0$ and the proof of statement $(*)$ is complete. □

Corollary 7.2.11. *If M is a connected hyperbolic complex manifold, then there is a C^∞ Hermitian metric on M such that every element of* $\text{Aut}\,(M)$ *acts as an isometry of the metric.*

Proof. This follows from Theorem 7.2.10 and the result of [Palais 1961] already discussed in Section 1.3, that every Lie group of diffeomorphisms that acts properly has a C^∞ Riemannian metric, invariant under the action of the group. If $g_0(\ ,\ )$ is such an $\text{Aut}\,(M)$-invariant Riemannian metric, then the metric g_0 defined by

$$g_0(v, w) = g(v, w) + g(Jv, Jw), \qquad v, w \in T_pM,$$

(here T_pM denotes the real tangent space of M at $p \in M$) is C^∞ Hermitian and Aut (M)-invariant. $\square$

Corollary 7.2.11 gives a second proof of the injectivity part of Corollary 7.2.9 since Riemannian metric isometries that fix a point are determined by their differential at that point.

Of course Aut (M) acts as isometries of the (infinitesimal) Kobayashi "metric," but this metric is seldom C^∞ Hermitian. The Wu metrics to be discussed later (Section 7.5) are Hermitian but not in general C^∞. The Bergman metric, when defined, is C^∞ Hermitian, indeed Kählerian, and also automorphism-invariant, but the Bergman metric may not exist on a hyperbolic complex manifold. For example, $\mathbb{C} - \{0, 1\}$ is hyperbolic (see Section 7.3), but every L^2-holomorphic $(1, 0)$-form there extends to be holomorphic (and L^2) on $\mathbb{C}$ and hence vanishes identically; so the Bergman kernel is identically zero and the Bergman metric is undefined. Thus Corollary 7.2.11 really does add something new to the invariant metric picture.

It is worthwhile to round out these considerations by noting that the reasoning used to prove Theorem 7.2.10 also shows that, for any Riemannian manifold M, the group Isom (M) acts properly (cf. [Yau 1977b]). Thus the properness-of-action condition in [Palais 1961] for the existence of an invariant Riemannian metric is not only sufficient but is also necessary (for group actions closed in the compact-open topology on the diffeomorphism group).

7.3 Riemann Surfaces and Curvature Conditions for Kobayashi Hyperbolicity

In classical Riemann surface theory, a Riemann surface is called *hyperbolic* if its simply connected covering surface is biholomorphic to the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. In this section, we shall see that this terminology is consistent with the idea of hyperbolicity in the sense of Kobayashi: a Riemann surface is hyperbolic in the classical sense if and only if it is Kobayashi hyperbolic. We shall also interpret this in terms of the curvature of naturally arising metrics and consider the extension of this relationship to curvature conditions for Kobayashi hyperbolicity for higher-dimensional complex manifolds.

To discuss first the Riemann surface case, note that if M is a Riemann surface such that there is a nonconstant holomorphic mapping $f : \mathbb{C} \to M$, then M cannot be Kobayashi hyperbolic. (This is actually true for a complex manifold of arbitrary dimension.) Thus $\mathbb{C}$ and the Riemann sphere $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ are not Kobayashi hyperbolic, nor is $\mathbb{C} \setminus \{0\}$. Also, a compact surface M of genus 1, that is, a torus, is not hyperbolic in the Kobayashi sense, because the simply connected cover $\widehat{M}$ of M is biholomorphic to $\mathbb{C}$, according to Chapter 2; and of course the holomorphic covering map $\widehat{M} \to M$ is nonconstant. All these examples are of course also not hyperbolic in the classical sense, because their simply connected covers are: $\widehat{\mathbb{C}} = \mathbb{C}$, $\widehat{\mathbb{CP}^1} = \mathbb{CP}^1$, $(\mathbb{C} \setminus \{0\})\widehat{\ } = \mathbb{C}$, and $\widehat{M} = \mathbb{C}$

if M is a compact Riemann surface of genus 1, all these being either truly obvious or discussed in Chapter 2.

Now, according again to Chapter 2, all other Riemann surfaces are covered by the unit disc. A few of these are topologically the same as one of the non-hyperbolic examples, first the unit disc itself (which is topologically the same as $\mathbb{C}$) and regions in $\mathbb{C}$ that are biholomorphic to $\{z \in \mathbb{C}\colon 0 < |z| < 1\}$ or $\{z \in \mathbb{C}\colon 0 < r_1 < |z| < r_2\}$. These are topologically the same as $\mathbb{C} \setminus \{0\}$, which is holomorphically covered by $\mathbb{C}$ via exponentiation. All other hyperbolic surfaces are, one might say, topologically hyperbolic: for example, no complex structure on a compact surface of genus 2 can fail to be hyperbolic.

To complete the proof of the equivalence of classical hyperbolicity of Riemann surfaces and Kobayashi hyperbolicity of such surfaces, it remains only to see that these classically hyperbolic surfaces are Kobayashi hyperbolic. This result follows from the following about complex manifolds in general.

Theorem 7.3.1. *If $\pi : M_1 \to M_2$ is a holomorphic covering map of a connected complex manifold M_1, onto a (connected) complex manifold M_2, then π is an isometry for the infinitesimal Kobayashi metrics of M_1 and M_2.*

Proof. Let D represent the unit disc as usual. If $F : D \to M_1$ is a holomorphic mapping with $F(0) = p$, then $\pi \circ F$ is a holomorphic mapping of D into M_2 with $(\pi \circ F)(0) = \pi(p)$. Every holomorphic mapping $G : D \to M_2$ with $G(0) = \pi(p)$ arises in this way. This follows from the simple connectivity of D and the standard "lifting" or "monodromy" argument. That π acts as a local isometry now follows by tracing through the definition of the infinitesimal Kobayashi (Kobayashi–Royden) metric. □

This result should be contrasted with the Bergman metric situation: While the *pullback* of a holomorphic $(n, 0)$-form by a holomorphic covering map is a holomorphic $(n, 0)$-form, the pullback of an L^2-form need not be L^2 so nothing like Theorem 7.3.1 holds for the Bergman metric situation.

It follows that, if a Riemann surface is covered by D, it must be Kobayashi hyperbolic. Actually, it also follows that, if a Riemann surface is Kobayashi hyperbolic, then its simply connected cover must be D, not $\mathbb{C}$ or $\mathbb{CP}^1$. So we could have omitted the earlier explicit enumeration of nonhyperbolic examples if we had wished to do so.

We noted in Chapter 2 that hyperbolic Riemann surfaces inherited Hermitian metrics of constant Gauss curvature -1 by "pushing down" the Poincaré metric of D. But $\mathbb{CP}^1$ cannot have an Hermitian metric of curvature ≤ 0; this follows from the Gauss–Bonnet theorem or, alternatively from the Cartan–Hadamard theorem and the simple connectivity of $\mathbb{CP}^1$ (cf. [Petersen 2006]). And $\mathbb{C}$, while it can have an Hermitian metric of negative curvature everywhere, cannot have an Hermitian metric of curvature everywhere ≤ -1. This latter follows from Ahlfors's well-known generalization of the Schwarz lemma ([Ahlfors 1938]).

Theorem 7.3.2 (Ahlfors's Schwarz Lemma). *If M is a Riemann surface with an Hermitian metric H of Gauss curvature ≤ -1 everywhere, then every holomorphic mapping $f : D \to M$ is length-nonincreasing for the Poincaré metric on D and the Hermitian metric H on M.*

This Ahlfors's Schwarz lemma (for Riemann surfaces) is proved by a straightforward maximum principle argument: let h be the pullback by f of H, and consider the quotient $Q = h/(\text{Poincaré metric})$. This quotient makes sense as an $\mathbb{R}$-valued function because both metrics have the form

$$(\text{function}) \cdot (\text{coordinate metric } dx^2 + dy^2)$$

in a local coordinate $z = x + iy$. Suppose that p_0 in D is a point where Q has a maximum. If $Q(p_0) = 0$, then the conclusion of the Schwarz lemma is obvious: $Q \equiv 0$ in this case. Assume that $Q(p_0) > 0$; the maximum principle then gives that $\Delta(\log Q)|_{p_0} \leq 0$, where Δ represents the ordinary Laplacian in some holomorphic coordinate system $z = x + iy$ around p_0. Now the Gauss curvature of a metric $\lambda(x, y)^2(dx^2 + dy^2)$ is, by a classical formula, equal to $-\frac{1}{\lambda^2}\Delta \log \lambda$ (cf. Chapter 2).

Write $h = h_1(dx^2+dy^2)$ and the Poincaré metric $= h_2(dx^2+dy^2)$. So, at p_0,

$$\begin{aligned} 0 &\geq \Delta \log Q \\ &= \Delta \log h_1 - \Delta \log h_2. \end{aligned}$$

Hence

$$\frac{1}{2h_1}\Delta \log h_1 \leq \frac{h_2}{h_1} \cdot \left(\frac{1}{2h_2}\Delta \log h_2\right).$$

Now the condition on the curvatures implies that $Q \leq 1$ at p_0. Since Q attains its maximum at p_0, $Q \leq 1$ everywhere.

If Q has no maximum, one applies the same argument to a slightly shrunken disc with Poincaré-type (constant negative curvature) metric that goes to $+\infty$ at radius $r < 1$, getting a similar estimate by the maximum principle. Then let $r \nearrow 1$. Details can be found in, e.g., [Ahlfors 1973], [Kobayashi 1970], [Greene/Wu 1977], [Krantz 2004], or [Kim/Lee 2010]. □

This actually holds for Hermitian metrics on complex manifolds of any dimension, provided that the holomorphic sectional curvature[2] of the Hermitian metric is everywhere ≤ -1. The proof is essentially the same as before (cf. [Kobayashi 1970]). The Ahlfors' Schwarz lemma in this generalized form immediately implies:

Corollary 7.3.3 (Kobayashi). *If M is a complex manifold which admits an Hermitian metric of holomorphic section curvature ≤ -1 everywhere, then M is hyperbolic.*

[2] See Section 3.4 for the concept of holomorphic sectional curvature.

There has been considerable investigation in Riemann surface theory of what curvature conditions on Hermitian metrics suffice to force the Riemann surface to be hyperbolic: curvature ≤ -1 is a considerably stronger condition than is actually required. The natural "boundary" between hyperbolic and nonhyperbolic behavior for Riemann surfaces, e.g., according to the classical work of Huber and Blanc/Fiala (See [Huber 1957]; cf. [Milnor 1977]), is more along the lines of negative with decay of the absolute value of curvature at distance r on the order of $1/r^2$, i.e., curvature $\leq -C/r^2$ for large r, C a positive constant. This classical line of thought was extended in [Greene/Wu 1977] to hyperbolicity conditions for complex manifolds of arbitrary distance, the following result illustrating the essential point. (See [Greene/Wu 1977] also for detailed references to the classical Riemann surface literature on this subject.)

Theorem 7.3.4 ([Greene/Wu 1977], Theorem E, p. 83). *Suppose that M is a complex manifold with an Hermitian metric G, the holomorphic sectional curvature K of which satisfies, for some $p_0 \in M$ and constant $A > 0$:*

$$K(q) \leq -A\big(1 + dis\ (p_0, q)\big)^{-2},$$

where the inequality is to be satisfied by every holomorphic sectional curvature at q. Then there is a positive constant B such that, for every $x \in M$, the infinitesimal Kobayashi metric $F(x, v)$, $v \in T_x M$, satisfies

$$F(x, v) \geq \frac{B}{\sqrt{1 + dis\ (x, p_0)^2}}\ \|v\|_G,$$

where $\|\ \|_G$ is the norm for the Hermitian metric G. In particular, M is hyperbolic and, if the Hermitian metric is complete, then M is complete in the Kobayashi metric, i.e., complete hyperbolic.

This result is effectively the best possible: quadratic decay in this sense is the "boundary" between hyperbolic and nonhyperbolic, as already mentioned. A detailed discussion of this matter can be found in [Greene/Wu 1977].

It is interesting to note that, in Riemannian geometry, quadratic decay of (sectional) curvature's negative part is also a boundary between two fundamentally different types of topological behavior. Suppose, for some $\epsilon, C > 0$ and some point p_0 in a complete Riemannian manifold M, the sectional curvature K at each $p \in M$ satisfies

$$\max(0, -K) \leq \frac{C}{\big(1 + \mathrm{dis}\ (p, p_0)\big)^{2+\epsilon}}.$$

We say for short that the negative part of the sectional curvature decays faster than quadratically. Then the manifold M has finite topology in the sense that M is homeomorphic (actually even diffeomorphic) to the interior of a compact manifold-with-boundary. This result of [Abresch 1985] is a generalization of Greene and Wu ([Greene/Wu 1974]) that a complete Riemannian manifold with sectional curvature nonnegative outside a compact set has finite topology. This latter result was in turn an extension of the well-known "Soul theorem" of [Cheeger/Gromoll 1972] that a complete manifold of nonnegative sectional curvature is diffeomorphic to the total space of a vector bundle over a compact manifold. On the other hand, there are complete noncompact manifolds with the negative part of the sectional curvature decaying quadratically which do not have finite topology. Indeed, even in dimension 2, there is a complete Riemannian manifold M such that, for each fixed $p_0 \in M$, there is a constant $C > 0$ such that the sectional curvature $K(p)$ at each $p \in M$ satisfies

$$|K(p)| \leq \frac{C}{\big(1 + \text{dis }(p, p_0)\big)^2}$$

and yet M fails to have finite topology. Such a manifold can be constructed as follows: start with the (half) cone $\{(x, y, z) \in \mathbb{R}^3 \colon z \geq 0,\ x^2 + y^2 = z^2\}$ and round off the vertex $(0, 0, 0)$ to give a smooth surface without altering the cone at points where $z > 1/2$. Next add a "tube," a cylinder, connecting a small disc removed around $(0, 3/2, 3/2)$ to a small disc removed around $(0, -3/2, 3/2)$ and smooth out the connections. Thus the points with $1 \leq z \leq 2$ form, topologically, a torus with two discs removed, the discs being the points with $z = 1$ and the points with $z = 2$. Next do similar tube constructions on the point with $2^k \leq z \leq 2^{k+1}$, $k = 1, 2, 3, \ldots$, in such a way that one finally obtains a C^∞ surface S such that, for each $k = 1, 2, 3, \ldots$, $\{(x, y, z) \in S \colon 2^k \leq z \leq 2^{k+1}\}$ is congruent exactly to the set $\{(x, y, z) \in S \colon 1 \leq z \leq 2\}$ scaled by the fact 2^k; first, for each $k = 1, 2, \ldots$, $\{(x, y, z) \in S \colon 2^k \leq z \leq 2^{k+1}\} = \{(2^k x, 2^k y, 2^k z) \in S \colon 1 \leq z \leq 2\}$. This surface is clearly complete and it is easily checked to have quadratic decay of the absolute value of sectional curvature and hence also quadratic decay of the negative part of sectional curvature. But S does not have finite topology since there are infinitely many of the tube connections across the cone. See Figure 7.1.

The fundamental significance of quadratic decay has to do with curvature itself being quadratic in scaling of the metric—cf. [Greene/Wu 1982], [Greene/Wu 1982a], [Greene/Petersen/Zhu 1994] for another situation involving this principle in the Riemannian case and [Greene/Wu 1977] (cf. Chapter 3)

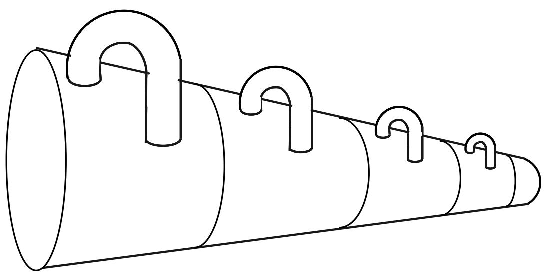

Fig. 7.1. Infinite topology with quadratic curvature decay.

on the significance of quadratic decay for the existence of the Bergman metric. (See also the remarks in [Greene 1987a] for more on curvature decay examples in general in the Riemannian setting.)

Finally, it should be noted that, somewhat surprisingly, a compact complex manifold M is Kobayashi hyperbolic if every holomorphic map of $\mathbb{C}$ into M is constant ([Brody 1978]). But this does not hold in general for M noncompact. For example, the domain

$$\Omega := \{(w_1, w_2) \in \mathbb{C}^2 \colon |w_2|(1 + |w_1|^2) < 1\} \setminus \{(1,0), (2,0)\}$$

admits no nonconstant holomorphic map $f : \mathbb{C} \to \Omega$: if $f(z) = (f_1(z), f_2(z))$, then $f_2(z)$ is bounded, hence constant, hence $\equiv 0$, and $f_1 : \mathbb{C} \to \mathbb{C} \setminus \{1, 2\}$ must also be constant. But $f(z) = (kz, z^2/(1 + k^2))$ maps the unit disc into Ω for every $k > 0$, and has $f_*\left(\frac{\partial}{\partial z}\right) = k\frac{\partial}{\partial w_1}$ at 0, so that the Kobayashi length of $\partial/\partial w_1$ at 0 is 0.

7.4 Remarks on Finsler Metrics and the CRF System

Considerable research has been devoted to Finsler geometry recently. On a smooth differentiable manifold M, a *Finsler metric* is a function $F : TM \to \mathbb{R}$ that satisfies:

(1) $F \geq 0$,
(2) $F(p, \lambda v) = |\lambda| F(p, v)$,
(3) F is smooth on TM except at the points on the zero section of TM, and
(4) In each tangent space the set of unit vectors forms a strongly convex smooth hypersurface.

The infinitesimal Kobayashi metric of Royden is not a Finsler metric in general. Property (4) does not hold in general: the Kobayashi–Royden metric is not necessarily even subadditive. However, in case the manifold M is a bounded strongly convex domain in $\mathbb{C}^n$ with C^∞ boundary, the infinitesimal Kobayashi metric (=the Kobayashi–Royden metric) is indeed Finsler, but not Hermitian: if the Kobayashi metric of a bounded strongly convex domain with C^∞ boundary is Hermitian, then the inverse map of the Lempert representation map (cf. [Lempert 1981]) defines a C^∞ diffeomorphism from the unit ball onto the domain that is holomorphic along each complex disc passing through the origin, and hence is a biholomorphism by Forelli's theorem ([Patrizio 1983]). In this sense the bounded, strongly convex domains with C^∞ boundary, not biholomorphic to the ball, equipped with their Kobayashi–Royden metric are good examples for Finsler geometry ([Bao 2004]).

The concept of *CRF metric system* has also been considered by some researchers. First consider functions $F : TM \to \mathbb{R}$ satisfying only the conditions (1) and (2) above. Such a function F that is also upper semicontinuous is called a *length function*. Then a *CRF system* is an assignment of a length function μ_M to each complex manifold M satisfying the following additional conditions:

(1) μ_M coincides with the Poincaré metric if M is the unit disc D

and

(2) $\mu_N(f(p), df|_p(v)) \leq \mu_M(p, v)$ for any $f \in \mathrm{Hol}\ (M, N)$, $p \in M$ and $v \in T_pM$.

Not only do the infinitesimal Carathéodory metric and the infinitesimal Kobayashi metric belong to the collection of CRF systems,[3] they are known to be extremal in the following sense.

Proposition 7.4.1. *The Carathéodory metric is the smallest of CRF systems, whereas the Kobayashi metric is the largest. first,* $\mathsf{C}_M \leq \mu_M \leq \mathsf{K}_M$ *whenever* μ *is a CRF system.*

Other CRF systems have been discovered, including infinitesimal pseudometrics defined by Azukawa and Sibony. We refer the interested reader to [Jarnicki/Pflug 1993].

7.5 The Wu Metric

Several invariant metrics have already been introduced. The reader may wonder whether it is necessary at this point to introduce yet another metric such as the Wu metric. Would the list of Bergman, Carathéodory, and Kobayashi metrics not be sufficient?

The reasonable answer is that it depends upon what one needs to use the metrics for. In contrast to their distinctive merits, each invariant metric discussed so far has weaknesses. The Bergman metric as well as the canonical Einstein–Kähler metric (which is the Calabi–Yau metric), which will be discussed later, are Kählerian and smooth, but they do not possess the distance-nonincreasing properties with respect to arbitrary holomorphic mappings. On the other hand, the Carathéodory and Kobayashi metrics enjoy the distance-nonincreasing property but neither of them is Hermitian in general. The Kobayashi-Royden metric typically does not even satisfy subadditivity (property (4) of the previous section).

Here enters the Wu metric, introduced by H. Wu in 1987 in a conference at the Mittag-Leffler Institute, although the first paper [Wu H. 1993] on this metric did not appear until 1993 in the conference proceedings volume. This metric is defined on all complex manifolds, is Hermitian, and is distance-nonincreasing up to a constant multiplier depending only on the dimension of the domain manifold. In fact, we are going to discuss two types of Wu metrics here with some separate analysis of their properties.

[3] In the terminology CRF, C = Carathéodory, R = Reiffen, and F = Finsler.

7.5.1 The Wu Metric of the First Kind

Although the original definition of this metric given by Wu in [Wu H. 1993] depends only upon the complex linear structure of the complex tangent space of the complex manifold (and hence the metric is automatically invariant under the action of biholomorphic mappings), here we choose to deal with the complex manifolds that are Kobayashi hyperbolic in the sense that the Kobayashi–Royden (infinitesimal) metric is positive definite at every point.

Let M be a complex manifold, and let $k_M(p, v)$ denote the Kobayashi–Royden length of the vector $v \in T_pM$. The notation T_pM denotes the complex tangent space of M at p.

Fix the point p momentarily. Then consider the set

$$\mathcal{I}(M, p) := \{v \in T_pM : k_M(p, v) \leq 1\}$$

that is often called the *Kobayashi indicatrix*. Introduce a basis $\{u_1, \ldots, u_n\}$ for the complex vector space T_pM and then identify T_pM with $\mathbb{C}^n$ via this basis. Then consider a complex ellipsoid, say E_g, in $\mathbb{C}^n$ defined by the inequality

$$\sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}} u_\alpha \overline{u_\beta} \leq 1$$

for a positive definite Hermitian $n \times n$ matrix $g = (g_{\alpha\bar{\beta}})$.

Staying with such a basis-dependent setting, we shall consider the unique ellipsoid with the minimum volume (i.e., the value of $\det g$ is the greatest) among such E_gs with $\mathcal{I}(M, p) \subset E_g$, following F. John's theorem of 1948 ([John 1948]), a result which has become well known in Banach space theory and control theory. For completeness, we present the argument for the existence and uniqueness of this ellipsoid explicitly here. This re-working of John's theorem will also be useful later, when we discuss the Wu metric of the second kind.

Lemma 7.5.1. *Let A, B be $n \times n$ positive definite Hermitian matrices with $\det A = \det B$. Then, for every t with $0 < t < 1$, $\det[tA + (1-t)B] \geq \det A$. Moreover, equality holds here if and only if $A = B$.*

Proof. Choose a nonsingular matrix W such that $A = W^* W$, where W^* represents the conjugate transpose of W. Let I denote the identity matrix. Let $V = W^{-1}$, $y = -1 + \frac{1}{t}$ and $H = V^*BV$. Then

$$\begin{aligned}
\det[tA + (1-t)B] &= \det A \cdot \det[tI + (1-t)V^*BV] \\
&= t^n \det A \cdot \det\left[I + \left(\frac{1}{t} - 1\right)H\right] \\
&= (1+y)^{-n} \cdot \det A \cdot \det[I + yH].
\end{aligned}$$

Hence the claim that $\det[tA + (1-t)B] \geq \det A$ for $0 < t < 1$ follows as soon as we establish:

(†) Let H be a positive definite $n \times n$ Hermitian matrix. Then the inequality $\det(I + yH) \geq (1+y)^n$ holds for every $y > 0$.

We verify this statement. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of H. These are positive numbers with $\lambda_1 \cdots \lambda_n = 1$.

Now consider the symmetric function s_k of degree k in n variables defined by

$$s_k(x_1, \ldots, x_n) = \sum_{\sigma \in \mathcal{S}(k,n)} x_{\sigma(1)} \cdots x_{\sigma(k)}$$

where $\mathcal{S}(k,n)$ denotes the set of injective maps from $\{1, \ldots, k\}$ into $\{1, \ldots, n\}$. Then observe that the product of all monomial terms in the expression of $s_k(\lambda_1, \ldots, \lambda_n)$ is equal to 1, since it is a power of $\lambda_1 \cdots \lambda_n$. Therefore one deduces from the obvious comparison of arithmetic and geometric means that

$$s_k(\lambda_1, \ldots, \lambda_n) \geq \binom{n}{k}$$

for every $k = 0, 1, \ldots, n$. Now, comparing the coefficients, one immediately obtains

$$\begin{aligned} \det(I + yH) &= (1 + \lambda_1 y) \cdots (1 + \lambda_n y) \\ &\geq (1+y)^n \end{aligned}$$

as desired.

Finally, consider the case when equality holds. The equality means, in the argument above, that $\lambda_1 = \cdots = \lambda_n = 1$. Hence H has to be similar to I. Since the only matrix similar to I is actually I, we must have $H = I$. This implies $A = B$, which completes the proof. □

As a consequence, one realizes that the minimum ellipsoid is uniquely determined. On the other hand, our argument so far is still dependent on the choice of basis, at least in its construction. That dependence is what we are going to treat now.

For two complex ellipsoids E_g and E_h associated with the positive definite Hermitian forms g and h, respectively, suppose that the volume of E_g is larger than the volume of E_h. This means that

$$\det\left(g(u_\alpha, u_\beta)\right) \leq \det\left(h(u_\alpha, u_\beta)\right) \tag{7.1}$$

with a choice of basis $u_1, \ldots, u_n$ for T_pM. What about with another choice of basis? Let $v_1, \ldots, v_n$ be another set of basis vectors for T_pM. Then of course there is a nonsingular $n \times n$ matrix $Q \in GL(n, \mathbb{C})$ such that $Q(u_\alpha) = v_\alpha$ for every $\alpha = 1, \ldots, n$. But then we know that, as matrices,

$$\left(g(v_\alpha, v_\beta)\right) = Q\left(g(u_\alpha, u_\beta)\right) Q^*.$$

This shows that (7.1) above holds regardless of the choice of the basis by the multiplicative property of determinant. Thus it is clear that, for every bounded set containing an open ball centered at the origin, the minimum volume ellipsoid containing this bounded set exists and is unique. This ellipsoid is often called the *best fitting ellipsoid*.

We are now ready to introduce the definition of the *Wu metric of the first kind* on a Kobayashi hyperbolic complex manifold M. For each point $p \in M$, let h_p be the positive definite Hermitian form on T_pM that defines the best fitting complex ellipsoid for the Kobayashi indicatrix. Then the correspondence

$$p \mapsto h_p$$

is the *Wu metric of the first kind* on M.

7.5.2 Properties of the Wu Metric of the First Kind

The Wu metric of the first kind is essentially automatically invariant under biholomorphic mappings: let M, N be complex manifolds and suppose that they admit a biholomorphism $f : M \to N$. Denote by h_M, h_N the Wu metric of the first kind on M, N respectively. Then

$$f^* h_N = h_M.$$

The proof is straightforward: as the Kobayashi metric is invariant under biholomorphisms, the Kobayashi indicatrices are preserved. Then the unique best fitting ellipsoids that define the Wu metrics are preserved. Hence the invariance statement follows.

On account of the distance-nonincreasing property of the Kobayashi metric, for a holomorphic map $f : M \to N$ from a Kobayashi hyperbolic complex manifold M into another such N, we see that

$$df|_p(\mathcal{I}(M,p)) \subset \mathcal{I}(N, f(p)).$$

On the other hand, the best fitting ellipsoids need not in general satisfy a similar inclusion. The Wu metric of the first kind, however, turns out to be distance-nonincreasing for holomorphic mappings up to a constant factor with this fact depending only on dimension.

According to [John 1948], the best fitting ellipsoid E and the closed convex set V containing 0 in $\mathbb{C}^m$ share the property that

$$V \subset E \subset \sqrt{m+1}\ V.$$

In case V is circular in each coordinate direction, then $E \subset \sqrt{m}V$. Therefore in the case of the Wu metric

$$\begin{aligned} df|_p(E_{h_M|_p}) &\subset df|_p\left(\sqrt{\dim_{\mathbb{C}} M}\ \mathcal{I}(M,p)\right) = \sqrt{\dim_{\mathbb{C}} M}\ df|_p(\mathcal{I}(M,p)) \\ &\subset \sqrt{\dim_{\mathbb{C}} M}\ \mathcal{I}(N, f(p)) \subset \sqrt{\dim_{\mathbb{C}} M}\ E_{h_N|_{f(p)}}. \end{aligned}$$

This implies the following distance-nonincreasing property of the Wu metric up to factors.[4]

In summary:

Proposition 7.5.2 (Wu). *Let M and N be complex Kobayashi hyperbolic manifolds, and let h_M and h_N represent their Wu metrics of the first kind. Then:*

(i) *For every biholomorphic mapping $f : M \to N$, $f^* h_N = h_M$.*
(ii) *If $f : M \to N$ is a holomophic mapping, then $f^* h_N \leq (\dim M)\ h_M$.*

As already noted, the Wu metric of the first kind can be defined independently of the Kobayashi metric. The Wu metric will then be only nonnegative instead of being positive definite. However, we choose not to introduce the full details here. Interested readers should consult [Wu H. 1993] for more properties.

One of the distinctive merits of the Kobayashi metric is that its infinitesimal metric is preserved by holomorphic covering mappings (Theorem 7.3.1). As before, if $f : M \to N$ is a holomorphic mapping that is also a covering map in the algebraic topological sense, then the Kobayashi–Royden metrics satisfy

$$f^* k_N = k_M .$$

There is a corresponding result for the Kobayashi distance which says in effect that the distance of two points in the base is the infimum of the distances of their respective pre-images in the covering space ([Kobayashi 1998], p. 61): this follows by a similar lifting argument.

The uniqueness of the minimum volume ellipsoid immediately implies:

Proposition 7.5.3. *If $f : M \to N$ is a holomorphic covering map from a Kobayashi hyperbolic complex manifold M onto another Kobayashi hyperbolic complex manifold N, then f is an isometry of their Wu metrics, i.e., $f^* h_N = h_M$.*

This type of theorem plays an important role later when we characterize generic analytic polyhedra in $\mathbb{C}^2$ with non compact automorphism group (see Section 9.4).

As already noted, the Kobayashi metric is upper semicontinuous. The Carathéodory metric is always continuous ([Royden 1971]). The Cheng–Yau metric on domains, which will be discussed later in this chapter, is always smooth. The Bergman metric is always real analytic. Then what about the Wu metric of the first kind? Contrary to what Wu says in his article [Wu H. 1993], this metric turns out not to even be upper semicontinuous on some complex manifolds. On the other hand, it is always continuous whenever the Kobayashi metric is continuous (cf. [Jarnicki/Pflug 1993]). In case the manifold is homogeneous under the action of a Lie group, the Wu metric is automatically real analytic, as is any invariant Hermitian metric in this case.

[4] The Wu metric is not a CRF system.

7.5.3 The Wu Metric of the Second Kind

The original Carathéodory metric was constructed from the family of holomorphic mappings into the unit disc. Thus any compact complex manifold should have Carathéodory metric identically zero. Because of the Riemann removable singularities theorem, the situation is as poor even if a finite number of punctures are made. Thus we recall the concept of (generalized) C-hyperbolicity from Section 7.1.2: a complex manifold M is called C-hyperbolic if its universal covering space $\widetilde{M}$ has a positive definite Carathéodory metric.

Thus a complex manifold M of complex dimension n is C-hyperbolic if and only if every point $\widetilde{p}$ in the universal covering space $\widetilde{M}$ of M admits a holomorphic mapping $\widetilde{f} : \widetilde{M} \to B^n$ such that $d\widetilde{f}|_{\widetilde{p}}$ is nonsingular (cf. [Wu H. 1993]).

Given a complex manifold M of complex dimension n, let $\pi : \widetilde{M} \to M$ be the holomorphic covering map from its universal covering space $\widetilde{M}$ onto M, and let $\widetilde{x} \in \widetilde{M}$. Denote by

$\mathcal{Q}_{\widetilde{x}}$ the set of positive semidefinite Hermitian inner products on the complex tangent space $T_{\widetilde{x}}\widetilde{M}$,
$\mathcal{F}_{\widetilde{x}}$ the set of holomorphic maps $f : \widetilde{M} \to B^n$ with $f(\widetilde{x}) = 0$,
β_0 the Bergman metric of the unit ball B^n at the origin normalized so that its holomorphic sectional curvature is identically -1,
$\Phi_{\widetilde{x}} = \{f^*\beta_0 \mid f \in \mathcal{F}_{\widetilde{x}}\}$.

In case M is C-hyperbolic, the covering map $\widetilde{\pi} : \widetilde{M} \to M$ has the property that $\Phi_{\widetilde{x}}$ contains an interior point of $\mathcal{Q}_{\widetilde{x}}$, with respect to the induced subspace topology from the set of Hermitian symmetric bilinear forms.

As before, the concept of the element of $\Phi_{\widetilde{x}}$ having determinant greater than or equal determinant to another is independent of basis choice. Applying Lemma 7.5.1 (in fact, applying the proof of John's theorem mentioned above) to the elements of $\Phi_{\widetilde{x}}$ one obtains the unique element of $\Phi_{\widetilde{x}}$ having the maximal determinant value. Denoting this element by $\widetilde{\gamma}_{\widetilde{x}}$, the assignment

$$\widetilde{x} \mapsto \widetilde{\gamma}_{\widetilde{x}}$$

defines a continuous Hermitian metric $\widetilde{\gamma}$ on $\widetilde{M}$. This is the *Wu metric of the second kind* on $\widetilde{M}$. Since the definition of this metric depends only upon the complex structure of the complex tangent space, it is invariant under biholomorphisms. Thus one may push down the metric using the covering map to M. Call this metric γ; this is the *Wu metric of the second kind* on M.

Proposition 7.5.4. *The Wu metric γ of the second kind for C-hyperbolic complex manifolds satisfies the following properties:*

(i) *$x \to \gamma_x$ is a continuous, positive definite Hermitian metric;*
(ii) *The metric is invariant under biholomorphic mappings.*

7.5.4 An Inequality Property and Why C-hyperbolic Manifolds Are Algebraic

Any "metric"—an assignment to each point of a complex manifold M a positive definite Hermitian form h—induces an Hermitian metric H on the dual of the canonical bundle of M, where the canonical bundle is the holomorphic line bundle of forms of type $(n, 0)$ on M. first, if $v_1, \ldots, v_n$ is an h-orthonormal basis for the holomorphic tangent space at p, one defines $v_1 \wedge \cdots \wedge v_n$ to have length 1 at p. Informally, H is the "determinant" of h. In the situation where h is the Wu metric of the second kind, so that h need not be smooth as a function of the point of M, the line bundle metric H need not be smooth and hence need not have the usual "curvature" form defined. Recall that, when H is smooth, this curvature form Θ_H is globally defined via a local construction: first, if $(z_1, \ldots, z_n)$ is a local holomorphic coordinate system, then the real $(1,1)$-form

$$i\,\partial\overline{\partial} \log H\Big(\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}, \overline{\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}}\Big)$$

is in fact independent of the local coordinate choice, since $\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}$ changes, under a local coordinate change, by a holomorphic Jacobian factor (cf. [Greene 1987]).

In case H is not smooth—which can happen if h is not smooth—then the curvature form Θ_H is not as such defined, since it is defined by differentiation (twice). But it is defined in a distribution sense, as a "current" of type $(1,1)$. (See, for example, [Harvey/Lawson 1975] for details of this concept.) In particular, there is a well-defined distributional sense in which Θ_H or $-\Theta_H$ can be positive definite: this amounts to Θ_H or $-\Theta_H$ being distribution-subharmonic on each (smooth) complex submanifold of complex dimension 1.

In practice, as was pointed out in [Wu H. 1993], the (distributional) curvature form Θ_H attached to the Hermitian metric on the dual of the canonical bundle already discussed, is always positive definite in the distributional sense already indicated in the case of the metric h arising on a C-hyperbolic manifold. This is proved by a support-function argument (we continue the notation of the previous section here): without loss of generality we consider the case that M is simply connected so that $\widetilde{M} = M$. Note that each $x \in M$ admits a holomorphic mapping $f : M \to B^n$ with $df|_x$ nonsingular. For each $q \in B^n$, choose $\mu_q \in \operatorname{Aut}(B^n)$ with $\mu_q(q) = 0$. Then, for the Bergman metric β, $\beta_y = \mu_y^* \beta_0$. Therefore, for an arbitrary holomorphic mapping $g : M \to B^n$,

$$\det g^* \beta_{g(x)} = \det(\mu_{g(x)} \circ g)^* \beta_0 \le \det \gamma_x$$

for every $x \in M$.

Let $p \in M$. By a standard normal families argument, one may choose a holomorphic mapping $f \colon M \to B^n$ such that $f(p) = 0$ and $\gamma_p = f^* \beta_0$. Since $df|_p$ is nonsingular, one can use f as a local coordinate system at p. Then, in an open neighborhood, say U, of p in the coordinate system:

$$p = 0, \log \frac{\det \beta}{\det \gamma}\bigg|_0 = 0 \qquad \text{and} \qquad \log \frac{\det \beta}{\det \gamma}\bigg|_x \leq 0, \quad \forall x \in U.$$

Thus H is supported at p by a smooth Hermitian metric with strictly positive curvature form, as desired. See [Wu H. 1993] for details.

The characterization of the positivity of Θ_H by distribution subharmonicity on complex 1-manifolds shows that the positivity is preserved by convolution smoothing in holomorphic local coordinates. It then follows that, in the C-hyperbolic situation, the (possibly nonsmooth) metric H on the dual of the canonical bundle gives rise to a C^∞ Hermitian metric $\widehat{H}$ which has $\Theta_{\widehat{H}}$ a C^∞ positive form (cf. [Greene/Wu 1979]). It is a famous result of Kodaira that a compact complex manifold with a line bundle with signed curvature is algebraic (cf. [Miyaoka 1977]).

Therefore one obtains:

Theorem 7.5.5 ([Wu H. 1993]). *A compact C-hyperbolic complex manifold is necessarily algebraic.*

There is a related result in which C-hyperbolicity is replaced as a hypothesis by a Bergman metric condition. first:

(*) *If M is a compact complex manifold such that some normal covering $\widetilde{M}$ of M has a positive definite Bergman metric, then M is projective algebraic.*

Here we are saying, as in Section 3.2, that a complex manifold has a positive definite Bergman metric if the space of L^2 holomorphic $(n, 0)$-forms, $n =$ the complex dimension, has its associated (n, n)-form $K(z, z)$ nowhere vanishing and if the Levi form of $\log K(z, z)$ (which is well defined) is positive definite everywhere. These definitions were discussed in detail in Section 3.2. (By *normal covering* we mean as usual that M is a quotient of $\widetilde{M}$ by the covering transformations of $\widetilde{M}$ over M.)

The proof of the result $(*)$ is obtained again by applying the Kodaira embedding theorem as in the proof of the C hyperbolicity result, as follows. With M and $\widetilde{M}$ as in $(*)$ and for each $p \in \widetilde{M}$, a pointwise Hermitian inner product on $(n, 0)$-forms at p can be defined by

$$\langle \omega, \theta \rangle := \frac{\omega \wedge \overline{\theta}}{K(p, p)}.$$

This is clearly invariant under automorphisms of $\widetilde{M}$ and hence "pushes down" to the normal covering quotient M to give an Hermitian metric H on the canonical line bundle (of $(n, 0)$-forms) on M. This metric H exhibits the canonical bundle of M as a positive line bundle: for this, recall that a line bundle is positive if it admits a metric G such that the Levi form of $\log G$ (which is well defined independent of local holomorphic trivialization) is negative definite (cf. [Greene 1987] for details). The Levi form of $\log H = -$ the

Levi form of $\log K(p,p)$, and the Levi form of $\log K(p,p)$ is positive definite by the hypothesis in $(*)$ on the existence and positive definiteness of the Bergman metric.

Thus M admits a positive line bundle and hence, by the Kodaira embedding theorem, is algebraic. $\square$

This result $(*)$ of course applies in particular if some normal cover is a bounded domain in $\mathbb{C}^n$. Theorem 7.5.5 also applies, but the proof of $(*)$ is easier. [For the application of Theorem 7.5.5, note that, as in Section 7.1, if any holomorphic covering of M has positive definite Carathéodory distance, then so does the universal covering by composition of mappings.]

7.6 The Cheng–Yau Invariant Einstein–Kähler Metric

If Ω is a bounded domain in $\mathbb{C}^n$ with C^∞ strongly pseudoconvex boundary, then the Bergman metric of Ω is a complete Kähler metric on Ω with holomorphic sectional curvatures that approach the negative constant $-4/(n+1)$ near the boundary of Ω. This was discussed in detail in Chapter 3 (cf. also Chapter 10). In particular, the results in Section 3.6 show that this metric has negative Ricci curvature near the boundary of Ω, and indeed is asymptotically Ricci-negative Einstein–Kähler in the obvious sense that the eigenvalues of the Ricci tensor converge to a negative constant (cf. Theorem 3.6.3).

It is natural to ask whether such an Ω admits some complete metric for which the curvature in some sense is negative over the whole of Ω. It cannot be expected in general that Ω would admit even a Riemannian metric that was complete and had negative sectional curvature: For a complete Riemannian manifold with nonpositive sectional curvature that is simply connected must be real diffeomorphic to a Euclidean space, while, for example, a small enough tubular neighborhood of the 2-dimensional sphere S^2 $(\subset \mathbb{R}^3 \subset \mathbb{C}^3)$ defines a C^∞ strongly pseudoconvex and simply connected domain in $\mathbb{C}^3$. But this domain is not diffeomorphic to a Euclidean space; it is not even homeomorphic.

On the other hand, it is rather easy to modify the Bergman metric of a smoothly bounded strongly pseudoconvex domain so that the modified metric has negative, bounded-from-zero, holomorphic sectional curvature over all of Ω, with the modified metric equal to the Bergman metric near the boundary so that the modified metric remains complete. This is an immediate consequence of the usual formula for the holomorphic sectional curvature of the sum of two Hermitian metrics (cf. [Grauert/Reckziegel 1965]): if g_b is the (constant negative holomorphic sectional curvature) Bergman metric of a large ball containing Ω and g_Ω is the Bergman metric of Ω, then $\lambda g_b + g_\Omega$, for sufficiently large positive number λ, will do the job (cf. [Klembeck 1978]).

Between these two extremes—the generally not achievable negative sectional curvature and the always achievable (strictly bounded from 0) negative holomorphic sectional curvature—lies then a natural intermediate question: Is there a complete Kähler metric with constant negative Ricci curvature, that is, a negative Einstein–Kähler metric?

For later purposes, we introduce specific notation and a normalization convention: if g is a Kähler metric and $(z_1, \dots, z_n)$ is a holomorphic local coordinate system, then we write $g = \sum_{i,j=1}^{n} g_{i\bar{j}}\, dz_i \otimes d\bar{z}_j$, where $g_{i\bar{j}} = g\big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\big)$. Then the (Hermitian) Ricci tensor $R_{i\bar{j}}$ is given by the standard formula

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det(g_{\alpha\bar{\beta}})$$

(cf. [Kobayashi/Nomizu 1963], Vol. II, Chapter XI. It is worth noting that this formula is very much a matter of complex geometry: there is no genuine analogue in Riemannian geometry in general.) If a Kähler metric g is negative Einstein–Kähler in the sense that $R_{i\bar{j}} = -c\ g_{i\bar{j}}$ for some constant $c > 0$, then the metric $h_{i\bar{j}} := \sqrt{c} g_{i\bar{j}}$ has Ricci tensor $S_{i\bar{j}}$ satisfying $S_{i\bar{j}} = -h_{i\bar{j}}$. This is an immediate consequence of the usual scaling properties of curvature: multiplication of the metric by a constant $b > 0$ multiplies sectional (and hence Ricci) curvature by $1/b$. We shall hereafter call such a Kähler metric, with Ricci tensor = the negative of a metric tensor, a *normalized Ricci-negative Einstein–Kähler metric.* [A similar normalization is made for the positive Ricci case. Normalization in this sense is of course not applicable for the Ricci flat, Ricci $\equiv 0$, case.]

No obvious topological restriction arises here from negativity of the Ricci curvature. All manifolds of real dimension > 2 admit complete Riemannian metrics of negative Ricci curvature ([Lohkamp 1994]). The Chern class condition that occurs in the compact case (the Calabi conjecture situation) is not an obstruction for domains in $\mathbb{C}^n$: the "canonical bundle" of $(n, 0)$-forms is trivial since $\Omega \subset \mathbb{C}^n$, but on an open manifold there is no reason why a negative closed $(1, 1)$-form cannot be cohomologous to 0. And in fact, the answer to the question about the existence of complete negative-constant Einstein–Kähler metrics is "yes," not only for C^∞ strongly pseudoconvex domains but for bounded domains of holomorphy in general.

Theorem 7.6.1 ([Cheng/Yau 1980], extended in [Mok/Yau 1983]). *If Ω is a bounded domain of holomorphy in $\mathbb{C}^n$ (i.e., a bounded, connected pseudoconvex open subset of $\mathbb{C}^n$), then there is a C^∞ complete Einstein–Kähler metric on Ω, which necessarily has constant negative Ricci curvature. And this metric is unique if the metric is normalized by multiplication by a constant to have the eigenvalue(s) of the Ricci tensor identically -1.*

This result was proved by Cheng and Yau in [Cheng/Yau 1980] for domains with C^2 boundary and for intersections of such domains. This was extended to bounded domains of holomorphy in [Mok/Yau 1983]. The proof in [Cheng/Yau 1980] involves solution by the continuity method of a Monge–Ampère equation, similarly to the proof of the negative-curvature Calabi conjecture proof but with compactness replaced by suitable conditions "at infinity" for the case of an open set Ω. Even an outline of this result would be somewhat beyond the intended scope of this book—and rather long if carried out in detail—so the reader is simply referred to [Cheng/Yau 1980].

The uniqueness part of the theorem immediately implies the automorphism invariance of the normalized complete Ricci-negative Einstein–Kähler metric under biholomorphic maps.

As already discussed in the C-hyperbolic and Bergman cases, special interest is attached to cases in which a pseudoconvex Ω admits a compact (normal) covering-space quotient, and in particular when Ω is the universal cover of some compact complex manifold. For a normal covering space in general, the covering transformations are (by definition) transitive on the preimages of such points in the quotient. Automorphism invariance then gives that the negative-constant Einstein–Kähler metric "pushes down" to the compact manifold that is covered. The compact manifold is then again algebraic with $c_1 < 0$ (c_1 denotes the first Chern class), and the push-down of the covering space's Einstein–Kähler metric is exactly the Einstein–Kähler metric on the compact manifold, the existence of which is guaranteed by the affirmative solution of the negative-case Calabi conjecture by [Aubin 1976] and [Yau 1977a] (cf. [Yau 1978]).

We have of course already observed at the end of the last section that this quotient has in fact positive canonical bundle and hence has $c_1 < 0$: this came from Bergman kernel considerations.

It is worth noting that, if Ω is a bounded domain with a compact covering-space quotient, then Ω is automatically a domain of holomorphy. This need not be assumed separately, and thus Theorem 7.6.1 automatically applies, to give the existence of a canonical Einstein–Kähler metric to be pushed down.

Proposition 7.6.2. *If Ω is a bounded domain in $\mathbb{C}^n$ such that, for some compact complex manifold M, there is a holomorphic covering map $F : \Omega \to M$, then Ω is a domain of holomorphy.*

In case Ω has C^1 boundary, the fact that Ω is a domain of holomorphy follows from Ohsawa's result (Theorem 3.4.2): the Bergman metric of Ω is necessarily complete because any locally isometric covering space of a compact manifold is complete (cf. [Kobayashi/Nomizu 1963]). [One sees this immediately from lifting geodesics.] The fact that the Bergman metric is invariant yields that it can be pushed down to the compact quotient so that the covering is then locally isometric.

But in fact, Ω is necessarily a domain of holomorphy whether or not it has C^1 boundary; this holds independently of any boundary smoothness at all. The idea to be used here is essentially the same as that used in Section 1.5, but in this case it is most efficient to apply the behavior of the Jacobian to the Bergman kernel (cf. e.g., [Akhiezer 1995], pp. 61–62).

Proof of Proposition 7.6.2. By the solution of the Levi problem, it suffices to show that the C^∞ strictly plurisubharmonic function $K(z,z)$, $z \in \Omega$, is proper, that is, that it goes to $+\infty$ at the boundary of Ω. Here $K(z,z)$ is the diagonal Bergman kernel function as usual. For this, choose a compact set $\mathcal{C} \subset \Omega$ such that every point of the compact quotient is the covering map image of some point of $\mathcal{C}$. Suppose a sequence $\{p_j\}$ in Ω "diverges to infinity." Choose $z_j \in \mathcal{C}$ and covering transformations $\varphi_j \colon \Omega \to \Omega$ such that $\varphi_j(z_j) = p_j$. Passing to a subsequence if necessary, we can assume by normal families that the sequence φ_j converges uniformly on compact subsets of Ω to a map $\varphi_0 \colon \Omega \to \mathrm{cl}(\Omega)$, where $\mathrm{cl}(\Omega)$ is the closure of Ω. Again passing to a subsequence, we can suppose that $\{p_j\}$ converges to some point $p_0 \in \mathrm{cl}(\Omega) \setminus \Omega$. It follows then from Theorem 1.3.4 that $\varphi_0(\Omega) \subset \mathrm{cl}(\Omega) \setminus \Omega$. In particular, the Jacobian determinant $\mathcal{J}_{\varphi_0}$ is identically zero, since $\mathrm{cl}(\Omega) \setminus \Omega$ has empty interior. Hence $\mathcal{J}_{\varphi_0}$ converges uniformly on compact subsets of Ω to 0. Now,

$$K(p_j, p_j) = K(\varphi_j(z_j), \varphi_j(z_j)) = |\mathcal{J}_{\varphi_j}(z_j)|^{-2} K(z_j, z_j)$$

by Proposition 3.1.1. Since $\{\mathcal{J}_{\varphi_j}\}$ goes to 0 uniformly on $\mathcal{C}$, since $z_j \in \mathcal{C}$, and since $K(z, z)$ is bounded away from 0 uniformly for $z \in \mathcal{C}$, it follows that $K(p_j, p_j)$ goes to $+\infty$, as required. □

Thus, Theorem 7.6.1 does indeed apply to every bounded domain which has a compact quotient.

It is worth noting that Ricci-negative is the only possibility for a complete Einstain-Kähler metric on a complex manifold covered by any bounded domain in $\mathbb{C}^n$. For this, it is enough to show that Ricci-negative is the only possibility for a complete Einstein–Kähler metric on any bounded domain, say Ω, in $\mathbb{C}^n$. (This is the "necessarily negative" part of Theorem 7.6.1.) This can be seen as follows.

Completeness and constant Ricci-positive would imply compactness of Ω by Myers's theorem [Petersen 2006], a contradiction. Ricci-zero is also not possible, for the following reason. The real part of a holomorphic function is a harmonic function relative to any Kähler metric. Thus with any complete Kähler metric, Einstein or not, Ω admits many nonconstant harmonic functions, since Ω is a bounded domain in $\mathbb{C}^n$. But on a complete manifold of nonnegative Ricci curvature, every harmonic function is constant according to [Yau 1975]. [Of course this second argument rules out a Ricci-positive Einstein–Kähler metric for bounded domains as well, but the earlier argument involving Myers's theorem for this case is more elementary. It is also worth noting that, in any case where a Ricci-negative complete Einstein-Kahler metric exists, Yau's Schwarz lemma [Yau 1978a] in its volume version implies that there is no Ricci-nonnegative complete Kähler metric possible. This follows from Theorem 3 of [Yau 1978a], since that theorem yields in this case that the identity map from the Ricci-nonnegative to the Ricci-negative metrics would be volume-degenerate: the Jacobian would be everywhere zero, a contradiction. The volume behavior given by this Theorem 3 is also the vital point in the establishment of uniqueness for the negative case (Proposition 7.6.3).]

We turn now to a detailed statement and the proof of the general uniqueness result that in particular yields the uniqueness part of Theorem 7.6.1 (see [Cheng/Yau 1980], Proposition 5.5).

Proposition 7.6.3. *If M_1 and M_2 are two complex manifolds with a complete normalized Ricci-negative Einstein–Kähler metric and if $F: M_1 \to M_2$ is a biholomorphic mapping, then F is an isometry.*

Proof. According to Yau's generalization of Schwarz's lemma for volume forms ([Yau 1978a], Theorem 3), the map F is volume-preserving, in the sense that the pullback of the volume form M_2 to M_1 by F equals the volume form of M_1. (Apply the theorem to F and F^{-1}.)

Now, for Riemannian metrics in general, it is of course very far from true that a volume-preserving diffeomorphism is necessarily an isometry (cf. [Moser 1965], [Greene/Shiohama 1979] for the full extent to which this fails). But, for Einstein–Kähler metrics, the volume form determines the metric and such a result does apply.

To see this, note first that, if ω is a nowhere-zero form of type (n, n) on a complex manifold of complex dimension n, then there is an Hermitian form L_ω as follows: in holomorphic local coordinates $(z_1, \ldots, z_n)$, write $\omega = f\ dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z_1} \wedge \cdots d\overline{z_n}$ and set

$$L_\omega := \sum_{i,j=1}^{n} \frac{\partial^2(\log f)}{\partial z_i \partial \overline{z_j}}\ dz_i \otimes d\overline{z_j}.$$

This is easily checked to be independent of choice of local coordinate systems (and of choice of local branch of log): this is analogous to the calculations earlier for why the Bergman metric for the manifold is well defined (see Section 3.2). In a different coordinate system, f is replaced by $f\mathcal{J}\overline{\mathcal{J}}$ where $\mathcal{J}$ is a "holomorphic Jacobian" factor so that $\log f$ is replaced by $\log f + \log \mathcal{J} + \log \overline{\mathcal{J}}$ and the latter two terms are annihilated by the z_i- and $\overline{z_j}$-derivatives. We call the form L_ω the Levi form of ω. In this terminology, the Ricci tensor $R_{\alpha\bar{\beta}}$ of a Kähler metric $g_{\alpha\bar{\beta}}$ is in fact (interpreting $R_{\alpha\bar{\beta}}$ as an Hermitian form) exactly equal to the negative of the Levi form of the volume form ω of g: $R = -g$. This is simply a restatement of the standard formula already noted:

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det(g_{\alpha\bar{\beta}}).$$

Now let g be the metric of M_1 and h the metric of M_2, R and S their respective Ricci tensors, and ω_1 and ω_2 respective volume forms. Then $g = -R = L_{\omega_1}$ while $h = -S = L_{\omega_2}$. But it is just in effect a restatement of coordinate invariance of the Levi form that

$$F^* L_{\omega_2} = L_{F^*\omega_2}.$$

Thus, from the volume-preserving formula $F^*\omega_2 = \omega_1$, it follows that

$$F^* h = F^* L_{\omega_2} = L_{F^*\omega_2} = L_{\omega_1} = g.$$

Hence F is an isometry as desired. □

This general uniqueness result, Proposition 7.6.3, makes it possible to extend the "push-down" considerations that were discussed just after Theorem 7.6.1. In particular, the normality of the coverings there can be dispensed with.

Corollary 7.6.4. *If $\pi\colon M_1 \to M_2$ is a holomorphic covering map of one complex manifold M_1 to another M_2 with complex dimension ≥ 2, and if M_1 has a complete Ricci-negative Einstein–Kähler metric g_1, then there is a complete Ricci-negative Einstein–Kähler metric g_2 on M_2 with the property that π is a local isometry relative to g_1 and g_2.*

In particular, every compact complex manifold M that is covered by some bounded (necessarily pseudoconvex) domain Ω in $\mathbb{C}^n$, $n \geq 2$, admits a Ricci-negative Einstein–Kähler metric.

Proof. Let $\pi_0\colon M_0 \to M_1$ be the universal holomorphic covering of M_1. Then $g_0 := \pi_0^* g_1$ is a complete Ricci-negative Einstein–Kähler metric on M_0, and by Proposition 7.6.3, every biholomorphic map $\gamma\colon M_0 \to M_0$ is an isometry of $\pi_0^* g_1$. The composition, call it F, of π_0 followed by π_1 taking M_0 to M_2 is the universal covering of M_2. Thus M_2 is the quotient space of M_0 by the group of holomorphic covering transformations of M_0 over M_2. For each $p \in M_2$, this group is transitive on the pre-image $F^{-1}(\{p\})$ in M_0. Since the group consists of biholomorphic mappings which are, as noted, necessarily isometries of g_0, the metric g_0 "pushes down" to M_2. first, given $p \in M_2$, choose any $q \in F^{-1}(\{p\})$ and assign a metric g_2 to T_pM_2 by declaring $dF|_q\colon T_qM_0 \to T_pM_2$ to be isometric for g_0 at q (and g_2 at p). The transitivity of the group of (isometric) covering transformations on the points of $F^{-1}(\{p\})$ shows that this is well defined. That this metric "factors" through $\pi_0\colon M_0 \to M_1$ is clear since π_0 is already locally g_0 isometric to g_1. That the push-down metric g_2 thus defined is Ricci-negative Einstein–Kähler is clear since the condition of Ricci-negative Einstein–Kähler is a local-isometry invariant. Completeness of the push-down follows from standard Riemannian geometry: geodesics for g_2 are the π_1-images of the infinitely extendable geodesics in M_1 and hence themselves infinitely extendable. □

The second aspect of Corollary 7.6.4, that a compact complex manifold covered by a bounded domain admits a Ricci-negative Einstein–Kähler metric, is of course also deducible from the affirmative solution of the Ricci-negative Calabi conjecture ([Aubin 1976], [Yau 1977a]). As was already discussed in connection with Theorem 7.5.5 and the statement $(*)$ thereafter, a compact complex manifold arising in this way has necessarily a positive canonical line bundle. Thus, by the solution of the Calabi conjecture in the case of negative curvature, it admits a Ricci-negative Einstein–Kähler metric. Thus, this part of Corollary 7.6.4 is simply an independent verification, a different way from the Calabi conjecture solution itself, to find the Einstein–Kähler metric. This relationship is also reflected in the similar general methods of [Cheng/Yau 1980] and [Yau 1978]. The compact and noncompact cases are clearly related, though by no means equivalent.

These ideas of the relationship between pullback and push-down of complete Einstein–Kähler metrics in the presence of uniqueness will play an interesting explicit role later, at the end of Chapter 10 (Theorem 10.4.3).

In complex dimension 1, the natural analogue of Einstein–Kähler metrics will be the metrics of constant Gauss curvature. [It is a classical theorem of I. Schur that, in higher dimensions, the Ricci tensor being at each point a multiple of the metric tensor implies that the multiple is constant from point to point. This of course does not apply in real dimension 2, so one assumes constancy as the natural analogue of the higher dimensional case.] In this sense, the "Einstein–Kähler metric" for bounded domains in $\mathbb{C}$ should be thought of as the metric (up to a constant multiple) obtained by "pushing down" the Poincaré metric of the disc from the universal cover of the domain by the disc, as discussed in Chapter 2. Lu Qi-Keng's result (Theorem 4.2.2) shows that this coincides (up to a constant multiple) with the Bergman metric (if and) only if the domain in biholomorphic to the unit disc.

It is natural to ask if some corresponding result holds for domains in $\mathbb{C}^n$, $n \geq 2$. In particular, it is natural to ask whether the Bergman metric of a C^∞ strongly pseudoconvex bounded domain in $\mathbb{C}^n$ can be Einstein–Kähler if the domain is not biholomorphic to the ball. The affirmative answer to this question is sometimes called *the Cheng conjecture*.[5] For $n = 2$, the affirmative answer is known to be correct: the Bergman metric is Einstein–Kähler (if and) only if the domain is biholomorphic to the ball B^2 (see [Fu/Wong 1997] for the simply connected 2-dimensional case, [Nemirovskii/Shafikov 2006] for the general 2-dimensional case). The proof involves careful analysis of the Fefferman expansion and its relationship to the Tanaka–Chern–Moser boundary invariants of the domain. The corresponding question for $n > 2$ seems to involve considerable additional difficulties.

[5]Sometimes also attributed to S.T. Yau.

8

Automorphism Groups and Classification of Reinhardt Domains

This chapter will give a brief survey of results about the automorphisms of domains that possess circular symmetries. They are a rich source of examples in the study of invariant geometry and automorphism groups.

8.1 Reinhardt Domains

We begin with definitions. A domain D in $\mathbb{C}^n$ is called *circular* if it is invariant under the rotations

$$(z_1, \ldots, z_n) \mapsto (e^{i\varphi} z_1, \ldots, e^{i\varphi} z_n), \qquad \varphi \in \mathbb{R}.$$

It is possible for a domain to possess further circular symmetries. A domain D in $\mathbb{C}^n$ is called a *Reinhardt domain* if it is invariant under the rotations

$$(z_1, \ldots, z_n) \mapsto (e^{i\varphi_1} z_1, \ldots, e^{i\varphi_n} z_n), \qquad (\varphi_1, \ldots, \varphi_n) \in \mathbb{R}^n.$$

A domain D in $\mathbb{C}^n$ is called a *complete Reinhardt domain* if, whenever the point $z = (z_1, \ldots, z_n) \in D$, then $(\alpha_1 z_1, \ldots, \alpha_n z_n) \in D$ for all complex constants α_j satisfying $|\alpha_j| \le 1$ for all $j = 1, \ldots, n$.

Originally, the concept of Reinhardt domain arose together with the concept of *region of convergence*: for a (formal) power series $\sum_\beta a_\beta z^\beta$ (about the origin) in several complex variables, its *region of convergence* is the unique open set $\mathcal{R}$ such that the power series converges on $\mathcal{R}$ and diverges outside the closure $\mathrm{cl}(\mathcal{R})$. It is well known and easy to check that, for any formal power series, the region of convergence is a complete Reinhardt domain. (See for instance Theorem 1.4, page 7 of [Grauert/Fritzsche 1976].)

8.2 Sunada's Work

A classification of Reinhardt domains (up to a biholomorphic equivalence) with a description of their automorphism groups was established in

R. E. Greene et al., *The Geometry of Complex Domains,* Progress in Mathematics,
DOI 10.1007/978-0-8176-4622-6_8,

[Sunada 1974] (further detaila can be found in [Sunada 1978]) for the case when the Reinhardt domains in $\mathbb{C}^n$ under consideration are bounded and contain the origin. Then the case of Reinhardt domains that do not contain the origin was treated in [Shimizu 1987] (further details are in [Shimizu 1989]).

The work of Sunada which is about to be discussed was based upon Lie-theoretic understanding of (the Lie algebra of) the automorphism groups of Reinhardt domains. One might wonder how this method can be successful in establishing a classification because, in general, holomorphically inequivalent domains can have the same automorphism group. So what is special about Reinhardt domains?

We give a heuristic explanation. Since we are dealing with bounded Reinhardt domains, the rotations in each variable separately already tell us that one single boundary point gives rise to a real n-dimensional "round" torus in the boundary. If there were other automorphisms that create a real k-dimensional set (in the boundary) transverse to the torus just mentioned, then one obtains an $(n+k)$-dimensional subset of the boundary. Thus one can imagine that the matter boils down to understanding what kinds of automorphism groups are possible. Sunada tells us that the crucial part can be largely understood from the structure theory of Lie algebras of the automorphisms group of Reinhardt domains, because the Reinhardt property gives a good place to start.

8.2.1 Theorem of Thullen in Complex Dimension 2

In complex dimension 1, the only Kobayashi hyperbolic Reinhardt domains containing the origin are discs. It should be mentioned that, in complex dimension 2, P. Thullen classified the Reinhardt domains containing the origin.

Theorem 8.2.1 ([Thullen 1931]). *Let Ω be a bounded Reinhardt domain in $\mathbb{C}^2$ containing the origin. Then it is biholomorphic to one of the following types of domains:*

(i) *The bidisc $\{(z,w) \in \mathbb{C}^2 \colon |z| < 1, |w| < 1\}$;*
(ii) *The Thullen domain $\{(z,w) \in \mathbb{C}^2 \colon |z|^2 + |w|^r < 1\}$ for some $r > 0$;*
(iii) *A domain whose automorphism orbit of the origin consists of the origin only.*

8.2.2 Decomposition of the Lie Algebra of $\mathrm{Aut}^0(\Omega)$

Sunada's work generalizes the preceding theorem of Thullen (Theorem 8.2.1) to all dimensions. We shall sketch this work now. The details can be found in [Sunada 1978].

Recall that $\mathrm{Aut}\,(\Omega)$ is a (finite-dimensional) Lie group, since our domain Ω is bounded (see Section 1.3). More generally, the automorphism groups of Kobayashi hyperbolic domains are (finite-dimensional) Lie groups (cf. Section 7.2.3).

Let D be a Kobayashi hyperbolic Reinhardt domain in $\mathbb{C}^n$ containing the origin. Denote by G the identity component (=the connected component $\mathrm{Aut}^0(\Omega)$ of $\mathrm{Aut}(\Omega)$ that contains the identity) of $\mathrm{Aut}(\Omega)$. So both G and $\mathrm{Aut}(\Omega)$ are Lie groups of the same dimension. The Reinhardt property implies that G contains the n-dimensional torus subgroup consisting of coordinate-by-coordinate rotations, as in the preceding section.

Sunada considers the Lie algebra $\mathbf{g}$ of the Lie group G, and then studies its structure. The results from this study give normal forms for the Reinhardt domains containing the origin, and this provides a description of their automorphism groups. In outline:

Notice that $\mathbf{g}$ consists of complete holomorphic vector fields on D. Thus it is natural to consider the subalgebra $\mathbf{k}$ consisting of vector fields vanishing at the origin. This subalgebra $\mathbf{k}$ is of course isomorphic to the Lie algebra of the isotropy subgroup hereafter denoted by K of G at the origin (cf. Section 1.6). In particular, the complex Euler vector field

$$E := \sum_{j=1}^{n} \sqrt{-1}\; z_j \frac{\partial}{\partial z_j}$$

is an element of $\mathbf{k}$ and corresponds to the 1-parameter subgroup of automorphisms

$$k_\theta(z_1, \ldots, z_n) = (e^{i\theta} z_1, \ldots, e^{i\theta} z_n), \qquad \theta \in \mathbb{R}.$$

Notice that every element of K must be complex linear by Cartan's result Corollary 1.3.2. Furthermore, the 1-parameter subgroup just mentioned is contained in the *center* of the group K, again by Corollary 1.3.2.

In case $G = K$, there is not much one can do. However, if $K \subsetneq G$, then one may try further, using the structure theory of Lie algebras, to understand the situation. One notices that every $X \in \mathbf{g}$ has a Taylor expansion

$$X = \sum_{\lambda=0}^{\infty} X_\lambda,$$

where the coefficients of each X_λ are homogeneous polynomials of total degree λ. (Here we use the standard Euclidean coordinate system of $\mathbb{C}^n$.) Sunada uses the methods developed by [Kaup/Matsushima/Ochiai 1970] and considers the Lie algebra automorphism $J(X) = [E, X]$ where E is the complex Euler vector field introduced above. Then the kernel of J is actually $\mathbf{k}$. Define

$$\mathbf{p} := \{X \in \mathbf{g} \colon J^2(X) = -X\};$$

then $\mathbf{p}$ is a subalgebra of $\mathbf{g}$ and the decomposition

$$\mathbf{g} = k + p$$

holds ([Kaup/Upmeier 1976]). Then Sunada goes on to decompose $\mathbf{p}$ and try to understand what the structure of $\mathbf{g}$ can be. First he proves that $X_\lambda = 0$

for $\lambda > 2$, whenever $X \in \mathbf{p}$. This is where the bounded circularity of D enters again. Then $\mathbf{p} \otimes \mathbb{C}$ decomposes naturally by J, as

$$\begin{aligned} &X_0 \in \mathbf{p}^- := \{X \in \mathbf{p} \otimes \mathbb{C} \colon JX = -\sqrt{-1}X\},\\ &X_1 \in \mathbf{k},\\ &X_2 \in \mathbf{p}^+ := \{X \in \mathbf{p} \otimes \mathbb{C} \colon JX = \sqrt{-1}X\}. \end{aligned}$$

Sunada then investigates how the Lie bracket multiplication table can "traverse" between the elements of $\mathbf{k}, \mathbf{p}, \mathbf{p}^{\pm}$. Such an investigation is important for understanding which automorphism groups are possible for a Reinhardt domain, because much of the Lie group structure of G can be understood from the structures of its Lie algebra.

Further understanding of the structure of $\mathbf{g}$, from the Reinhardt property of D, turns out to be obtained by considering the polynomial vector fields $iz_k(\partial/\partial z_k)$ $(k = 1, \ldots, n)$ and the vector space of linear combinations of these vector fields with coefficients in $\mathbb{R}$. Denote this space by $\mathbf{h}_0$; this is the maximal abelian Lie subalgebra of $\mathbf{g}$. Its complexification $\mathbf{h} := \mathbf{h}_0 \otimes \mathbb{C}$ is a maximal abelian subalgebra of $\mathbf{g}^{\mathbb{C}} := \mathbf{g} \otimes \mathbb{C}$ such that, for each element H of $\mathbf{h}$, *ad* H is a semi-simple endomorphism of $\mathbf{h}$. (Such a subalgebra is usually called a *Cartan subalgebra*.) This allows application of the concept called the *root system* in Lie algebra theory,[1] and we arrive eventually at a stage where understanding the possibilities for the automorphism group reduces to understanding the possibilities for the (complexified) Lie algebra of the identity component of the automorphism group (Lemmas 2–16, in Sections 2, 3 and 4 of [Sunada 1978]). For instance,

$$\mathbf{g}^{\mathbb{C}} = \mathbf{h} + \sum_{\alpha \in \Delta} \mathbf{g}^{\alpha} \qquad \text{(direct sum)}$$

and each $\mathbf{g}^{\alpha}$ is contained in $\mathbf{k}$, $\mathbf{p}^+$ or $\mathbf{p}^-$. (Here Δ is the root system.) Furthermore, the generators of each root space are described rather explicitly. This determines the complete possibilities for the normal forms for Reinhardt domains, and their automorphism groups.

In case the root system (i.e., the set of nonzero roots) is nonempty, and $\mathbf{g}^{\mathbb{C}} \supsetneqq \mathbf{h}$, then the $\mathbf{g}^{\alpha}$s give rise to a vector field (that in turn gives rise to a family of automorphisms). A typical (complexified) vector field turns into a differential equation

$$\begin{cases} \dfrac{dx_i}{dt} = 1 - x_i^2 \\[2mm] \dfrac{dx_k}{dt} = -a_k^i x_i x_k \end{cases}$$

[1]The roots are defined as follows: for a Cartan subalgebra $\mathbf{h}$, let $H \in \mathbf{h}$. Then the map adH is semi-simple. Now consider $\alpha : h \to \mathbb{C}$ a linear functional and set $\mathbf{g}^{\alpha} := \{Y \in \mathbf{g}^{\mathbb{C}} \colon ad\ H(Y) = \alpha(H)Y, \forall H \in \mathbf{h}\}$. If $\mathbf{g}^{\alpha}$ is nontrivial, then the functional α is called a *root*. Thus one can see that the collection of nonzero roots (called the *root system*) will result in a decomposition (called the *root decomposition* of $\mathbf{g}^{\mathbb{C}}$). The Jacobi identity implies that $[\mathbf{g}^{\alpha}, \mathbf{g}^{\beta}] \subset \mathbf{g}^{\alpha+\beta}$.

which defines a real curve (with parameter t) in

$$D_{ik} := \{(z_i, z_k) \in \mathbb{C}^2 \colon (0, \ldots, 0, z_i, 0, \ldots, 0, z_k, 0, \ldots, 0) \in D\}.$$

The Kobayashi hyperbolicity of D implies that $x_k(t)$ has to be bounded for all t, and hence $a_k > 0$. Furthermore, this gives rise to

$$D_{ik} = \left\{(z_i, z_k) \in \mathbb{C}^2 \colon |z_i| < 1, \frac{z_k}{(1 - |z_i|^2)^{a_k/2}} \in D_k\right\},$$

where D_k is a disc of bounded radius. We shall not go into the detailed analysis in Sunada's work any further, but we hope that this discussion gives some flavor of Sunada's analysis to the reader. Instead, while we refer to [Sunada 1978] for details, we simply jump to Sunada's main theorems in the next section.

8.2.3 Sunada's Theorems

Let

$$Z_i = (z_{n_1+\cdots+n_{i-1}+1}, \ldots, z_{n_1+\cdots+n_{i-1}+n_i})$$

and

$$|Z_i|^2 = |z_{n_1+\cdots+n_{i-1}+1}|^2 + \cdots + |z_{n_1+\cdots+n_{i-1}+n_i}|^2,$$

where $i \in \{1, \ldots, s\}$ with $n_1 + \cdots + n_s = n$.

Theorem 8.2.2 (Sunada). *Let D be a bounded (or Kobayashi hyperbolic) Reinardt domain containing the origin. Then there are positive integers $n_1, \ldots, n_s$ with $n_1 + \cdots + n_s = n$, a number r with $1 \le r \le s$ and a matrix (p^i_j) such that D is biholomorphic (by a biholomorphism of type $z \mapsto (r_1 z_{\sigma(1)}, \ldots, r_n z_{\sigma(n)})$, where σ is a permutation of indices and $r_i > 0$) to a Reinhardt domain*

$$\begin{aligned}\widetilde{D} = \Bigg\{&(Z_1, \ldots, Z_s) \in \mathbb{C}^n \colon |Z_1| < 1, \ldots, |Z_r| < 1;\\ &\left(\frac{Z_{r+1}}{\prod_{i=1}^r (1 - |Z_i|^2)^{p^i_{r+1}/2}}, \ldots, \frac{Z_s}{\prod_{i=1}^r (1 - |Z_i|^2)^{p^i_s/2}}\right) \in \widetilde{D}_1\Bigg\}.\end{aligned}$$

Here

$$\begin{aligned}\widetilde{D}_1 = \widetilde{D} \cap (\mathbb{C}^p \times \mathbf{0}) = \{&(Z_1, \ldots, Z_s) \in \mathbb{C}^n \colon\\ &|Z_1|^2 < 1, \ldots, |Z_r|^2 < 1, Z_{r+1} = 0, \ldots, Z_s = 0\}\end{aligned}$$

(with $p = n_1 + \cdots + n_r$) and $\widetilde{D} \cap \{(0, \ldots, 0)\} \times \mathbb{C}^{n-p}$ is a bounded Reinhardt domain in $\mathbb{C}^{n-p}$ (when projected).

This theorem in the case $n = 2$ recovers Thullen's theorem (Theorem 8.2.1). Let us see how that follows. Since $n = 2$, there are only four cases.

Case 1. $s = 1$: One sees immediately that D is equivalent to the ball.
Case 2. $s = 2$ and $r = 2$: D is equivalent to the bidisc.
Case 3. $s = 2$ and $r = 1$: D is biholomorphic to

$$\widetilde{D} = \left\{ (z_1, z_2) \in \mathbb{C}^2 \colon |z_1| < 1, \frac{z_2}{(1 - |z_1|^2)^{p/2}} \in \widetilde{D}_1 \right\}.$$

Since any bounded Reinhardt domain in $\mathbb{C}$ containing the origin is a disc, the second condition is simply $|z_2| < (1 - |z_1|^2)^{p/2}$, and hence $\widetilde{D}$ is defined by $|z_1|^2 + |z_2|^{2/p} < 1$.

Case 4. The generic case: all automorphisms are linear.

The following theorem describes the structure of the identity component G of the automorphism group of the Reinhardt domain $\widetilde{D}$ (of course isomorphic to G for the original D). In what follows, ${}^tA$ denotes the transpose of the matrix A. The automorphisms of the unit ball B^n in $\mathbb{C}^n$ can be written as follows.

$$z \to {}^t\big((A^t z + b)(c^t z + d)^{-1}\big),$$

where A, b, c, d are matrices of type $n \times n, n \times 1, 1 \times n$, and 1×1, respectively, satisfying the relations

$${}^t\bar{A}A - {}^t\bar{c}c = I, \quad {}^tbb - d^2 = -1, \quad {}^t\bar{b}A = \bar{d}c, \quad \det \begin{pmatrix} A & b \\ c & d \end{pmatrix} = 1.$$

For short-hand notation, let $Z_i = {}^tZ_i$.

Theorem 8.2.3 (Sunada). *The identity component G of the automorphism group of $\widetilde{D}$ consists of transformations of the following type:*

$$\begin{cases} Z_i \mapsto (A_i Z_i + b_i)(c_i Z_i + d_i)^{-1}, & i = 1, \dots, r, \\ Z_j \mapsto B_j \prod_{i=1}^r (c_i Z_i + d_i)^{-p_i^j} Z_j, & j = r+1, \dots, s, \end{cases}$$

where $\left(\begin{smallmatrix} A_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$ is an $n_{i+1} \times n_{i+1}$ matrix satisfying the relation above, and B_j is a unitary $n_j \times n_j$ matrix.

8.3 Shimizu's Theorems

There are of course Reinhardt domains that do not contain the origin. In [Shimizu 1987], S. Shimizu obtained a general result encompassing Sunada's work sketched above and also dealing with Reinhardt domains not containing the origin. When the Reinhardt domain does not contain the origin, Cartan's uniqueness theorem (Theorem 1.3.1) is not available. Hence an alternative is needed. One of the key facts Shimizu uses is the following.

Proposition 8.3.1 (Shimizu). *Let $\varphi : D \to D'$ be a biholomorphic mapping of bounded Reinhardt domains D and D' in $\mathbb{C}^n$. Then φ is equivariant with respect to the n-dimensional torus action (i.e., for every rotation ρ of D there exists a rotation ρ' of D' such that $\varphi \circ \rho \circ \varphi^{-1} = \rho'$) if and only if*

$$\varphi(z_1, \ldots, z_n) = (\ldots,\ \alpha_i z_1^{a_{1i}} \cdot \ldots \cdot z_n^{a_{ni}}, \ldots),$$

where $(a_{ij}) \in GL(n, \mathbb{Z})$, and where $\alpha_1, \ldots, \alpha_n$ are nonzero complex numbers.

It is shown in [Shimizu 1988] that every automorphism of a bounded Reinhardt domain can be written as the composition of automorphisms of the above type and an element of the identity component. Since the arguments of [Shimizu 1988] are purely Lie-group-theoretic, the result naturally extends to Kobayashi hyperbolic Reinhardt domains.

Theorem 8.3.2 (Shimizu). *To each bounded (or Kobayashi hyperbolic) Reinhardt domain D in $\mathbb{C}^n$, there is an associated Reinhardt domain $\widetilde{D}$ in $\mathbb{C}^n$ which is the image of D under an algebraic biholomorphism in the preceding proposition for which, with the block decomposition*

$$\begin{aligned} z &= (Z_1, \ldots, Z_r, Z_{r+1}, \ldots, Z_s, \ldots, Z_t) \in \mathbb{C}^n \\ &= \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r} \times \mathbb{C}^{n_{r+1}} \times \cdots \times \mathbb{C}^{n_s} \times \mathbb{C}^{n_{s+1}} \times \cdots \times \mathbb{C}^{n_t}, \end{aligned}$$

the following hold:

(i) $\widetilde{D}_1 = \pi(\widetilde{D}) = B^{n_1} \times \cdots \times B^{n_r} \times \mathbb{C}^{n_{r+1}} \times \cdots \times \mathbb{C}^{n_s}$, *where* $\pi : \mathbb{C}^n \to \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_r} \times \mathbb{C}^{n_{r+1}} \times \cdots \times \mathbb{C}^{n_s}$ *is the obvious projection. (Of course,* B^k *denotes the open unit ball in* $\mathbb{C}^k$*.)*

(ii) $\widetilde{D}_2 = \widetilde{D} \cap \{0\} \times \cdots \times \{0\} \times \mathbb{C}^{n_{s+1}} \times \cdots \times \mathbb{C}^{n_t}$.

(iii) $\widetilde{D}$ *can be written in the form*

$$\begin{aligned} \widetilde{D} = \Bigg\{ & z \in \mathbb{C}^n \colon (Z_1, \ldots, Z_s) \in \widetilde{D}_1, \\ & \left(\frac{Z_{s+1}}{\prod_{i=1}^r (1 - |Z_i|^2)^{p_i^{s+1}/2} \prod_{j=r+1}^s \exp(-q_j^{s+1}|Z_j|^2)}, \right. \\ & \left. \ldots, \frac{Z_t}{\prod_{i=1}^r (1 - |Z_i|^2)^{p_i^t/2} \prod_{j=r+1}^s \exp(-q_j^t |Z_j|^2)} \right) \in \widetilde{D}_2 \Bigg\}, \end{aligned}$$

where P_j^k, q_j^k, $i = 1, \ldots, r$, $j = r+1, \ldots, s$, $k = s+1, \ldots, t$ are nonnegative real constants, and for each index j with $r + 1 \le j \le s$, there is an index k with $s + 1 \le k \le t$ such that $q_j^k > 0$, $n_k = 1$, and $\widetilde{D} \cap \{Z_k = 0\} = \emptyset$.

There is an associated characterization of the automorphism groups as follows (with the same notation as above).

Theorem 8.3.3 (Shimizu). *The identity component G of* $\operatorname{Aut}(\tilde{D})$ *consists of the transformations* $(Z_1, \ldots, Z_t) \to (W_1, \ldots, W_t)$ *of the form*

$$\begin{cases} W_i = (A_i Z_i + b_i)(c_i Z_i + d_i)^{-1}, & i = 1, \ldots, r \\ W_j = B_j Z_j + e_j, & j = r+1, \ldots, s \\ W_k = C_k \prod_{i=1}^{r} (c_i Z_i + d_i)^{-p_i^k} & \\ \quad \cdot \prod_{j=r+1}^{s} \exp[-q_j^k \{2 {}^t\bar{e}_j B_j Z_j + |e_j|^2\}] Z_k, & k = s+1, \ldots, t, \end{cases}$$

where

$$\begin{cases} \begin{pmatrix} A_i & b_i \\ c_i & d_i \end{pmatrix} \in SU(n,1), & & i = 1, \ldots, r, \\ B_j \in U(n_j), & e_j \in \mathbb{C}^{n_j}, & j = r+1, \ldots, s, \\ C_k \in U(n_k), & & k = s+1, \ldots, t. \end{cases}$$

The detailed arguments can be found in [Shimizu 1987], [Shimizu 1988] and [Shimizu 1989].

8.4 Non-Reinhardt Circular Domains

A natural question that can arise is whether there are circular domains that cannot be holomorphically equivalent to any Reinhardt domain.

This question was studied by [Fu/Isaev/Krantz 1996]. They came up with the following

Example 1 *The domain*

$$D = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon |z_1|^2 + |z_2|^4 + |z_3|^4 + (z_2\bar{z}_3 + \bar{z}_2 z_3)^2 < 1\}$$

is not biholomorphic to any Reinhardt domain.

This domain is circular, but not a Reinhardt domain as defined. But the question is whether it is or is not biholomorphic to some Reinhardt domain. For this purpose Fu, Isaev and Krantz computed the automorphism group. They started with the observation that D is a bounded, pseudoconvex domain with a real analytic boundary. Therefore, by [Diederich/Fornæss 1978] for instance, every automorphism of D extends to a diffeomorphism (in fact a holomorphic mapping defined on an open neighborhood) of the closure of D. In particular, this shows that the maximal order of contact by smooth analytic varieties to each boundary point (related to, though not the same as the D'Angelo type) has to be preserved. With this information, together with Cartan's theorem (Corollary 1.3.2), they were able to compute the entire automorphism group which is generated by the rotations of type

$$(z_1, z_2, z_3) \mapsto (e^{it} z_1, e^{is} z_2, \pm e^{is} z_3), \quad s, t \in \mathbb{R},$$

and the Möbius-type transform

$$(z_1, z_2, z_3) \mapsto \left(\frac{z_1 - \alpha}{1 - \bar{\alpha} z_1}, \left(\frac{\sqrt{1 - |\alpha|^2}}{1 - \bar{\alpha} z_1} \right)^{1/2} z_2, \left(\frac{\sqrt{1 - |\alpha|^2}}{1 - \bar{\alpha} z_1} \right)^{1/2} z_3 \right),$$

where α is a complex number with modulus less than 1. Thus the automorphism group has dimension 4. They then appeal to Sunada's classification theory which implies that the automorphism group of a Kobayashi hyperbolic Reinhardt domain is always odd dimensional. Thus the assertion follows.

This conclusion can be obtained without Sunada's theory. Since the Möbius-type transform has no fixed points, it is easy to see that the isotropy subgroup at any point of D is of dimension 2 or smaller. So one immediately realizes that it cannot be biholomorphic to any Reinhardt domain in $\mathbb{C}^3$, as they possess a (at least) 3-dimensional isotropy subgroup at the origin. (Notice that the above example has a noncompact automorphism group.)

It is not so difficult to find other examples. Consider for instance

$$\Omega = \{(z, w) \in \mathbb{C}^2 \colon |z|^4 + |w|^8 + (z\bar{w} + \bar{z}w)^8 < 1\}.$$

This is also a pseudoconvex bounded domain with a real analytic boundary. Now the circle $\{(e^{it}, 0) \colon t \in \mathbb{R}\}$ has to be preserved by its automorphisms, and so does the circle $(0, e^{is}) \colon s \in \mathbb{R}\}$, due to the type considerations already discussed. Thus the linear discs bounded by these circles, being the only holomorphic discs bounded by them, respectively, must also be preserved. Thus the origin, the unique intersection point of these two discs, must be preserved and all automorphisms must thus be linear maps (Corollary 1.3.2). Then it is easy to see that these linear maps can be only

$$(z, w) \mapsto (e^{it} z, e^{it} w)$$

so that $\dim_{\mathbb{R}} \operatorname{Aut}(\Omega) = 1$. But the automorphism group of a Reinhardt domain in $\mathbb{C}^2$ contains a real 2-dimensional torus. This implies that Ω is (biholomorphically) a non-Reinhardt circular domain.

9

The Scaling Method, I

If a bounded domain Ω in $\mathbb{C}^n$ has a noncompact automorphism group, then all the orbits of the automorphism group are noncompact (Proposition 1.3.10). Thus each orbit must "go out to the boundary" of the domain Ω, since orbits are closed in Ω (Corollary 1.3.6). Such boundary orbit accumulation points are pseudoconvex, when they are "one-sided," e.g., when the boundary is C^1 smooth near the point (Theorem 1.5.1, cf. also the discussion in Chapter 7 following Theorem 7.6.1). Under some reasonable hypothesis on the domain as a whole, e.g., that it is a domain of holomorphy, one expects in general terms that localization properties of the $\overline{\partial}$ operator would imply that the essentials of the situation would be localized. first, if a sequence of automorphisms $\varphi_j \in \mathrm{Aut}\,(\Omega)$, $j = 1, 2, \ldots$, has, for some $p \in \Omega$, $\lim \varphi_j(p) = q_0 \in \partial\Omega$, then the structure of Ω as a whole should be controlled by the nature of $\partial\Omega$ near q_0. The guiding principle is "*What's behind is not important*" ([Bail 1976]), what's behind in this case being anywhere except near q_0.

It has been conjectured (by two of the authors, Greene and Krantz—see Section 9.5 and [Greene/Krantz 1991]) that, when such a q_0 is a C^∞ boundary point, it must be of "finite type" in the sense of D'Angelo. In this case, rather precise information on $\overline{\partial}$ localization is available (cf. [Catlin 1983]; also [Catlin 1989]).

It has turned out that for many purposes the "What's behind is not important" principle can be made explicit more easily and efficiently by a kind of re-normalized normal families process rather than by looking at $\overline{\partial}$ results as such. The collection of techniques and results of this sort has become known as the *scaling method*. This chapter and the following one are devoted to exploring this method and its results in some detail. In particular, we shall present a scaling-method proof of a new result, the asymptotic constancy of holomorphic sectional curvature for C^2 strongly pseudoconvex domains. This result improves Theorem 3.4.3 of Klembeck [Klembeck 1978], which used the Fefferman expansion and hence required C^∞ boundary. This C^2 result (see Section 10.1) is also shown to be stable in an appropriate sense extending Theorem 3.5.1. As discussed in Chapters 3 and 4, this stable asymptotic constancy

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yields important consequences about automorphism groups. The extension of these results to more general situations will be presented in Section 10.2, Chapter 10.

Although we shall not discuss any of its details, a theorem of Kodama should be mentioned—it also follows a similar principle but is an even more striking example of the localization idea. For example his result implies

Theorem ([Kodama 1999]). If a bounded domain Ω in $\mathbb{C}^2$ has a boundary point $p = (1, 0)$ satisfying

(1) $\Omega \cap U = E \cap U$ for some open neighborhood of p, where $E = \{(z,w) \in \mathbb{C}^2 \colon |z|^2 + |w|^{2m} < 1\}$ for some integer $m > 0$, and
(2) there exists $\varphi_j \in \operatorname{Aut}(\Omega)$ and $q \in \Omega$ such that $\lim_{j\to\infty} \varphi_j(q) = p$,

then $E = \Omega$ as sets.

See also [Dini/Selvaggi 1997] for related results.

Since the scaling method thus occupies a central place in our overall picture and since it is not widely available in a systematic form, the original developments being somewhat scattered in the literature, we shall give a somewhat leisurely presentation beginning with the case of complex dimension 1. References to the literature will be provided as we go along.

9.1 A Basic Example: Scaling Method in Dimension 1

9.1.1 The Scaling of the Unit Disc

Let D be the open unit disc in the complex plane $\mathbb{C}$. Choose a sequence a_j in D satisfying the conditions

$$0 < a_j < a_{j+1} < 1, \quad \forall j = 1, 2, \ldots,$$

and

$$\lim_{j\to\infty} a_j = 1.$$

Consider the sequence of dilations

$$L_j(z) = \frac{1}{1 - a_j}(z - 1).$$

Let us write $\lambda_j = 1 - a_j$ for a moment. Then one sees immediately that

$$\begin{aligned} L_j(D) &= \left\{\zeta \in \mathbb{C} \mid (1 + \lambda_j\zeta)(1 + \lambda_j\bar{\zeta}) < 1\right\} \\ &= \left\{\zeta \in \mathbb{C} \mid 2\operatorname{Re}\zeta < -\lambda_j|\zeta|^2\right\}. \end{aligned}$$

It follows that the sequence of sets $L_j(D)$ *converges* to the left half-plane $H = \{\zeta \in \mathbb{C} \mid \operatorname{Re}\zeta < 0\}$ in the sense that

$$L_j(D) \subset L_{j+1}(D), \quad \forall j = 1, 2, \ldots,$$

and

$$\bigcup_{j=1}^{\infty} L_j(D) = H.$$

Now we combine this simple observation with the fact that there exists a sequence of maps

$$\varphi_j(z) = \frac{z + a_j}{1 + a_j z}$$

that are automorphisms of D satisfying $\varphi_j(0) = a_j$. Consider the sequence of composite maps

$$\sigma_j \equiv L_j \circ \varphi_j : D \to \mathbb{C}.$$

A direct computation yields that

$$\begin{aligned} L_j \circ \varphi_j(z) &= \frac{1}{1 - a_j} \left(\frac{z + a_j}{1 + a_j z} - 1 \right) \\ &= \frac{z - 1}{1 + a_j z}. \end{aligned}$$

Hence in fact we see that the sequence of holomorphic mappings $L_j \circ \varphi_j$ converges uniformly on compact subsets of D to the mapping

$$\widehat{\sigma}(z) = \frac{z - 1}{z + 1}$$

which is a biholomorphic mapping from the open unit disc D onto the left half-plane H. (We have exhibited a means to discover the Cayley map by way of scaling.)

9.1.2 A Generalization

We now expand the simple observations of the preceding subsection to yield the statement and the proof of the following one-dimensional version of the Wong–Rosay theorem (Theorem 9.2.1).

Proposition 9.1.1. *Let Ω be a domain in the complex plane $\mathbb{C}$ admitting a boundary point p such that*

(i) *there exists an open neighborhood U of p in $\mathbb{C}$ such that $U \cap \partial\Omega$ is a C^1 curve, and*
(ii) *there exists a sequence f_j of automorphisms of Ω and a point $q \in \Omega$ such that*

$$\lim_{j\to\infty} f_j(q) = p.$$

Then Ω is biholomorphic to the open unit disc.

Note that we proved this result in an earlier part of the book by a different technique (cf. Theorem 2.7.1 combined with the Riemann mapping theorem). The point now is just to illustrate the technique of scaling.

Plan of the proof. Let $q_j = f_j(q)$ for each j. Choose the closest point in the boundary to q_j and call it p_j. If the closest boundary point p_j to q_j is not unique, then choose one arbitrarily. As j tends to infinity, p_j converges to p because q_j converges to p. Then we select θ_j and apply the map $r_j(z) \equiv e^{i\theta_j}(z - p_j)$ so that

$$r_j(p_j) = 0 \quad \text{and} \quad r_j(q_j) > 0$$

for each j. Now consider the sequence of mappings

$$\sigma_j(z) = \frac{1}{r_j(q_j)}(r_j \circ f_j(z)).$$

Notice that $\sigma_j(\Omega) = \frac{1}{r(q_j)}\, r_j(\Omega)$ for each j. Thus we expect that $\sigma_j(\Omega)$ is almost the right half-plane as j becomes very large. At least every $\sigma_j(\Omega)$ is contained in $\mathbb{C} \setminus \ell$ for some line segment of positive length ℓ and for every j. (Note that ℓ can be chosen independently of j.) Therefore one can select a subsequence from $\{\sigma_j\}$ that converges uniformly on compact subsets of Ω. Let $\widehat{\sigma}$ be the limit mapping. Then we expect $\widehat{\sigma} : \Omega \to \mathbb{C}$ to be an injective holomorphic mapping, and furthermore $\widehat{\sigma}(\Omega)$ is equal to the right half-plane. Thus we hope to conclude that Ω is biholomorphic to the right half-plane, which in turn is biholomorphic to the open unit disc. See Figure 9.1. $\square$

This plan actually works, but it is evident that there are several points that need clarification. We shall now present the precise proof, which will show much of the essence of the scaling method. We shall use the common notation $z = x + iy$ for the complex variable z and its real and imaginary parts x and y.

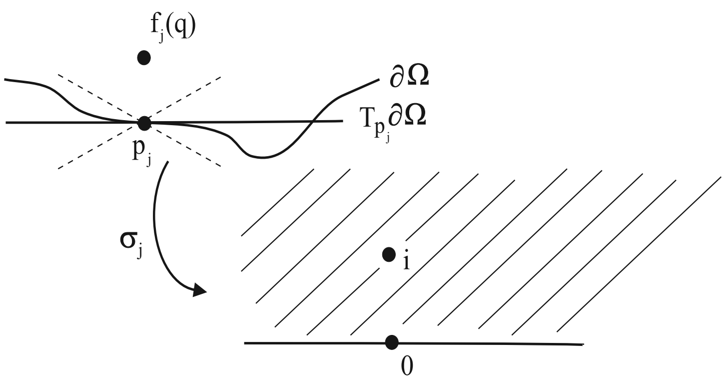

Fig. 9.1. The scaling process.

Proof of Proposition 9.1.1. Keeping the "plan of the proof" in mind, we present the precise proof in several steps. Let $p \in \partial\Omega$ be as in the hypothesis of the proposition. Write $D(p,r) = \{z \in \mathbb{C} \mid |z-p| < r\}$. Transforming Ω by a biholomorphic map $z \mapsto e^{i\alpha}(z-p)$, we may assume the following with no loss of generality:

(a) $p = 0$
(b) $\Omega \cap D(p,r) = \{z = x+iy \mid y > \psi(x), |z-p| < r\}$ and $\partial\Omega \cap D(p,r) = \{z \mid y = \psi(x), |z-p| < r\}$ for a real-valued C^1 function ψ in one real variable satisfying $\psi(0) = 0$ and $\psi'(0) = 0$.

Step 1. *Construction of the Scaling Map.* Notice that the sequence $f_j(q)$ now converges to 0 as $j \to \infty$. For each j, we choose a point $p_j \in \partial\Omega$ that is the closest to $f_j(q)$. Since p_j also converges to 0, replacing f_j by a subsequence if necessary, we may assume that every $p_j \in D(p, r/4)$. Now, for each j, set

$$A_j(z) = i\,\frac{|f_j(q) - p_j|}{f_j(q) - p_j}\,(z - p_j).$$

Notice that $f_j(q) - p_j$ is a positive scalar multiple of the inward unit normal vector to $\partial\Omega$ at p_j. Thus $\frac{f_j(q)-p_j}{|f_j(q)-p_j|}$ converges to the inward unit normal vector to $\partial\Omega$ at 0. This implies that A_j in fact converges to the identity map. Consequently, there exist positive constants r_1, r_2 independent of j such that, for each j, there exists a C^1 function $\psi_j(x)$ defined for $|x| < r_1$ satisfying

$$A_j(z) \cap ([-r_1, r_1] \times [-r_2, r_2]) = \{x + iy \mid |x| < r_1, |y| < r_2, y > \psi_j(x)\}.$$

Furthermore, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\psi_j(x) < \epsilon |x| \text{ whenever } |x| < \delta$$

regardless of j.

Next, let $\lambda_j = |f_j(q) - p_j|$ for each j. Consider the dilation map

$$L_j(z) = \frac{z}{\lambda_j}.$$

Then the sequence of holomorphic mappings we want to construct is given by

$$\sigma_j \equiv L_j \circ A_j \circ f_j : \Omega \to \mathbb{C}.$$

Before starting the next step, we make a few remarks. The automorphism f_j preserves the domain Ω but moves q to $f_j(q)$ so that $f_j(q)$ converges to the origin—recall that we made changes so that p became the origin at the beginning of the proof. Then the affine map A_j adjusts Ω so that the direction vector $\frac{f_j(q)-p_j}{|f_j(q)-p_j|}$ is transformed to a purely imaginary number. The final component L_j in the construction simply magnifies the domain $A_j(\Omega)$, while the map L_j itself diverges.

Step 2. *Convergence of* σ_j. We shall actually choose a subsequence from $\{\sigma_j\}$ that converges uniformly on compact subsets of Ω. Observe first that

$$\sigma_j(\Omega) = L_j \circ A_j \circ f_j(\Omega) = L_j \circ A_j(\Omega)$$

since $f_j(\Omega) = \Omega$. Choosing a subsequence of σ_j we may assume that $\lambda_j < 1$ for every j. Then, since L_j is a simple dilation by a positive number, and since $A_j(\Omega)$ will miss a line segment

$$E = \{-iy \mid 0 \le y \le b\}$$

for some constant b independent of j, we see immediately that

$$\sigma_j(\Omega) \subset \mathbb{C} \setminus E$$

for every $j = 1, 2, \ldots$. Therefore Montel's theorem implies that every subsequence of $\{\sigma_j\}$ admits a subsequence, which we again (by an abuse of notation) denote by σ_j, that converges uniformly on compact subsets of Ω. Denote by $\widehat{\sigma}$ the limit of the sequence σ_j.

Step 3. *Analysis of* $\widehat{\sigma}(\Omega)$. We want to establish that

$$\widehat{\sigma}(\Omega) = \mathcal{S},$$

where $\mathcal{S} \equiv \{z \in \mathbb{C} \mid \ \operatorname{Im} z > 0\}$.

Let ϵ be a positive real number and let K be an arbitrary compact subset of Ω. We will show that $\widehat{\sigma}(K) \subset C_\epsilon$, where $C_\epsilon \equiv \{z \in \mathbb{C}\colon\ -\epsilon < \arg z < \pi + \epsilon\}$.

Choose $R > 0$ such that $\widehat{\sigma}(K)$ is contained in the disc $D(0, R)$ of radius R centered at 0.

The sequence $f_j : \Omega \to \Omega$ is a normal family since $\mathbb{C} \setminus \Omega$ contains a line segment with a positive length. Every subsequence of f_j contains a subsequence that converges uniformly on compact subsets, since $f_j(q)$ converges to p. Let $g : \Omega \to \operatorname{cl}(\Omega)$ be a subsequential limit map. Then $g(q) = p$. Recall that $p \in \partial\Omega$. Hence the open mapping theorem yields that $g(z) = p$ for every $z \in \Omega$. Thus the sequence f_j itself converges uniformly on compact subsets to the constant map with value p. Therefore we may choose $N > 0$ such that $f_j(K)$ is contained in a sufficiently small neighborhood of the origin for every $j > N$, and hence $A_j \circ f_j(K) \subset C_\epsilon$ for every $j > N$. Then it follows immediately that $\sigma_j(K) \subset C_\epsilon$ for every $j > N$, and consequently that

$$\widehat{\sigma}(K) \subset C_\epsilon.$$

Since K is an arbitrary compact subset of Ω, it follows that $\widehat{\sigma}(\Omega) \subset \operatorname{cl}(\mathcal{S})$. We also have $\widehat{\sigma}(q) = i$, since $\sigma_j(q) = L_j \circ A_j \circ f_j(q) = i$ for every $j = 1, 2, \ldots$. Therefore $\widehat{\sigma}(\Omega) \subset \mathcal{S}$.

Step 4. *Convergence of* σ_j^{-1}. Let $\widetilde{K}$ be an arbitrary compact subset of the upper half-plane $\mathcal{S}$. Then choose $\epsilon > 0$ so that $\widetilde{K} \subset C_\epsilon$. Choose then $r > 0$ such that

$$D(0, r) \cap C_\epsilon \subset \Omega \cap D(0, r).$$

Shrinking $r > 0$ if necessary, since A_j converges to the identity map uniformly on compact subsets of $\mathbb{C}$, there exists $N > 0$ such that

$$D(0, r) \cap C_\epsilon \subset A_j(\Omega) \cap D(0, r)$$

for every $j > N$. Hence we see that σ_j^{-1} maps K into Ω. Since $\Omega \subset \mathbb{C} \setminus E$ as observed before, we may again choose a subsequence of σ_j, which we again denote by σ_j, so that σ_j^{-1} converges to a holomorphic map, say $\tau : \mathcal{S} \to \mathrm{cl}(\Omega)$. Since τ is holomorphic and $\tau(i) = q$, we see that τ maps the upper half-plane $\mathcal{S}$ into Ω.

Step 5. *Synthesis.* We are ready to complete the proof. By the Cauchy estimates, the derivatives $d\sigma_j$ of σ_j as well as the derivatives $d[\sigma_j^{-1}]$ both converge. Therefore, $d\widehat{\sigma}(q) \cdot d\widehat{\tau}(i) = 1$. This means that $\widehat{\sigma} \circ \tau : \mathcal{S} \to \mathcal{S}$ is a holomorphic mapping satisfying $\widehat{\sigma} \circ \tau(i) = i$ and $(\widehat{\sigma} \circ \tau)'(i) = 1$. Then, by Schwarz's lemma, one concludes that $\widehat{\sigma} \circ \tau = \mathrm{id}$, where id is the identity mapping. Likewise, the same reasoning applied to $\tau \circ \widehat{\sigma} : \Omega \to \Omega$ implies that $\tau \circ \widehat{\sigma} = \mathrm{id}$. So $\widehat{\sigma} : \Omega \to \mathcal{S}$ is a biholomorphic mapping. $\square$

Remark 9.1.2. The sequence of mappings σ_j constructed above is often called a *scaling sequence*. It is constructed from a composition of

(1) the automorphisms carrying one fixed interior point successively to a boundary point,
(2) certain affine adjustments, and
(3) the stretching dilation map.

The proof given above is a good example of the scaling technique. The main thrust of the method is that the image of the limit mapping is determined solely by the affine adjustments and the dilations, while the scaling sequence converges to a biholomorphic mapping.

Remark 9.1.3. As observed earlier, Proposition 9.1.1 can be proved in a much simpler way. first, one may conclude immediately from the argument on the shrinking of $f_j(K)$ into a simply connected subset of Ω, that Ω must be simply connected. Then the conclusion follows by the Riemann mapping theorem. However, we chose not to do so, because the goal of this section is to provide a basis for the scaling method which can be applied to the higher dimensional cases. We shall see further developments in higher dimensions in subsequent sections.

9.2 Higher Dimensional Scaling and the Wong–Rosay Theorem

9.2.1 Nonisotropic Scaling

We now continue our discussion in complex dimension 2. It appears to be appropriate to demonstrate the scaling of the complex two-dimensional ball

$$B^2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$$

at the boundary point $(1, 0)$.

Denote by a_j a sequence of real numbers satisfying

$$0 < a_j < a_{j+1} < 1 \quad \forall j = 1, 2, \ldots$$

and

$$\lim_{j\to\infty} a_j = 1,$$

and let $q_j = (a_j, 0)$ for each $j = 1, 2, \ldots$. Then consider the translation

$$T(z_1, z_2) = (z_1 - 1, z_2).$$

The domain $T(B^2)$ is now defined by the inequality

$$|\zeta_1 + 1|^2 + |\zeta_2|^2 < 1$$

or, equivalently, by

$$2\,\mathrm{Re}\,\zeta_1 < -|\zeta_1|^2 - |\zeta_2|^2.$$

Notice that the mapping

$$\varphi_j(z_1, z_2) = \left(\frac{z_1 + a_j}{1 + a_j z_1}, \frac{\sqrt{1 - |a_j|^2}}{1 + a_j z_1} z_2\right)$$

is an automorphism of B^2 satisfying $\varphi_j(0) = q_j$ for every j. Finally consider

$$L_j(z_1, z_2) = \left(\frac{z_1}{\lambda_j}, \frac{z_2}{\sqrt{\lambda_j}}\right)$$

where $\lambda_j = 1 - a_j$ for each j. Imitating the one-dimensional case, we consider the scaling sequence

$$\sigma_j(z_1, z_2) = L_j \circ T \circ \varphi_j(z_1, z_2).$$

Notice here that L_j is a dilation but, unlike the one-dimensional case, it is nonisotropic in the sense that the eigenvalues are not uniformly comparable.

We now compute the limit map $\widehat{\sigma}(z_1, z_2) \equiv \lim_{j\to\infty} \sigma_j(z_1, z_2)$, and the set $\widehat{\sigma}(B^2)$. A direct computation yields the following:

$$\begin{aligned}\sigma_j(z_1, z_2) &= \left(\frac{1}{\lambda_j}\left(\frac{z_1 + a_j}{1 + a_j z_1} - 1\right), \frac{1}{\sqrt{\lambda_j}} \cdot \frac{\sqrt{1 - a_j{}^2}}{1 + a_j z_1} \cdot z_2\right) \\ &= \left(\frac{z_1 - 1}{1 + a_j z_1}, \frac{\sqrt{1 + a_j}\; z_2}{1 + a_j z_1}\right).\end{aligned}$$

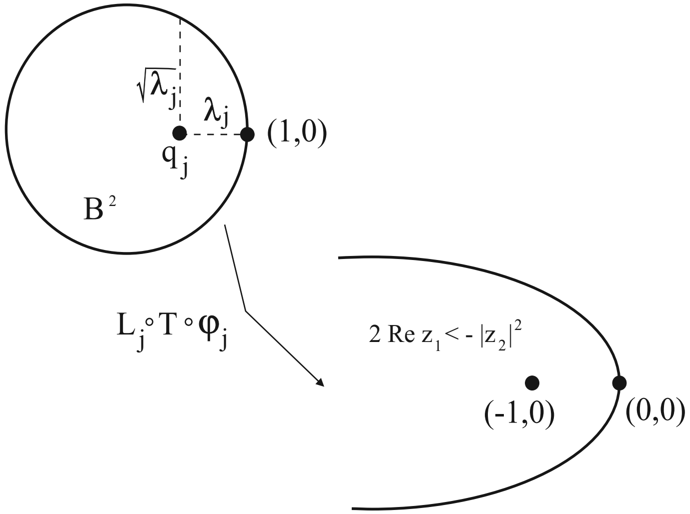

Fig. 9.2. Scaling of the ball B^2.

Therefore we see immediately that

$$\widehat{\sigma}(z_1, z_2) = \left(\frac{z_1 - 1}{z_1 + 1}, \frac{\sqrt{2}\, z_2}{z_1 + 1} \right)$$

and that σ_j converges to $\widehat{\sigma}$ uniformly on compact subsets of B^2.

Observe that the map $\widehat{\sigma} : B^2 \to \mathbb{C}^2$ is an injective holomorphic mapping, and that its image coincides with the Siegel upper half-space

$$\mathcal{S} = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid 2 \operatorname{Re} z_1 < -|z_2|^2 \right\}.$$

Therefore $\widehat{\sigma} : B^2 \to \mathcal{S}$ is in fact a biholomorphic mapping. Refer to Figure 9.2.

Observe also that one can see the convergence of the sets $\sigma_j(B^2)$ here. A direct argument yields

$$\begin{aligned}
\sigma_j(B^2) &= L_j \circ A_j \circ \varphi_j(B^2) \\
&= L_j \circ A_j(B^2) \\
&= L_j \left(\left\{ z \in \mathbb{C}^2 \mid |z_1 + 1|^2 < 1 - |z_2|^2 \right\} \right) \\
&= L_j \left(\left\{ z \in \mathbb{C}^2 \mid 2 \operatorname{Re} z_1 < -|z_1|^2 - |z_2|^2 \right\} \right) \\
&= \left\{ z \in \mathbb{C}^2 \mid 2 \operatorname{Re} \lambda_j z_1 < -{\lambda_j}^2 |z_1|^2 - \lambda_j |z_2|^2 \right\} \\
&= \left\{ z \in \mathbb{C}^2 \mid 2 \operatorname{Re} z_1 < -\lambda_j |z_1|^2 - |z_2|^2 \right\}.
\end{aligned}$$

Since $\lambda_j \searrow 0$, it follows immediately that

$$\sigma_j(B^2) \subset \sigma_{j+1}(B^2) \quad \forall j = 1, 2, \ldots$$

and

$$\bigcup_{j=1}^{\infty} \sigma_j(B^2) = \mathcal{S}.$$

In this sense, it seems sensible to say that $\widehat{\sigma}(B^2)$ is in fact the limit domain of the sequence of domain $\sigma_j(B^2)$.

This simple example already illustrates an important aspect of the scaling technique in complex dimension 2, as well as in higher complex dimensions.

In light of the exposition of the one-dimensional scaling, the following theorem may come now with no surprise.

Theorem 9.2.1 (Wong 1977, Rosay 1979). *Let Ω be a bounded domain in $\mathbb{C}^n$ with a boundary point $p \in \partial\Omega$ satisfying the following:*

(i) *$\partial\Omega$ is C^2 smooth and strongly pseudoconvex near p, and*
(ii) *there exists a sequence $\varphi_j \in \mathrm{Aut}\,\Omega$ and an interior point $q \in \Omega$ such that $\lim_{j\to\infty} \varphi_j(q) = p$.*

Then the domain Ω is biholomorphic to the unit ball in $\mathbb{C}^n$.

We shall present a proof of this result, which illustrates the scaling method in detail, in subsequent sections. First we shall present a detailed exposition starting with the notion of *normal set-convergence.*

9.2.2 Normal Convergence of Sets

We first describe the concept of normal convergence of domains.

Definition 9.2.2. Let Ω_j be domains in $\mathbb{C}^n$ for each $j = 1, 2, \ldots$. The sequence Ω_j is said to *converge normally* to a domain $\widehat{\Omega}$ if the following two conditions hold:

(i) For any compact set K contained in the interior (i.e., the largest open subset) of $\bigcap_{j>m} \Omega_j$ for some positive integer m, $K \subset \widehat{\Omega}$.
(ii) For any compact subset K' of $\widehat{\Omega}$, there exists a constant $m > 0$ such that $K' \subset \bigcap_{j>m} \Omega_j$ for every $j > m$.

This notion of normal convergence is essentially equivalent to the notion of the *Carathéodory kernel convergence* ([Carathéodory 1912]; cf. [Duren 1983], p. 77): for a sequence $\{\Omega_j : j = 1, 2, \ldots\}$ of domains in $\mathbb{C}^n$ with $p_0 \in \bigcap_{j=1}^{\infty} \Omega_j$, the *Carathéodory kernel* is defined to be the largest subdomain containing p_0 of $\bigcap_{j=1}^{\infty} \Omega_j$, when it is nonempty. If p_0 is not an interior point of $\bigcap_{j=1}^{\infty} \Omega_j$, then the *Carathéodory kernel* is defined to be $\{p_0\}$. Then the sequence $\{\Omega_j : j = 1, 2, \ldots\}$ is said to *converge in the sense of Carathéodory kernel convergence* if every subsequence admits the same Carathéodory kernel.

The reason for introducing such notions of convergence of sets is because they are used for the scaling methods and normal families with source and target domains varying.

Proposition 9.2.3. *If Ω_j is a sequence of domains in $\mathbb{C}^n$ that converges normally to the domain $\widehat{\Omega}$, then:*

(1) *If a sequence of holomorphic mappings $f_j : \Omega_j \to \Omega'$ from Ω_j to another domain Ω' converges uniformly on compact subsets of $\widehat{\Omega}$, then its limit is a holomorphic mapping from $\widehat{\Omega}$ into the closure of the domain Ω'.*
(2) *If a sequence of holomorphic mappings $g_j : \Omega' \to \Omega_j$ converges uniformly on compact subsets of G, then its limit is a holomorphic mapping from the domain Ω' into the closure of $\widehat{\Omega}$.*

9.2.3 Localization

Local Holomorphic Peak Functions

Definition 9.2.4. Let Ω be a domain in $\mathbb{C}^n$. A boundary point $p \in \partial\Omega$ is said to admit a *holomorphic peak function* if there exists a continuous function $h : \mathrm{cl}(\Omega) \to \mathbb{C}$ that satisfies the following properties:

(i) h is holomorphic on Ω,
(ii) $h(p) = 1$, and
(iii) $|h(z)| < 1$ for every $z \in \mathrm{cl}(\Omega) \setminus \{p\}$.

Such a function h is called a *holomorphic peak function* for Ω at p.

Furthermore, we say that a boundary point p of Ω admits a *local holomorphic peak function* if there exists an open neighborhood U of p such that there exists a holomorphic peak function for $\Omega \cap U$ at p.

Proposition 9.2.5. *Let Ω be a bounded domain in $\mathbb{C}^n$ with a C^2 smooth, strongly pseudoconvex boundary point p. Let B^n be the unit open ball in $\mathbb{C}^n$. Let η be a positive real number satisfying $0 < \eta < 1$. Then, for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|f(z) - p| < \epsilon, \quad \forall z \text{ with } |z| < \eta,$$

for every holomorphic mapping $f : B^n \to \Omega$ with $|f(0) - p| < \delta$.

Proof. Assume to the contrary that there exist holomorphic mappings $f_j : B^n \to \Omega$ satisfying the following two conditions:

(a) $\lim_{j\to\infty} f_j(0) = p$.
(b) $\exists\epsilon > 0$ for which there exists a sequence $z_j \in B^n$ such that $|z_j| < \eta$ and $|f_j(z_j) - p| \geq \epsilon$ for every $j = 1, 2, \ldots$.

Let U be an open neighborhood of p such that there exists a local holomorphic peak function $h : \mathrm{cl}(\Omega) \cap U \to \mathbb{C}$ at p. [Here we use the fact that a strongly pseudoconvex boundary point always admit a local holomorphic peak function—see [Graham 1975].]

Since Ω is bounded, Montel's theorem yields that f_j admits a subsequence that converges uniformly on compact subsets. By an abuse of notation, we

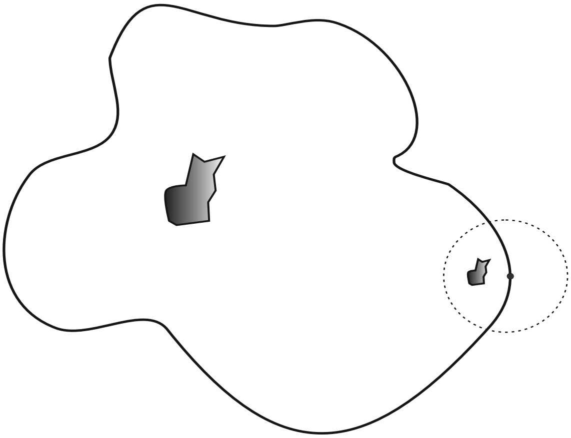

Fig. 9.3. The localization argument.

denote the subsequence by the same notation f_j, and then the subsequential limit mapping by $F : B^n \to \mathrm{cl}(\Omega)$.

Take an open neighborhood V of 0 sufficiently small so that it satisfies the properties:

(1) $\mathrm{cl}(V) \subset B^n$.
(2) There exists $N > 0$ such that $f_j(\mathrm{cl}(V)) \subset U \cap \Omega$ for every $j > N$.

Consider the sequence of mappings $h \circ f_j|_V : V \to D$, where D is the open unit disc in $\mathbb{C}$. Apply Montel's Theorem again to this sequence. Choosing a subsequence from f_j again, we may assume that $h \circ f_j|_V$ converges uniformly on compact subsets of V to a holomorphic map $G : V \to \mathrm{cl}(D)$. Since $G(0) = 1$ and $|G(\zeta)| < 1$ for every $\zeta \in V$, the maximum principle implies that $G(\zeta) \equiv 1$ for every $\zeta \in V$.

By the properties of the local holomorphic peak function h at p, this implies that $F(\zeta) = p$ for every $\zeta \in V$. Since V is open, and since F is holomorphic, it follows that $F(z) = p$ for every $z \in B^n$. Since the convergence of f_j to F is uniform on compact subsets, it is impossible to have z_j with $|z_j| \leq \eta$ such that $f_j(z_j)$ stays away from p for every j. (Refer to Figure 9.3.) This contradiction completes the proof. $\square$

Plurisubharmonic Peak Functions

There is an effective method of localization in a more general setting (Sibony 1981). A main point of this method is that it avoids Montel's theorem altogether. Thus, for instance, the assumption that Ω is bounded is no longer needed.

Definition 9.2.6. Let Ω be a domain in $\mathbb{C}^n$ and let p be a boundary point. If there exists a continuous function $h : \mathrm{cl}(\Omega) \to \mathbb{R}$ satisfying:

(i) h is plurisubharmonic on Ω, and
(ii) $h(p) = 0$ and $h(z) < 0$ for every $z \in \mathrm{cl}(\Omega) \setminus \{0\}$,

then we call h a *plurisubharmonic peak function at p for Ω*. In such a case, p is called a *plurisubharmonic peak point for Ω*.

Likewise, a boundary point p of the domain Ω is called a *local plurisubharmonic peak point* if there exists an open neighborhood of p in $\mathbb{C}^n$ such that p is a plurisubharmonic peak point for $\Omega \cap U$.

We present first the following lower bound estimate for the Kobayashi metric near a local plurisubharmonic peak boundary point.

Proposition 9.2.7 ([Sibony 1981]). *Let Ω be a bounded domain in $\mathbb{C}^n$ with a boundary point $p \in \partial\Omega$ which admits a local plurisubharmonic peak function for Ω. Then, for every open neighborhood U of p in $\mathbb{C}^n$, there exists an open neighborhood V with $p \in V \subset U$ such that we have the inequality*

$$k_\Omega(z,\xi) \geq \frac{1}{2}\, k_{\Omega\cap U}(z,\xi), \quad \forall (z,\xi) \in (\Omega \cap V) \times \mathbb{C}^n,$$

where k_Ω denotes the infinitesimal Kobayashi pseudo-metric of a domain Ω.

Proof. Denote by D_r the open disc in $\mathbb{C}$ of radius r centered at the origin. For the unit open disc, write $D = D_1$.

By the definition of the Kobayashi metric, it suffices to prove the following statement:

(†) *It is possible to choose V so that the following holds: given $(z,\xi) \in (\Omega \cap V) \times \mathbb{C}^n$, every holomorphic mapping $f : D \to \Omega$ from the unit disc D into Ω satisfying $f(0) = z, df\big|_0(\lambda) = \xi$ for some $\lambda > 0$ enjoys the property that $f(D_{1/2}) \subset U$.*

Replacing U by a smaller neighborhood of p if necessary, let $\psi_1 : U \cap \mathrm{cl}(\Omega)$ be a local plurisubharmonic peak function at p. Choose an open neighborhood U_1 of p inside U and a constant $c_1 > 0$ such that

$$\sup\{\psi_1(z) \mid z \in \mathrm{cl}(\Omega) \cap \partial U_1\} = -c_1.$$

Choose a neighborhood V_1 of p inside U_1 such that

$$V_1 = \left\{z \in \Omega \cap U_1 \mid \psi_1(z) > -\frac{c_1}{2}\right\}.$$

Then we can extend ψ_1 to a new function $\psi_2 : \mathrm{cl}(\Omega) \to \mathbb{R}$ by

$$\psi_2(z) = \begin{cases} \psi_1(z) & \text{if } z \in \mathrm{cl}(\Omega) \cap \mathrm{cl}(V_1) \\ \max\{\psi_1(z), -3c_1/2\} & \text{if } z \in \mathrm{cl}(\Omega) \cap (U_1 \setminus \mathrm{cl}(V_1)) \\ -3c_1/2 & \text{if } z \in \mathrm{cl}(\Omega) \setminus U_1. \end{cases}$$

Notice that ψ_2 is a global plurisubharmonic peak function for Ω at p.

Toward the proof of (†), there is no harm in assuming (by a simple dilation) that the analytic disc f is holomorphic in a neighborhood of the closed unit disc $\mathrm{cl}(D)$.

Let $a > 0$ be such that $\psi_2 \circ f(0) > -a$. Consider

$$E_a = \left\{\theta \in [0, 2\pi] \mid \psi_2 \circ f(e^{i\theta}) \geq -2a\right\}.$$

By the sub-mean value inequality, we see that

$$\begin{aligned} -a &< \psi_2 \circ f(0) \\ &\leq \frac{1}{2\pi} \int_{[0,2\pi]} \psi_2 \circ f(e^{i\theta})\, d\theta \\ &\leq \frac{1}{2\pi} \int_{[0,2\pi]\setminus E_a} (-2a)\, d\theta \\ &\leq -\frac{a}{\pi}(2\pi - |E_a|), \end{aligned}$$

where $|E_a|$ denotes the Lebesgue measure of E_a. Hence we see that

$$|E_a| > \pi.$$

Now consider a plurisubharmonic function at p given by

$$v_\epsilon(z) = \epsilon \log \|z - p\|,$$

where ϵ is a certain positive constant to be chosen shortly. This is often called an *anti-peak function* as it satisfies $v_\epsilon(p) = -\infty$.

Let

$$\inf\left\{\psi_1(z) + v_\epsilon(z) \mid z \in \mathrm{cl}(\Omega) \cap \partial V_1\right\} = -c_2,$$

and

$$\sup\left\{\psi_1(z) + v_\epsilon(z) \mid z \in \mathrm{cl}(\Omega) \cap \partial U_1\right\} = -c_3.$$

Choose $\epsilon > 0$ so that

$$-c_3 < -c_2 < 0.$$

Extend $\psi_1 + v_\epsilon$ to the plurisubharmonic function $\Upsilon : \mathrm{cl}(\Omega) \to \mathbb{R}$ defined by

$$\Upsilon(z) = \begin{cases} \psi_1(z) + v_\epsilon(z) & \text{if } z \in \mathrm{cl}(\Omega) \cap \mathrm{cl}(V_1) \\ \max\left\{\psi_1(z) + v_\epsilon(z), -\dfrac{c_2 + c_3}{2}\right\} & \text{if } z \in \mathrm{cl}(\Omega) \cap (\mathrm{cl}(U_1) \setminus \mathrm{cl}(V_1)) \\ -\dfrac{c_2 + c_3}{2} & \text{if } z \in \mathrm{cl}(\Omega) \setminus \mathrm{cl}(U_1). \end{cases}$$

Observe that $\Upsilon^{-1}(-\infty) = \{p\}$.

For each $\zeta \in D_{1/2}$, apply the Poisson integral formula to obtain

$$\Upsilon \circ f(\zeta) \leq \frac{1}{10\pi} \int_0^{2\pi} \Upsilon \circ f(e^{i\theta})\, d\theta.$$

We now focus upon the peak function ψ_2 and the anti-peak function Υ. Since the sets

$$G_k = \{z \in \mathrm{cl}(\Omega) \mid \psi_2(z) \geq -1/k\},$$

for $k = 1, 2, \ldots$, form a neighborhood basis for p in $\mathrm{cl}(\Omega)$, we see that for each $L > 0$ there exists $a > 0$ with a arbitrarily small such that

$$\{z \in \mathrm{cl}(\Omega) \mid \psi_2(z) \geq -2a\} \subset \{z \in \mathrm{cl}(\Omega) \mid \Upsilon(z) < -L\}.$$

Then we present:

Claim. If a holomorphic function $f : cl(D) \to \Omega$ satisfies $\psi_2 \circ f(0) > -a$, then $\Upsilon \circ f(\zeta) \leq -L/10$ for every $\zeta \in D_{1/2}$.

The proof is immediate; simply check for each $\zeta \in D_{1/2}$ that

$$\begin{aligned}
\Upsilon \circ f(\zeta) &\leq \frac{1}{10\pi} \int_0^{2\pi} \Upsilon \circ f(e^{i\theta})\, d\theta \\
&\leq \frac{1}{10\pi} \int_{E_a} (-L)\, d\theta + \frac{1}{10\pi} \int_{[0,2\pi]\setminus E_a} 0\, d\theta \\
&= -\frac{L}{10}.
\end{aligned}$$

Finally we are ready to finish the proof. Observe that the sets

$$U_k = \left\{ z \in \mathrm{cl}(\Omega) \mid \Upsilon(z) < -\frac{k}{10} \right\}$$

for $k = 10, 11, \ldots$ also form a neighborhood basis for p in $\mathrm{cl}(\Omega)$. By the claim above, for each k we may choose $a_k > 0$ such that

(1) $\Upsilon(z) > -k$ whenever $\psi_2(z) > -2a_k$, and
(2) $a_{10} > a_{11} > \cdots \to 0$.

Consequently, if we choose $V_k = \{z \in \mathrm{cl}(\Omega) \mid \psi_2 > -a_k\}$ for each k, then it follows immediately that

$$f(0) \in V_k \Rightarrow f(D_{1/2}) \subset U_k$$

for every $k = 10, 11, \ldots$. This completes the proof. □

Proposition 9.2.8. *Let Ω be a bounded domain in $\mathbb{C}^n$ with a boundary point $p \in \partial\Omega$ which admits a local holomorphic peak function for Ω. Let K be a compact subset of Ω and let $q \in \Omega$. Then, for every open neighborhood U of p in $\mathbb{C}^n$, there exists an open set V with $p \in V \subset U$ such that $f(K) \subset U$ whenever $f : \Omega \to \Omega$ is a holomorphic mapping satisfying $f(q) \in V$.*

Proof. Note first that a local holomorphic peak function h at p generates the local plurisubharmonic peak function $\log |h|$ at p.

Since the Kobayashi pseudodistance $d_M : M \times M \to \mathbb{R}$ is continuous for any complex manifold M, we may select $R > 0$ such that the Kobayashi distance ball

$$B^K_\Omega(q, R) = \{z \in \Omega \mid d_\Omega(z, q) < R\}$$

contains K.

Then use the local holomorphic peak function $h : U \cap \mathrm{cl}(\Omega) \to D$ at p. The distance-decreasing property implies that

$$\lim_{\Omega \cap U \ni p_j \to p} d_{\Omega \cap U}(z, p_j) \geq \lim_{\Omega \ni p_j \to p} d_D(h(z), h(p_j)) = \infty.$$

Moreover, the local holomorphic peak function and the distance-decreasing property guarantee the existence of an open set U' with $p \in U' \subset \mathrm{cl}(U') \subset U$ and an open neighborhood V with $p \in V \Subset U'$ such that

$$d_{\Omega \cap U}(z, w) > 3R$$

for every $z \in V$ and every $w \in \partial U' \cap \Omega$.

Now let $\zeta \in \Omega \setminus U$. Then, by the definition of the Kobayashi metric, there exists a piecewise smooth "almost-the-shortest" connector $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = z, \gamma(1) = \zeta$ induced from the holomorphic chain in the definition of the Kobayashi metric such that

$$L^K_\Omega(\gamma) - \frac{R}{2} < d_\Omega(z, \zeta) < L^K_\Omega(\gamma),$$

where $L^K_\Omega(\gamma)$ denotes the length of γ measured by the Kobayashi metric of Ω. Since $\gamma([0, 1])$ has to cross $\partial U' \cap \Omega$, we let $t \in (0, 1)$ such that $\gamma([0, t)) \subset U' \cap \Omega$ and $\gamma(t) \in \partial U' \cap \Omega$. Then it follows that

$$L^K_\Omega(\gamma) > L^K_\Omega(\gamma|_{[0,t]}) > \frac{1}{2} L^K_{\Omega \cap U}(\gamma|_{[0,t]}) > \frac{1}{2} d_{\Omega \cap U}(z, \gamma(t)) > \frac{3}{2} R.$$

This therefore implies that

$$d_\Omega(z, \zeta) > \frac{3}{2} R - \frac{R}{2} = R.$$

In particular, $B^K_\Omega(z, R) \subset U$ whenever $z \in V$.

Since f given in the hypothesis is a holomorphic mapping, the distance-decreasing property of the Kobayashi distance yields that

$$f(K) \subset f(B^K_\Omega(q, R)) \subset B^K_\Omega(f(q), R) \subset U.$$

Since $f(q) \in V$, we see that $B^K_\Omega(f(q), R) \subset U$. This is what we wanted to establish. $\square$

Remark 9.2.9. This argument avoiding Montel's theorem is also useful in infinite dimensions.

9.2.4 Application: the Wong–Rosay Theorem

Here we state the most general version of the Wong–Rosay Theorem in $\mathbb{C}^n$, which is due to Efimov ([Efimov 1995]).

Theorem 9.2.10. *Let Ω be a domain in $\mathbb{C}^n$. If Ω has a C^2 strongly pseudoconvex boundary point p for which there exist a sequence f_j of automorphisms of Ω and a point $q \in \Omega$ such that $\lim_{j\to\infty} f_j(q) = p$, then Ω is biholomorphic to the unit open ball in $\mathbb{C}^n$.*

Notice that, unlike the original Wong–Rosay theorem (Theorem 9.2.1), there are no restrictions on Ω in this version; Ω is assumed neither bounded nor Kobayashi hyperbolic.

Proof. By what is called Narasimhan's lemma, there exists an open neighborhood U of p and a biholomorphic mapping $G : U \to B^n(0, r)$ from U onto the open ball $B^n(0, r)$ in $\mathbb{C}^n$ of radius r centered at the origin such that

$$\mathcal{O}_p = \mathbf{1} \equiv (1, 0, \ldots, 0)$$

and

$$G(\Omega \cap U) = \left\{z \in B^n(\mathbf{0}, r) \mid \|z\|^2 + E(z) < 1\right\}$$

where

$$E(z) = o(\|z\|^2).$$

Let ϵ be an arbitrarily given positive number. Then we may choose a smaller value for r such that $G(\Omega \cap U)$ is contained in the set

$$\left\{z \in \mathbb{C}^n \mid |z_1 + \epsilon|^2 + |z_2|^2 + \cdots + |z_n|^2 < (1 + \epsilon)^2\right\}.$$

Hence, if we apply the translation $\tau(z_1, \ldots, z_n) = (z_1 - 1, z_2, \ldots, z_n)$, then

$$\tau \circ G(\Omega \cap U) = \left\{z \in B^n(0, r) \mid 2 \operatorname{Re} z_1 < -\|z\|^2 + \widetilde{E}(z)\right\},$$

where $\widetilde{E}(z) = o(\|z\|^2)$ as $z \to 0$.

Define the domain

$$\mathcal{E}_\epsilon = \{z \in \mathbb{C}^n \mid 2(1 + \epsilon) \operatorname{Re} z_1 < |z_2|^2 + \ldots + |z_n|^2\}$$

and write

$$\Omega_U \equiv \tau \circ G(\Omega \cap U).$$

Then $\Omega_U \subset \mathcal{E}_\epsilon$.

Let K be an arbitrary compact subdomain of Ω. Since $f_j(q)$ converges to p, we may choose, by virtue of the localization methods of Section 9.2.3, a neighborhood V of p in $\mathbb{C}^n$ such that

(a) $p \in V \subset U$,

and

(b) $f_j(K) \subset U$ whenever $f_j(q) \in V$.

We now introduce the appropriate scaling method. Write $\widetilde{q}_j \equiv \tau \circ G \circ f_j(q)$. Notice that $\widetilde{q}_j$ approaches 0 as $j \to \infty$. Choose $\widetilde{p}_j \in \partial\Omega_U$ in such a way that it satisfies:

$$-\widetilde{q}_j + \widetilde{p}_j = (\lambda_j, 0, \ldots, 0).$$

Note that $\lambda_j > 0$ for every j.

Taking a subsequence, one can arrange that $\widetilde{p}_j \in \tau \circ G(\partial\Omega \cap U)$ for every j. Write $\widetilde{p}_j = (\widetilde{p}_{j,1}, \ldots, \widetilde{p}_{j,n})$ and $\widetilde{q}_j = (\widetilde{q}_{j,1}, \ldots, \widetilde{q}_{j,n})$ in components. Then $\widetilde{p}_{j,\ell} = \widetilde{q}_{j,\ell}$ for $\ell = 2, \ldots, n$.

Now we perform the process called "centering" of the orbit. (Refer to Figure 9.4.)

For $\theta_j \in \mathbb{R}$ and $\alpha_{j,k} \in \mathbb{C}$ $(k = 2, \ldots, n)$, define the complex affine transformation $\Psi_j : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Psi_j : \begin{cases} \zeta_1 = e^{i\theta_j}(z_1 - \widetilde{p}_{j,1}) + \alpha_{j,2}(z_2 - \widetilde{p}_{j,2}) + \cdots + \alpha_{j,n}(z_n - \widetilde{p}_{j,n}) \\ \zeta_2 = z_2 - \widetilde{p}_{j,2} \\ \vdots \\ \zeta_n = z_n - \widetilde{p}_{j,n} \end{cases} .$$

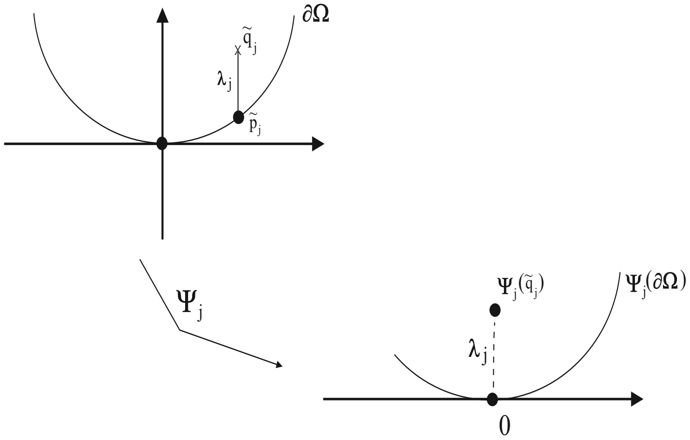

Fig. 9.4. The concept of centering.

It is possible to select the constants $\theta_j \in \mathbb{R}$ and $\alpha_{j,k} \in \mathbb{C}$ ($k = 2, \ldots, n$, $j = 1, 2, \ldots$) so that

(c) Ψ_j converges to the identity uniformly on compact subsets of $\mathbb{C}^n$,

and

(d) $\Psi_j(\Omega_U)$ is specified by the defining inequality

$$\operatorname{Re} \zeta_1 < -Q_j(\zeta_2, \ldots, \zeta_n) + \widehat{E}(\zeta),$$

where Q_j is a positive, quadratic, real analytic polynomial that converges to $|\zeta_2|^2 + \cdots + |\zeta_n|^2$, and where $\widehat{E}(\zeta) = o(|\zeta_1| + |\zeta_2|^2 + \cdots + |\zeta_n|^2)$ for $\zeta \sim 0$.

Therefore, selecting a smaller value again for $r > 0$ if necessary, we may actually find a positive constant $N > 0$ such that

$$\Psi_j(\Omega_U) \subset \mathcal{E}_\epsilon$$

for every $j > N$. Then apply the stretching map

$$L_j(z) = \left(\frac{z_1}{\lambda_j}, \frac{z_2}{\sqrt{\lambda_j}}, \ldots, \frac{z_n}{\sqrt{\lambda_j}} \right)$$

to arrive at the scaling sequence

$$\sigma_j \equiv L_j \circ \Psi_j \circ \tau \circ G \circ f_j,$$

which maps K injectively into $\mathbb{C}^n$.

Observe that $\sigma_j(K) \subset L_j(\mathcal{E}_\epsilon)$ according to our construction. On the other hand, $L_j(\mathcal{E}_\epsilon) = \mathcal{E}_\epsilon$ for every j. Since $\mathcal{E}_\epsilon$ is biholomorphic to the ball by an explicit linear fractional transformation, it follows that $\sigma_j|_K : K \to \mathcal{E}_\epsilon$ is a normal family. Furthermore, notice that

$$\sigma_j(q) = -\mathbf{1} \in \mathcal{E}_\epsilon.$$

Since $\mathcal{E}_\epsilon$ is pseudoconvex, it follows that σ_j admits a subsequence that converges uniformly on K to a holomorphic mapping $\widehat{\sigma}$ that maps K into $\mathcal{E}_\epsilon$. By an abuse of notation, denote this convergent subsequence by the same notation σ_j.

Since K is an arbitrarily chosen compact subdomain of Ω, we may select a subsequence again to conclude that the sequence σ_j converges uniformly on compact subsets to the holomorphic mapping

$$\widehat{\sigma} : \Omega \to \mathcal{E}_\epsilon.$$

Since $\epsilon > 0$ is also arbitrary, we see that $\widehat{\sigma}(\Omega) \subset \mathrm{cl}(\mathcal{E}_0)$, where

$$\mathcal{E}_0 = \left\{ z \in \mathbb{C}^n \mid 2 \operatorname{Re} z_1 < -(|z_2|^2 + \cdots + |z_n|^2) \right\}.$$

Since $\widehat{\sigma}(q) = -\mathbf{1} \in \mathcal{E}_0$ and since $\mathcal{E}_0$ is pseudoconvex, it follows that $\widehat{\sigma}(\Omega) \subset \mathcal{E}_0$.

The final step is to show that the mapping $\widehat{\sigma} : \Omega \to \mathcal{E}_0$ is a biholomorphism.

For this, let $\widetilde{K}$ be an arbitrarily given compact subdomain of $\mathcal{E}_0$. Then we may choose a subsequence of σ_j, which we denote again by the same notation, such that

$$L_j^{-1}(\widetilde{K}) \subset \Psi_j \circ \tau \circ G(\Omega_U).$$

This is possible because $\widetilde{K}$ is a subset of

$$\mathcal{E}_{-\delta} = \left\{ z \in \mathbb{C}^n \mid 2(1-\delta) \operatorname{Re} z_1 < -(|z_2|^2 + \cdots + |z_n|^2) \right\}$$

for some sufficiently small $\delta > 0$. Thus, extracting a subsequence again if necessary, we are given the sequence

$$\sigma_j^{-1}|_{\widetilde{K}} : \widetilde{K} \to \Omega$$

of well-defined holomorphic mappings. At this stage, we need

Claim. The sequence $\sigma_j^{-1} : \widetilde{K} \to \Omega$ admits a subsequence that converges uniformly to a mapping from $\widetilde{K}$ into Ω.

We will present the proof assuming that this claim is true. By a similar convergence argument to that above, we arrive at the limit holomorphic mapping $\widetilde{\omega} : \mathcal{E}_0 \to \Omega$, with $\widetilde{\omega}(-\mathbf{1}) = q$.

By the Cauchy estimates, both the sequences $d\sigma_j(q)$ and $d\sigma_j^{-1}(-\mathbf{1})$, which are inverse to each other, converge. They must converge to nonsingular matrices. Indeed, we arrive at the conclusion that $d\widehat{\sigma}(q)$ and $d\widetilde{\omega}(-\mathbf{1})$ are inverse to each other. By the claim above, it follows from Cartan's uniqueness theorem (Theorem 1.3.1) that $\widehat{\sigma} \circ \widetilde{\omega} = \mathrm{id}_{\mathcal{E}_0}$ and $\widetilde{\omega} \circ \widehat{\sigma} = \mathrm{id}_{\Omega}$. This shows that $\widehat{\sigma} : \Omega \to \mathcal{E}_0$ is a biholomorphism. Since $\mathcal{E}_0$ is biholomorphic to the unit open ball in $\mathbb{C}^n$, the desired conclusion follows.

To complete the proof, we need only prove the claim. Denote by

$$\mathrm{cl}(B^K(q, R)) = \{ z \in \Omega \mid d_\Omega(q, z) \leq R \}$$

the closed Kobayashi distance ball of radius R centered at q. It suffices to show that this is a compact set for every positive number R. Let U be the open neighborhood of p in $\mathbb{C}^n$ chosen above. On the other hand, recall that the only condition imposed on U by far was that $U \cap \Omega$ was convexifiable. So it is obvious that we may require one more condition, using the argument in the proof of Proposition 9.2.8, that U satisfies that

$$\mathrm{cl}(B^K(q, R)) \subset \Omega \cap U,$$

and

$$d_\Omega(q, z) > R$$

for every $z \in \mathrm{cl}(\partial U \cap \Omega)$. Therefore it follows that $\mathrm{cl}(B^K(f_j(q), R)) \cap \mathrm{cl}(\partial U \cap \Omega) = \emptyset$. At the same time, $\mathrm{cl}(\partial U \cap \Omega) \cap U \cap \partial\Omega = \emptyset$ since every point in $\partial\Omega \cap U$ admits a local holomorphic peak function for $\Omega \cap U$. So $\mathrm{cl}(B^K(f_j(q), R))$ is compact. Since $\mathrm{cl}(B^K(q, R)) = f_j^{-1}(\mathrm{cl}(B^K(f_j(q), R)))$, the closed Kobayashi distance ball $\mathrm{cl}(B^K(q, R))$ is compact. The proof is now complete. $\square$

Corollary 9.2.11. *Let Ω in $\mathbb{C}^n$ be a domain with a C^2 strongly pseudoconvex boundary point. If Ω is not biholomorphic to the unit open ball in $\mathbb{C}^n$, then it cannot be a holomorphic covering space of a compact variety.*

9.2.5 What Is the Scaling Method?

The scaling method introduced in the preceding sections can be summarized roughly as follows.

Given a domain Ω in $\mathbb{C}^n$ with an interior point q and a sequence f_j of automorphisms of Ω such that $\lim_{j\to\infty} f_j(q) = p$, for some boundary point $p \in \partial\Omega$, one follows the steps below.

Step 1. *Localization:* Translate Ω if necessary so that p becomes the origin. Establish that, for any compact subset K of Ω, the sequence of sets $f_j(K)$ shrinks successively to the boundary point $p = 0$.

Step 2. *Centering:* Adjust the domain Ω by a sequence of complex affine maps, say Ψ_j, so that the new point sequence $\Psi_j(q_j)$ behaves as if it converges nontangentially to the boundary of the limit domain.

Step 3. *Stretching:* Find a divergent sequence of complex linear maps, say L_j, so that $\sigma_j = L_j \circ \Psi_j \circ f_j$ converges uniformly on compact subsets of Ω into $\mathbb{C}^n$.

Step 4. *Analysis of the Limit Domain:* Since $f_j(\Omega) = \Omega$, it follows that $\sigma_j(\Omega) = L_j \circ \Psi_j(\Omega)$ for every j. Since the maps L_j and Ψ_j are often explicit, take the limit domain $\widehat{\Omega}$ of $\sigma_j(\Omega)$ in the sense of normal convergence of domains.

Step 5. *Synthesis:* In case all the pieces are put together, it usually follows that σ_j converges to a map $\widehat{\sigma}$ that turns into a biholomorphic mapping from Ω onto $\widehat{\Omega}$.

The localization followed by centering and stretching constitutes the scaling method. The main thrust is that the limit domain becomes much simpler. For example, the Siegel upper half-space is biholomorphic to the ball in the case when p is a C^2 strongly pseudoconvex boundary point.

9.2.6 Another Illustration for Scaling

The centering procedure was somewhat mysterious even to the experts when the scaling method was first developed. We give another illustration through the following real two-dimensional example. We are indebted to Eric Bedford for this illustration.

An Illustration. In $\mathbb{R}^2$, consider the domain

$$\Omega = \big\{(x, y) \in \mathbb{R}^2 \mid y > \rho(x)\big\}$$

with $\rho(0) = 0$, $\rho'(0) = 0$. Keeping the Wong–Rosay theorem (Theorem 9.2.1) in mind, we take a specific function $\rho(x) = x^2 + x^4$.

Let $q_j = (a_j, b_j)$ be a sequence in Ω that converges to the origin $(0, 0)$. Then the centering (originally, complex affine) mapping is constructed as follows: freeze the index j momentarily and take $p_j = (a_j, \rho(a_j))$. Then it follows that

$$p_j \in \partial\Omega \quad \text{and} \quad q_j - p_j = (0, \lambda_j)$$

where $\lambda_j = b_j - \rho(a_j) > 0$. Thus the centering map $(u, v) = \Psi_j(x, y)$ should be given by

$$\begin{cases} u = x - a_j \\ v = y - \rho(a_j) - \rho'(a_j) \cdot (x - a_j). \end{cases}$$

Then the stretching map in this case will be

$$L_j(x, y) = \left(\frac{x}{\sqrt{\lambda_j}}, \frac{y}{\lambda_j}\right).$$

We now compute $L_j \circ \Psi_j(\Omega)$. First, the domain $\Psi_j(\Omega)$ is represented in the (u, v)-coordinates by the inequality

$$v + \rho(a_j) + \rho'(a_j)u > \rho(u + a_j).$$

Rewriting this we obtain

$$v > \rho(u + a_j) - \rho(a_j) - \rho'(a_j)u.$$

Thus the domain $L_j \circ \Psi_j(\Omega)$ is represented by

$$\lambda_j v > \rho(\sqrt{\lambda_j}\ u + a_j) - \rho(a_j) - \rho'(a_j)\sqrt{\lambda_j}\ u.$$

Notice that the right-hand side of this inequality has neither constant term nor linear term in the u-variable since the situation is the same with the right-hand side of the preceding inequality. For the current case of explicit $\rho(x) = x^2 + x^4$, one can actually compute the whole thing as follows.

$$\begin{aligned} v &> \rho(u + a_j) - \rho(a_j) - \rho'(a_j)u \\ &> (u + a_j)^2 + (u + a_j)^4 - {a_j}^2 - {a_j}^4 - 2a_j u - 4{a_j}^3 u \\ &> u^2 + u^4 + 4a_j u^3 + 6a_j^2 u^2. \end{aligned}$$

Thus $L_j \circ \Psi_j(\Omega)$ is represented by

$$\begin{aligned}\lambda_j v &> \rho(\sqrt{\lambda_j}\ u + a_j) - \rho(a_j) - \rho'(a_j)\sqrt{\lambda_j}\ u \\ &> (\sqrt{\lambda_j}\ u)^2 + (\sqrt{\lambda_j}\ u)^4 + 4a_j(\sqrt{\lambda_j}\ u)^3 + 6a_j^2(\sqrt{\lambda_j}\ u)^2.\end{aligned}$$

first,

$$v > u^2 + \lambda_j u^4 + 4a_j\sqrt{\lambda_j}\ u^3 + 6a_j^2\lambda_j u^2.$$

Since a_j, b_j, and λ_j converge to 0 as j tends to ∞, it is evident that the limit domain in the sense of local normal convergence should be represented by

$$v > u^2.$$

This set-convergence is analogous to what we observed in the proof of the Wong–Rosay theorem (Theorem 9.2.1). This is the end of the illustration.

Remark 9.2.12 (On the "centering" maps). We would like to discuss the role of Ψ_j: Would it not be better if we apply L_j alone, with appropriate choices for the stretching factors? Surely $L_j(\Omega)$ converges locally normally to the same limit, and it is even much easier.

Such queries and the answers are important for the scaling method. While it is true that $L_j(\Omega)$ converges to the domain represented by $v > u^2$, the limit $\lim_{j\to\infty} L_j(q_j)$ can be in the boundary of the limit domain $\widehat{\Omega} = \{(u,v) \mid v > u^2\}$, in case q_j approaches the origin very tangentially to the boundary. For example, consider the case

$$q_j = \left(\frac{1}{j},\ \frac{1}{j^2} + \frac{2}{j^4}\right).$$

This is clearly a sequence in Ω approaching the origin. With $L_j(x,y) = (\sqrt{\lambda_j}x, \lambda_j y)$, one observes that

$$L_j(q_j) = \left(\frac{\sqrt{\lambda_j}}{j}, \frac{\lambda_j}{j^2} + \frac{2\lambda_j}{j^4}\right).$$

Notice that, in order for $L_j(q_j)$ to admit a bounded limit, one has no other choice but to implement the condition that both sequences $\frac{\sqrt{\lambda_j}}{j}$ and $\frac{\lambda_j}{j^2}$ admit convergent sequences with bounded limits. Then it is clear that the limit is always of the form (a, a^2). Thus, $L_j(q_j)$ can only converge to a boundary point of $\widehat{\Omega}$.

This is not desirable for the scaling method, as the limit map from the complex scaling in the preceding proof of the Wong–Rosay theorem (Theorem 9.2.1) then has to be degenerate.

On the other hand, with Ψ_j, one observes immediately that $L_j \circ \Psi_j(q_j) = L_j(0, \lambda_j) = (0,1)$, which is an interior point of $\widehat{\Omega}$. This simple adjustment provides a base for the proof of the biholomorphicity of the limit map $\widehat{\sigma}$ in the proof of the Wong–Rosay theorem.

Remark 9.2.13. As one may have already observed, this centering map Ψ_j is not needed when the point sequence q_j approaches the boundary point nontangentially to the boundary $\partial\Omega$. However, in case the point sequence approaches the boundary tangentially, then the centering procedure is quite essential. Roughly speaking, the affine adjustment Ψ_j *centers* the orbit q_j to Ψ_j, even though in this case the domain Ω also transforms to $\Psi_j(\Omega)$. This adjustment is effective because the effect of Ψ_j on the point sequence q_j is much more decisive than its effect upon the domain Ω. The reader shall see this observation returning whenever the scaling method is exploited.

9.3 A Theorem of Bedford and Pinchuk

The next result takes the significant step of being able to go beyond strong pseudoconvexity.

Theorem 9.3.1 ([Bedford/Pinchuk 1989]). *Let Ω be a bounded domain with real analytic (C^ω) pseudoconvex boundary in $\mathbb{C}^2$. If its automorphism group is noncompact, then there exists a positive integer m such that Ω is biholomorphic to the domain*

$$E_m = \{(z, w) \in \mathbb{C}^2 \colon |z|^{2m} + |w|^2 < 1\}.$$

The rest of the section is devoted to presenting a sketch of the proof, broken up into four subsections below.

This section is entirely based upon [Bedford/Pinchuk 1989], but we also use some observations and results from [Berteloot 1994] and [Berteloot/Cœuré 1991].

9.3.1 Initial Scaling

The first step of the proof is an application of the scaling method. Since the automorphism group of Ω is noncompact and since Ω is bounded, there exist a boundary point $p \in \partial\Omega$ and an automorphism orbit $\{\varphi_\nu(q)\}$, where $q \in \Omega$ and $\varphi_\nu \in \mathrm{Aut}\ \Omega$ for every $\nu = 1, 2, \ldots$, with $\varphi_\nu(q) \to p$.

After a coordinate change, we may take p to be the origin. Since the domain Ω is bounded, and since its boundary is real analytic, then another holomorphic change (call it Γ for instance) of local coordinates at p gives that the defining function of the domain in a neighborhood, say U, of p can be rewritten as follows.

$$\mathrm{Re}\ z_1 + P_{2m}(z_2) + \mathcal{R}(\mathrm{Im}\ z_1, z_2) = 0,$$

where P_{2m} is a homogeneous subharmonic, but not harmonic, polynomial of degree ℓ in $z_2, \bar{z}_2$ without harmonic terms, and where $\mathcal{R}(x) = o(\|x\|^{2m})$. (See [Catlin 1989].)

Write $z_\nu = \varphi_\nu(q)$. Fix ν momentarily. Choose $\epsilon_\nu > 0$ so that $\epsilon_\nu > 0$ is the smallest positive number such that $z_\nu + (\epsilon_\nu, 0)$ is a boundary point, call it ζ_ν, of Ω. This boundary point will be unique. Then consider the "centered" equation given by the affine transformation Ψ_ν that satisfies:

(i) $\Psi_\nu(\zeta_\nu) = (0,0)$,
(ii) $\Psi_\nu(z_\nu) = (-\epsilon_\nu, 0)$,
(iii) $\Psi_\nu(\partial\Omega)$ near $(0,0)$ is tangent to the real hyperplane defined by Re $z_1 = 0$.

Then the local defining function near the origin of $\Psi_\nu \circ G(\Omega \cap U)$ is given by

$$\mathrm{Re}\ z_1 + \sum_{\ell=2}^{2m} P_{\ell,(\nu)}(z_2) + \mathcal{R}_\nu(\mathrm{Im}\ z_1, z_2) < 0,$$

where $\mathcal{R}_\nu = o(|z_1| + |z_2|^{2m})$. Here the $P_{\ell,(\nu)}$ are homogeneous polynomials of degree ℓ without harmonic terms. Then choose $\tau_\nu > 0$ by requiring that

$$\max \frac{1}{\epsilon_\nu} \cdot \Big|\text{coefficient of each monomial term of } P_{\ell,(\nu)}(\tau_\nu z_2)\Big| = 1,$$

where the maximum is taken over all monomial terms of $\sum_{\ell=2}^{2m} P_{\ell,(\nu)}$.

Now, with τ_ν to be selected (for each ν), we shall use the stretching map of Pinchuk given by

$$\Lambda_\nu(z_1, z_2) = \left(\frac{z_1}{\epsilon_\nu}, \frac{z_2}{\tau_\nu}\right).$$

We shall not include all the details, but it is important to notice that the sequence of holomorphic embedding mappings

$$\sigma_\nu := \Lambda_\nu \circ \Psi_\nu \circ \Gamma \circ \varphi_\nu$$

gives rise to a subsequential limit map, say $\widehat{\sigma}$, defined on Ω and mapping Ω into $\mathbb{C}^2$ so that the following holds:

- $\widehat{\sigma} : \Omega \to \widehat{\sigma}(\Omega)$ is a biholomorphic mapping,

  and

- $\widehat{\sigma}(\Omega) = \{(z_1, z_2) \in \mathbb{C}^2 \mid \mathrm{Re}\ z_1 + \widehat{P}(z_2) < 0\}$, where $P(z_2)$ is a real-valued subharmonic polynomial of degree $\leq 2m$.

In any event, the conclusion of this particular step is that Aut $(\widehat{\Omega})$ contains the obvious 1-parameter family

$$\gamma_t := (z_1, z_2) \mapsto (z_1 + it, z_2)$$

of automorphisms parameterized by $t \in \mathbb{R}$. Consequently, Aut (Ω) admits the 1-parameter family

$$\Upsilon := \{\widehat{\sigma}^{-1} \circ \gamma_t \circ \widehat{\sigma} \mid t \in \mathbb{R}\}.$$

The details pertaining to this part can be found in pp. 142–144 of [Bedford/Pinchuk 1989]. But a cleaner and better exposition on this part of scaling and other arguments can be found in pp. 620–624 of [Berteloot 1994]. [This has all so far been strongly analogous to the scaling method that we demonstrated in Section 9.2.4 to prove the Wong–Rosay theorem (Theorem 9.2.1). But the scalings in slightly different context show subtleties, which are often significant and in fact essential.]

9.3.2 Parabolic Flow

The next step is to analyze the 1-parameter family Υ acting on $\mathrm{Aut}\,(\Omega)$.

Proposition 9.3.2 (Proposition 2.4, p. 145, [Bedford/Pinchuk 1989]). *Let $h^t := \widehat{\sigma}^{-1} \circ \gamma_t \circ \widehat{\sigma}$ for every $t \in \mathbb{R}$. Then*

$$\lim_{t \to -\infty} h^t(z_1, z_2) = \lim_{t \to \infty} h^t(z_1, z_2) = p'$$

for some $p' \in \partial\Omega$.

The readers may consult the statement and the proof of Lemma 4.3, p. 629 of [Berteloot 1994].

At this juncture, the following reformulation of a theorem of Bochner and Montgomery plays a crucial role (cf. Theorem 1.3.11.):

Theorem 9.3.3 (Bochner–Montgomery). *Let G and $\widetilde{G}$ be open subsets of $\mathbb{R}^N$. Let $\{F_t \mid -1 < t < 1\}$ be a family of mappings defined on G such that*

(1) *$F_t : G \to F_t(G) \subset \widetilde{G}$ is a $\mathcal{C}^\infty$ diffeomorphism for every t with $|t| < 1$,*
(2) *$(-1, 1) \times G \ni (t, x) \mapsto F_t(x)$ is continuous,*
(3) *$F_0(x) = x$ for every $x \in G$,*
(4) *$F_{t+s} = F_t \circ F_s$ whenever $|t| + |s|$ is small enough.*

Then there exists $r > 0$ such that the correspondence

$$(-r, r) \times G \ni (t, x) \mapsto F_t(x)$$

is C^∞ smooth.

Thanks to the well-known reflection principle for holomorphic automorphisms of a bounded pseudoconvex domain with real analytic boundary ([Diederich/Fornæss 1978], [Bedford 1985], [Bell/Ligocka 1980]), every automorphism of Ω extends to a holomorphic mapping on an open neighborhood of the closure $\mathrm{cl}(\Omega)$. Hence we may deduce now that the mapping

$$(t; z_1, z_2) \mapsto h^t(z_1, z_2)$$

is a smooth map from $(-1, 1) \times \mathrm{cl}(\Omega)$ to $\mathrm{cl}(\Omega)$ and is holomorphic in (z_1, z_2) on an open neighborhood of $\mathrm{cl}(\Omega)$.

Therefore it makes sense to consider the vector field X on $\mathrm{cl}(\Omega)$ defined by the equation

$$X\psi(z_1, z_2) = \frac{d}{dt}\Big|_{t=0} \psi \circ h^t(z_1, z_2)$$

that should hold for any smooth function $\psi : \mathrm{cl}(\Omega) \to \mathbb{R}$. One notices now that $X = \mathrm{Re}\ H$, where

$$H = \sum_{j=1}^{2} a_j(z_1, z_2) \frac{\partial}{\partial z_j}$$

and where each a_j is holomorphic in both variables z_1 and z_2.

9.3.3 Analysis with a Parabolic Vector Field

Let ρ denote a real analytic defining function of Ω. Notice that the flow of the vector field X constructed above is parabolic in the following sense.

(1) $X\rho(z_1, z_2) = 0$ whenever $z = (z_1, z_2) \in \partial\Omega$. first, X is tangent to the boundary of Ω.
(2) $X = \mathrm{Re}\ H$, where H is a holomorphic vector field defined on an open neighborhood containing $\mathrm{cl}(\Omega)$.
(3) $X(p') = 0$, where $p' = \lim_{t\to-\infty} h^t(z_1, z_2) = \lim_{t\to\infty} h^t(z_1, z_2)$.
(4) The action of X is parabolic, in the sense that it is neither expanding nor attracting at p'.

Then, with elementary but quite involved and clever computations, Bedford and Pinchuk proved the following:

Lemma 9.3.4 (Corollary 3.5, p. 147 [Bedford/Pinchuk 1989]). *If one expands the defining function ρ at p', up to a holomorphic change of coordinates at p', it follows that*

$$\rho(z, \bar{z}) = 2\ Re\ z_2 + |z|^{2m} + o(|z_1|^{2m}) + Im\ z_2 b(z_1, \bar{z}_1 Im\ z_2),$$

where:

(1) *m is a positive integer*

and

(2) *b is a real analytic function.*

9.3.4 Final Scaling and End of Proof

Notice that the 1-parameter family h^t generates a noncompact automorphism orbit accumulating at p'.

At this point, one can apply the scaling sequence in Section 9.3.1. Following the arguments given by [Berteloot 1994], pp. 620–627, one deduces that Ω is biholomorphic to the domain in $\mathbb{C}^2$ defined by the inequality

$$\mathrm{Re}\ z_2 + |z_1|^{2m} < 0.$$

Via a Cayley-type transformation, it is known that this domain is biholomorphic to the domain defined by

$$|z_1|^{2m} + |z_2|^2 < 1$$

as desired. This ends our sketch of the proof of Theorem 9.3.1.

9.4 Analytic Polyhedra with Noncompact Automorphism Group

So far we have considered domains in which the automorphism group admits boundary orbit accumulation points that are not Levi flat. But one of the most standard examples, the polydisc in $\mathbb{C}^n$, has the property that every boundary point is an orbit accumulation point. Now every boundary point is either singular or is Levi flat. Therefore there should be a result about domains admitting a Levi flat boundary orbit accumulation point. This section presents the characterization of bounded domains that possess piecewise smooth Levi flat boundary and noncompact automorphism group. This is a result of Kim, Krantz, and Spiro. For the details, see [Kim/Krantz/Spiro 2005] and references therein.

9.4.1 Analytic Polyhedra

The typical domains whose boundary geometry is modeled after the polydisc are the objects called analytic polyhedra. They constitute an important class of regions to study in the complex analysis and geometry of several variables and are defined as follows.

Definition 9.4.1. A bounded domain P in $\mathbb{C}^n$ is said to be an *analytic polyhedron* if there is an open neighborhood U of the closure $\mathrm{cl}(P)$ of P and finitely many holomorphic functions

$$f_1, \ldots, f_s : U \to \mathbb{C}$$

such that $P = \{z \in U : |f_1(z)| < 1, \ldots, |f_s(z)| < 1\}$.

Notice that the boundary of such a domain is, in general, singular. Moreover the boundary is Levi flat at every smooth point. Needless to say, the n-dimensional polydisc Δ^n is a prime example of an analytic polyhedron.[1]

[1] From here on, we will use the notation $\Delta = \{z \in \mathbb{C}\colon |z| < 1\}$, and $\Delta^n = \Delta \times \cdots \times \Delta$ (n times).

9.4.2 Characteristic Decomposition

Since the analytic polyhedron is not in general defined by a single equation, there is always an unpleasant possibility that the list of inequalities in its definition may include unnecessary and inessential functions. In other words, there could be redundancies. An efficient way of handling this problem is known, by way of the concept of the characteristic foliations.

For each function f_α in the definition of analytic polyhedron P above, consider

$$\mathcal{L}_\alpha(t) := \{z \in U : f_\alpha(z) = t\} \cap P, \quad t \in \mathbb{C}.$$

Denote by ω_α the set whose elements are nonempty, connected components of $\mathcal{L}_\alpha(t)$ for some $t \in \mathbb{C}$. Call this set the *characteristic decomposition* of P by f_α. The same characteristic decomposition may be shared by several functions in the definition of the analytic polyhedron P.

Let $T = \{f_1, \ldots, f_s\}$ be the tuple of functions given above in the definition of P. For a characteristic decomposition ω, let T_ω denote the collection of functions in T whose characteristic decomposition coincides with ω. It is known that, given an analytic polyhedron, the characteristic decomposition does not depend on the choice of the set of defining functions.

Hence, even though we do not discuss the matter in detail, it is possible to restrict oneself to a reasonable subclass of analytic polyhedron if there is a need for choosing a minimal set of defining functions for analytic polyhedra. See for instance [Fridman 1979].

9.4.3 Generic Analytic Polyhedra with Noncompact Automorphism Group

Our goal is to identify and classify the analytic polyhedra with noncompact automorphism group. But that situation in full generality is yet to be fully understood. A reasonable class of analytic polyhedra may be the *generic* polyhedra, defined as follows.

Definition 9.4.2. Let P be an analytic polyhedron in $\mathbb{C}^n$ defined by the inequalities

$$|f_1| < 1, \ldots, |f_s| < 1.$$

Call P *generic* if the defining system $\{f_1, \ldots, f_s\}$ of holomorphic functions defined on an open neighborhood of the closure of P satisfies the following condition:

(∗) The gradient vectors $\nabla f_{i_1}(p), \ldots, \nabla f_{i_k}(p)$ are linearly independent over $\mathbb{C}$ for every $p \in \partial P$ satisfying $|f_{i_1}(p)| = \cdots = |f_{i_k}(p)| = 1$ whenever $\{i_1, \ldots, i_k\} \subset \{1, \ldots, s\}$ is a set of unrepeated indices.

Then we have the following theorem in dimension 2:

Theorem 9.4.3 ([Kim/Krantz/Spiro 2005]). *Let P be a generic analytic polyhedron in $\mathbb{C}^2$ with noncompact automorphism group. Then:*

(i) *If an automorphism orbit accumulates at a singular boundary point, then P is biholomorphic to the bidisc.*
(ii) *If an automorphism orbit accumulates at a smooth boundary point, then P is biholomorphic to the product of the unit open disc and the maximal analytic variety in the boundary ∂P passing through the orbit accumulating boundary point.*

For the case of n dimensions, $n > 2$, we do have a result but only for convex domains.

Theorem 9.4.4 ([Kim K.-T. 1992]). *Let P be a convex generic analytic polyhdron in $\mathbb{C}^n$ admitting a noncompact automorphism group. Then there exists a positive integer k and a convex domain D' in $\mathbb{C}^{n-k}$ such that P is biholomorphic to the product domain $\Delta^k \times D'$.*

9.4.4 Sketch of the Proof of Theorem 9.4.4

The proof is again an application of the scaling method. Since the automorphism group is noncompact and since P is a bounded domain, there exists a sequence of automorphisms $\varphi_\nu \in \text{Aut}\ (P)$ such that

$$\lim_{\nu \to \infty} \varphi_\nu(q) = p$$

for some $q \in P$ and $p \in \partial P$. It is important to understand the geometry of the boundary ∂P in a neighborhood of p.

Convex Levi Flat Hypersurfaces

Let f denote one of the holomorphic functions in the defining system of the convex, generic analytic polyhedron P. Then let Σ_f be the surface defined by the equation $|f| = 1$. Since $\nabla|f|$ is nowhere zero on Σ_f, the surface S is a smooth hypersurface in $\mathbb{C}^n$ by the implicit function theorem.

Let $p \in \Sigma_f$. Then consider the set $V_p = \{z \in U \mid f(z) = f(p)\}$. (Recall that U is the open neighborhood of the closure of P on which the functions $f_1, \ldots, f_s$ constituting the defining system of P are defined.) Since f is holomorphic, V_p is a smooth analytic variety and obviously $V_p \subset \Sigma_p$. When P is convex, there is a very special phenomenon. In particular, we see the following:

Lemma 9.4.5. *The variety V_p is flat, in the sense that it is an open subdomain of a complex $n-1$ dimensional affine subspace of $\mathbb{C}^n$.*

Proof. Apply a complex affine change of coordinates of $\mathbb{C}^n$ so that the gradient $\nabla|f|$ at p is parallel to the Re z_1-axis. Furthermore one can arrange that p becomes the origin. Let $u = \operatorname{Re} z_1$. Then one sees that

$$u(z) \le 0, \quad \forall z \in \Sigma_f \quad \text{and} \quad u(p) = 0.$$

Now apply the maximum modulus principle to e^{z_1} restricted to the variety V_p. Then it follows immediately that e^{z_1} is constant on V_p and hence $z_1 = 0$ identically on V_p. Hence we have

$$V_p \subset \{(0, z_2, \ldots, z_n) \mid z_2, \ldots, z_n \in \mathbb{C}\}.$$

Since V_p is a smooth $(n-1)$-dimensional manifold, the assertion of the lemma follows. $\square$

Therefore, for a convex analytic polyhedron P, we see that for every boundary point p, any analytic variety in ∂P passing through p must be an open subdomain (convex!) contained in some complex affine subspace of $\mathbb{C}^n$.

Scaling Sequences

With the preceding discussion, one learns how to apply the scaling method in order to generate the proof of Theorem 9.4.4.

Case 1. *There is no nontrivial variety in ∂P through p.*

In this case, there will be n functions in the defining system with their gradients at p linearly independent over $\mathbb{C}$. Let those functions be $f_1, \ldots, f_n$. Then there exists an open neighborhood W of p such that the mapping $F = (f_1, \ldots, f_n)$ maps $W \cap \mathrm{cl}(P)$ biholomorphically onto $\widetilde{W} \cap \mathrm{cl}(\Delta)^n$ for some open neighborhood $\widetilde{W}$ of some point, say $(e^{i\theta_1}, \ldots, e^{i\theta_n})$, in the Šilov boundary of the polydisc Δ^n. Now, consider the sequence

$$\widetilde{q}_\nu := F \circ \varphi_\nu(q) \quad (\nu = 1, 2, \ldots)$$

in $\widetilde{W}$.

Apply the usual linear fractional transformation, say G, sending $(e^{i\theta_1}, \ldots, e^{i\theta_n})$ to the origin and mapping the polydisc Δ_n biholomorphically onto the domain defined by

$$\operatorname{Re} z_1 > 0, \ldots, \operatorname{Re} z_n > 0.$$

Then let

$$\epsilon_{\ell,\nu} = \text{the real part of the } \ell\text{-th component of } G(\widetilde{q}_\nu)$$

for every $\ell \in \{1, \ldots, n\}$ and for every $\nu = 1, 2, \ldots$.

Now consider the linear mapping

$$\Lambda_\nu(z_1, \ldots, z_n) = \left(\frac{z_1}{\epsilon_{1,\nu}}, \ldots, \frac{z_n}{\epsilon_{n,\nu}} \right).$$

Then it is routine to check that the maps

$$\sigma_\nu := \Lambda_\nu \circ G \circ F \circ \varphi_\nu$$

give rise to a biholomorphic mapping from P onto Δ^n.

Case 2. *There is a nontrivial complex analytic variety in ∂P passing through p.*

Let $n - k$ be the dimension (hence $0 < k < n$) of the variety V_p in ∂P passing through p. Then there exist k functions $f_1, \ldots, f_k$ in the defining system with $\mathbb{C}$-linearly independent gradient vectors at p and where $|f_1(p)| = \cdots = |f_k(p)| = 1$. It is possible to find a local biholomorphic mapping H that sends an open neighborhood W_1 of V_p such that its image $H(W_1)$ satisfies the following:

(i) $H(p) = 0$.
(ii) $H(V_p)$ is a subdomain of $\{(0, \ldots, 0; z_{k+1}, \ldots, z_n) \mid z_{k+1}, \ldots, z_n \in \mathbb{C}\}$.
(iii) $dH_p(\nabla f_\ell(p))$ is parallel to the $\mathrm{Re} z_\ell$-axis, for every $\ell \in \{1, \ldots, k\}$.

Choose the point, for each ℓ, that is the closest intersection point $p_{\ell,\nu}$ of the real line parallel to the $\mathrm{Re} z_\ell$-axis passing through $H(\varphi_\nu(q))$ and the hypersurface $H(\Sigma_{f_\ell})$. Then define the mapping Λ_ν by

$$\Lambda_\nu(z_1, \ldots, z_n) = \left(\frac{z_1}{\|p_{1,\nu} - H \circ \varphi_\nu(q)\|}, \ldots, \frac{z_k}{\|p_{k,\nu} - H \circ \varphi_\nu(q)\|}, z_{k+1}, \ldots, z_n \right).$$

Then again, it is not hard to check that the sequence $\Lambda_\nu \circ H \circ \varphi_\nu$ gives rise to a biholomorphic map from P onto the product domain

$$\{(z_1, \ldots, z_n) \mid \mathrm{Re}\, z_1 > 0, \ldots, \mathrm{Re}\, z_k > 0; (0, \ldots, 0; z_{k+1}, \ldots, z_n) \in H(V_p)\}.$$

From these two cases, the proof follows immediately. □

9.4.5 Sketch of the Proof of Theorem 9.4.3

Now we will consider the general generic analytic polyhedron that is not necessarily convex. In this case, the situation is more complicated, and in fact, the theorem we prove is valid only in complex dimension 2.

Scaling at a Singular Point

In this situation analysis near the singular point is simpler. In fact, this situation is essentially the same as the convex case. The end result by scaling is that the domain P is biholomorphic to the domain in $\mathbb{C}^2$ defined by two inequalities:

$$\operatorname{Re} z_1 > 0, \quad \operatorname{Re} z_2 > 0$$

that is in turn biholomorphic to the bidisc. We shall not include any further details. The reader can fill in the particulars, or can find a detailed proof in [Kim/Krantz/Spiro 2005].

Scaling at a Smooth Point

The reason that this theorem restricts its validity to complex dimension 2 lies in this particular: in case the automorphism orbit accumulation point $p = \lim_{\nu\to\infty} \varphi_\nu(q)$ is a smooth boundary point, let f_1 be the holomorphic function in the defining system of P satisfying $|f_1(p)| = 1$. Then of course there exists a maximal complex analytic variety V_p defined by the equation $f_1(z) = f_1(p)$. If V_p is disconnected, we shall replace V_p by its connected component that contains p.

Now V_p is a smooth 1-dimensional variety. Since $V_p \subset \partial P$ and since P is a bounded domain, there exists a holomorphic covering map $\pi : \Delta \to V_p$ from the open unit disc Δ onto V_p, by the uniformization theorem of Riemann surfaces (Theorem 2.5.1). (This particular argument works because V_p is 1-dimensional; and this follows because $\dim P = 2$.)

Here we shall present an intuitive approach. This is not the actual proof, but it will give the reader a very good idea of how the verification unfolds.

Since $|\nabla f_1|$ is bounded away from zero, there is an open neighborhood W of the closure of V_p in $\mathbb{C}^2$ such that $W \cap P$ is biholomorphic, say by the biholomorphism ψ, to an open neighborhood of $V_p \times \{0\}$ in the open set $V_p \times \Delta_r$, where $\Delta_r = \{z \in \mathbb{C} \mid |z| < r\}$ for some $r > 0$. It can still be arranged, roughly speaking, that $W \cap P$ still contains $V_p \times \Delta_{r'}$. Now consider the map $\widetilde{\psi} := \psi^{-1} \circ \pi \times \mathrm{id}$ from $\Delta \times \Delta_{r'}$ into W (see Figure 9.5).

Once this is done, choose a lifted point-orbit accumulating at the origin selected from $\widetilde{\psi}^{-1}(\varphi_\nu(q))$ and then scale in such a way that the scaled map converges to a holomorphic mapping that sends $\{(z, w) \in \mathbb{C}^2 \mid |z| < 1, \operatorname{Re} w > 0\}$ into P. It is important to analyze this limit mapping.

The Wu metric and the Theorem of Kim–Pagano

Continuing from above, let h be the holomorphic mapping obtained as a subsequential limit of the scaling procedure previously described. Then this map is not only holomorphic, but it preserves the Wu metric of the first kind. (This uses the covering property of the Wu metric!) Notice that the Wu metric of the bidisc is a constant multiple of the Bergman metric, and hence is real analytic and Kähler.

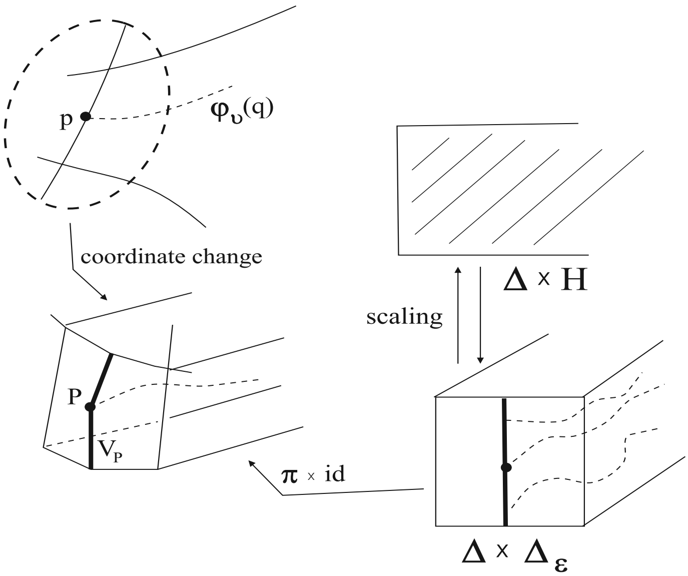

Fig. 9.5. Scaling of a normal polyhedron in $\mathbb{C}^2$.

Readers who are familiar with Riemannian geometry will recall the proof of the Cartan–Hadamard Theorem. That argument proves at this point that h is onto, and is actually a holomorphic covering map. Thus the analytic polyhedron P under current consideration has the bidisc as its holomorphic universal covering manifold. By the way, this was the conclusion of the theorem of Kim–Pagano ([Kim/Pagano 2001]). However, one sees that the conclusion of the theorem of Kim–Pagano is clearly not sufficient for the proof of Theorem 9.4.3. So we shall continue with the sketch.

Further Analysis and the Conclusion

In order to obtain more precise information from the holomorphic covering map, one needs to understand the structure of the small piece of the boundary of P near V_p. Thus Kim, Krantz, and Spiro analyze the set in an elementary but careful way, and they scale again with special care. (See pages 6–10, [Kim/Krantz/Spiro 2005] for details.) After this careful analysis, they were able to conclude that P is indeed biholomorphic to the product $V_p \times \Delta$ as desired. □

9.5 The Greene–Krantz Conjecture

A recurring theme in this book has been the role of finite type. A natural generalization of the concept of strong pseudoconvexity, finite type has served as the right geometric condition to make the scaling method converge, and to force other geometric constructs to behave in a tractable fashion. The Bedford–Pinchuk theorem (Theorem 9.3.1) discussed earlier in the present chapter depends crucially on finite type.

It is natural to wonder whether finite type is crucial to the theory. Are the most natural boundary points (of a bounded domain with smooth boundary at which an automorphism orbits accumulate) that we may study perforce of finite type? These considerations lead to an important conjecture which we treat briefly here.

First we review a definition that has arisen earlier in various contexts.

Definition 9.5.1. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain and $p \in \partial\Omega$. We say that p is a *boundary orbit accumulation point* if there are a point $q \in \Omega$ and automorphisms φ_j of Ω such that $\varphi_j(q) \to p$ as $j \to \infty$.

We have offered considerable evidence in this book that the (Levi) geometry of a boundary orbit accumulation point can yield considerable information about the global geometry of the domain in question. Now we have the following conjecture of Greene and Krantz (see [Greene/Krantz 1991]):

> **Conjecture:** Let Ω be a smoothly bounded domain in $\mathbb{C}^n$ and $p \in \partial\Omega$. If p is a boundary orbit accumulation point, then p is a point of finite type.

If this conjecture is true, then one can see right away that the Bedford/Pinchuk theorems will take a much more simple, natural, and elegant form. Many other parts of the subject will fit together very naturally. There are several partial results that lend credence to the Greene–Krantz conjecture. We describe a few of them here. The next theorem is contained in [Kim K.-T. 1992].

Theorem 9.5.2 (Kim). *Let Ω be a smoothly bounded, convex domain in $\mathbb{C}^2$. Suppose that $p \in \partial\Omega$ and that a 1-dimensional complex analytic manifold containing p lies in $\partial\Omega$. If p is a boundary orbit accumulation point, then Ω is biholomorphic to the bidisc.*

Since the bidisc is *not* smoothly bounded, this theorem supports the Greene–Krantz conjecture. See [Kim/Krantz 2001] for the next result, and for relevant examples as well.

Theorem 9.5.3 (Kim–Krantz). *Let Ω be a smoothly bounded, convex domain in $\mathbb{C}^2$. If $p \in \partial\Omega$ is a boundary orbit accumulation point, then p cannot be exponentially flat.*

Several researchers have contributed to the study of the Greene–Krantz conjecture and related topics. In addition to Bedford/Pinchuk, Greene/Krantz, and Kim, we should mention Berteloot, Bland, Catlin, Fu, Gaussier, Isaev, and Kodama. References to their work appear in our Bibliography, and also in the survey article [Isaev/Krantz 1999].

Scaling is one of the principal tools for studying the Greene–Krantz conjecture. Although the Bedford–Pinchuk theorems do not literally support or imply the conjecture, they certainly lend evidence to its probable correctness.

10

The Scaling Method, II

In the preceding chapter, we discussed theorems concerning the characterization of bounded domains in $\mathbb{C}^n$ by their boundary geometry and the noncompactness of their automorphism groups. There, the scaling method served as a medium that produces the "best" holomorphic re-embedding of the domain into $\mathbb{C}^n$. Thus the scaling method replaced the role of the study of asymptotic boundary behavior of holomorphic invariants.

But it has turned out that the *stretching* feature of the scaling alone (without the presence of boundary-approaching automorphism orbits) plays an important role for the study of asymptotic boundary behavior of holomorphic invariants. This chapter will present this particular aspect of the scaling method—*without* the presence of automorphism orbits.

10.1 Klembeck's Theorem with Stability in the C^2 Topology

10.1.1 The Main Goal

The current goal is to present Klembeck's theorem (Theorem 3.4.3) with stability in the C^2 topology.

The precise target should be described first. Denote by $\mathcal{D}_n$ the collection of all bounded domains in $\mathbb{C}^n$ with C^2 smooth, strongly pseudoconvex boundary. We impose the C^2 topology on $\mathcal{D}_n$ by invoking the C^2 topology on defining functions. See Section 3.5 for the definition in detail.

The result we seek is a strengthening of Theorem 3.4.3. first, denote by $S_\Omega(p;\xi)$ the holomorphic sectional curvature at p in the holomorphic 2-plane direction ξ of the Bergman metric of the domain Ω.

Theorem 10.1.1. *Let $\widehat{\Omega}$ be a bounded strongly pseudoconvex domain with C^2 boundary in $\mathbb{C}^n$. Then, for every $\epsilon > 0$, there exist $\delta > 0$ and an open neighborhood $\mathcal{U}$ of $\widehat{\Omega}$ in $\mathcal{D}_n$ such that, whenever $\Omega \in \mathcal{U}$,*

R. E. Greene et al., *The Geometry of Complex Domains,* Progress in Mathematics,
DOI 10.1007/978-0-8176-4622-6_10,

$$\sup\left\{\left|S_\Omega(p;\xi) - \left(-\frac{4}{n+1}\right)\right| : \Omega \in \mathcal{U}, \xi \in \mathbb{C}^n \setminus \{0\}\right\} < \epsilon$$

whenever $p \in \Omega$ *satisfies* $\operatorname{dis}(p, \partial\Omega) < \delta$.

We will present the proof by contradiction. To be more precise, we shall show that the following statement *cannot* hold:

(†) $\exists \epsilon_0 > 0$, $\exists \Omega_\nu \in \mathcal{D}_n$ *such that* $\Omega_\nu \to \widehat{\Omega}$ *in the* C^2 *topology as* $\nu \to \infty$ *and a sequence* $\{p_\nu \in \Omega_\nu\}$ *with* $\lim_{\nu\to\infty} dis(p_\nu, \partial\Omega_\nu) = 0$ *such that*

$$\left|S_{\Omega_\nu}(p_\nu, \xi_\nu) + \frac{4}{n+1}\right| \geq \epsilon_0,$$

for every $\nu = 1, 2, \ldots$.

10.1.2 Essential Components of the Proof

The method we are introducing here is due to K.T. Kim and J. Yu [Kim/Yu 1996]. This flexible method of proof that can be applied to a broader collection of domains consists of the following three components:

COMPONENT 1. LOCALIZATION

Let $\widehat{\Omega}, \Omega_\nu, p_\nu$ be as in Subsection 10.1.1. Since the goal is to show that

$$\lim_{\nu\to\infty}\left|S_{\Omega_\nu}(p_\nu, \xi_\nu) + \frac{4}{n+1}\right| = 0,$$

we may assume without loss of generality that $\lim_{\nu\to\infty} p_\nu$ exists. Denote this limit by $\widehat{p}$. Notice that $\widehat{p} \in \partial\widehat{\Omega}$.

Let $q_\nu \in \partial\Omega_\nu$ be the closest boundary point of Ω_ν to p_ν for every $\nu = 1, 2, \ldots$. Then consider a sequence $R_\nu : \mathbb{C}^n \to \mathbb{C}^n$ of complex rigid motions (i.e., unitary maps followed by translations) in $\mathbb{C}^n$ and another rigid motion $\widehat{R}$ satisfying:

(1) $\widehat{R}(\widehat{p}) = 0$ and $R_\nu(q_\nu) = 0$ for every ν;

(2) $R_\nu(\partial\Omega_\nu)$, for every ν, and $\widehat{R}(\partial\widehat{\Omega})$ are tangent at 0 to the hyperplane defined by $\operatorname{Re} z_1 = 0$;

and

(3) $\lim_{\nu\to\infty} \|R_\nu - \widehat{R}\|_{C^2} = 0$ where the norm here is the C^2-norm of mappings on an open neighborhood of the closure of $\widehat{\Omega}$ in $\mathbb{C}^n$.

Notice that $R_\nu(\Omega_\nu)$ converges to $\widehat{R}(\widehat{\Omega})$ in the C^2 topology on bounded domains with smooth boundaries. Therefore, without loss of generality, we may also assume the following:

(1′) $0 \in \partial\widehat{\Omega} \cap \left(\bigcap_{\nu=1}^{\infty} \partial\Omega_\nu\right)$.

(2′) $\partial\widehat{\Omega}$ and $\partial\Omega_\nu$ (for every $\nu = 1, 2, \ldots$) share the same outward normal vector $\mathbf{n} = (-1, 0, \ldots, 0)$ at the origin.
(3′) $p_\nu = (r_\nu, 0, \ldots, 0)$ with $r_\nu > 0$ for every ν.

The following gives in effect the localization of Bergman metric holomorphic sectional curvature.

Theorem 10.1.2. *There exists an open neighborhood U of the origin in $\mathbb{C}^n$ such that*

$$\lim_{\nu\to\infty} \sup_{\xi\in\mathbb{C}^n, |\xi|=1} \left| \frac{2 - S_{\Omega_\nu \cap U}(p_\nu; \xi)}{2 - S_{\Omega_\nu}(p_\nu; \xi)} - 1 \right| = 0.$$

The conclusion of this statement implies: as soon as $\lim_{\nu\to\infty} S_{\Omega_\nu\cap U}(p_\nu; \xi)$ exists, it will coincide with $\lim_{\nu\to\infty} S_{\Omega_\nu}(p_\nu; \xi)$. This theorem will be proved later.

COMPONENT 2. CONVERSION BY SCALING

We now demonstrate how the problem on boundary asymptotic behavior of the Bergman curvature (generally considered difficult) can be converted to the problem on the stability of the Bergman kernel function in the interior under perturbation of the boundaries (which is generally easier). This is done by the scaling method, and this conversion is the important, second component of the proof.

Theorem 10.1.3. *Let the sequence $\{(p_\nu; \xi_\nu) \in \Omega_\nu \times (\mathbb{C}^n \setminus \{0\})\}$ be chosen as above. Let B^n denote the open unit ball in $\mathbb{C}^n$. Then there exists a sequence of injective holomorphic mappings $\sigma_\nu : \Omega_\nu \cap U \to \mathbb{C}^n$ satisfying the following properties:*

(i) $\sigma_\nu(p_\nu) = 0$ *(the origin of $\mathbb{C}^n$).*
(ii) *For every $r > 0$, there exists $N > 0$ such that*

$$(1 - r)B^n \subset \sigma_\nu(\Omega_\nu \cap U) \subset (1 + r)B^n$$

for every $\nu > N$.

COMPONENT 3. INTERIOR STABILITY

The third component is the following theorem of Ramadanov.

Theorem 10.1.4 ([Ramadanov 1967]). *Let D be a bounded domain in $\mathbb{C}^n$ containing the origin 0. Let D_ν denote a sequence of bounded domains in $\mathbb{C}^n$ that satisfies the following convergence condition:*

given $\epsilon > 0$, there exists $N > 0$ such that

$$(1 - \epsilon)D \subset D_\nu \subset (1 + \epsilon)D$$

for every $\nu > N$.

Then, for every compact subset F of D, the sequence of Bergman kernel functions K_{D_ν} of D_ν converges uniformly to the Bergman kernel function K_D of D on $F \times F$.

10.1.3 Proof of Theorem 10.1.1

Assuming Theorems 10.1.2, 10.1.3, and 10.1.4, we prove Theorem 10.1.1.

Let $q_\nu, \xi_\nu, \widehat{\Omega}, \Omega_\nu$ be as above. Let U be an open neighborhood of the origin as in Theorem 10.1.2. Taking a subsequence, we may assume that $q_\nu \in \Omega_\nu \cap U$ for every ν. Select σ_ν as in Theorem 10.1.3.

We now apply Theorem 10.1.4 to our setting, with $D_\nu = \sigma_\nu(\Omega_\nu \cap U)$ and $D = B^n$. The conclusion of Theorem 10.1.4 merely states that the sequence $K_{D_\nu}(z, \zeta)$ converges uniformly to $K_D(z, \zeta)$ on $F \times F$. This of course implies that the sequence $K_{D_\nu}(z, \bar{\zeta})$ converges to $K_D(z, \bar{\zeta})$. Notice that the functions now involved are holomorphic functions in the z and ζ variables together. Therefore Cauchy estimates imply that $K_{D_\nu}(z, \zeta)$ converges uniformly to $K_D(z, \zeta)$ on $F \times F$ in the C^k sense for any positive integer k. Since the holomorphic sectional curvature of the Bergman metric involves derivatives of the Bergman kernel function up to the fourth order, we may conclude that $S_{\sigma_\nu(\Omega_\nu \cap U)}(0; \cdot)$ converges uniformly to $S_{B^n}(0; \cdot)$ on $\{\xi \in \mathbb{C}^n : \|\xi\| = 1\}$. Notice that the latter is the constant function with value $-4/(n+1)$.

Combining this result with Theorem 10.1.2, Theorem 10.1.3, and the fact that every biholomorphism is an isometry for the Bergman metric, we see that:

$$
\begin{aligned}
-\frac{4}{n+1} &= \lim_{\nu \to \infty} S_{\sigma_\nu(\Omega_\nu \cap U)}(0; d\sigma_\nu\big|_{q_\nu}(\xi_\nu)) \\
&= \lim_{\nu \to \infty} S_{\sigma_\nu(\Omega_\nu \cap U)}(\sigma_\nu(q_\nu); d\sigma_\nu\big|_{q_\nu}(\xi_\nu)) \\
&= \lim_{\nu \to \infty} S_{\Omega_\nu \cap U}(q_\nu; \xi_\nu) \\
&= \lim_{\nu \to \infty} S_{\Omega_\nu}(q_\nu; \xi_\nu).
\end{aligned}
$$

This proves the desired conclusion. □

It now remains to present the proofs of Theorems 10.1.2, 10.1.3, and 10.1.4. We do that in the subsequent sections.

10.1.4 Localization: Proof of the Theorem 10.1.2

Take a sufficiently small open neighborhood U of the origin in $\mathbb{C}^n$ so that there exists a holomorphic function $h : U \to \mathbb{C}$ such that $h(0) = 1$ and $|h(\zeta)| < 1$ for every $\zeta \in \left(\mathrm{cl}(\widehat{\Omega}) \cup \bigcup_{\nu=1}^{\infty} \mathrm{cl}(\Omega_\nu)\right) \cap (U \setminus \{0\})$. This is possible by taking a subsequence of Ω_ν, because the domains Ω_ν converge in the C^2 sense to $\widehat{\Omega}$, and because $\widehat{\Omega}$ is a bounded strongly pseudoconvex domain.

In general, for every bounded domain Ω in $\mathbb{C}^n$, the following holds: let $p \in \Omega$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ with $\|\xi\| = 1$. Let μ denote the standard

Lebesgue measure of $\mathbb{C}^n$. It is necessary to exploit the following formula by Bergman and Fuks (see [Bergman 1970], Chapter III) for the holomorphic sectional curvature:

$$S^{\Omega}(p;\xi) = 2 - \frac{(I_1^{\Omega}(p;\xi))^2}{I_0^{\Omega}(p) I_2^{\Omega}(p;\xi)},$$

where

$$I_0^{\Omega}(p) := \inf\Big\{ \int_{\Omega} |f|^2 d\mu : f \in A^2(\Omega), f(p) = 1 \Big\},$$

$$I_1^{\Omega}(p;\xi) := \inf\Big\{ \int_{\Omega} |f|^2 d\mu \ : \ f \in A^2(\Omega), f(p) = 0, \sum_{j=1}^{n} \xi_j \frac{\partial f}{\partial z_j}(p) = 1 \Big\},$$

and

$$I_2^{\Omega}(p;\xi) := \inf\Big\{ \int_{\Omega} |f|^2 d\mu \ : \ f \in A^2(\Omega), \\ f(p) = 0, df\big|_p = 0, \sum_{j,k=1}^{n} \xi_j \xi_k \frac{\partial^2 f}{\partial z_j \partial z_k}(p) = 1 \Big\}.$$

In general, if two domains D_1 and D_2 in $\mathbb{C}^n$ are related by $D_1 \subset D_2$, it follows by definition that $I_k^{D_1} \leq I_k^{D_2}$. The quantities I_j above are often called the *minimum integrals,* and the property we just observed is called the *monotonicity of minimum integrals.*

Notice that the proof will be completed as soon as we show that the quotients $\frac{I_0^{\Omega}(p)}{I_0^{\Omega\cap U}(p)}$, $\frac{I_1^{\Omega}(p;\xi)}{I_1^{\Omega\cap U}(p;\xi)}$, and $\frac{I_2^{\Omega}(p;\xi)}{I_2^{\Omega\cap U}(p;\xi)}$ converge to 1 uniformly on unit vectors ξ, independent of choices of domains Ω from the sequence $\{\Omega_\nu : \nu = 1, 2, \ldots\}$, as p tends to the origin 0.

Since the justification of the convergence of the preceding quotients follows similar arguments, we choose to establish that

$$\limsup_{p \to 0} \frac{I_2^{\Omega}(p;\xi)}{I_2^{\Omega\cap U}(p;\xi)} \leq 1.$$

Notice that, with the monotonicity observed above, this implies that $\frac{I_2^{\Omega}(p;\xi)}{I_2^{\Omega\cap U}(p;\xi)}$ converges to 1. Along the way, the uniformity of the convergence will be obtained as well.

Now let $k > 1$ be an integer. Take an open neighborhood V of 0 satisfying $0 \in V \subset\subset U$ so that h is nowhere zero in V. Let a be a constant satisfying $0 < a < 1$ and $|h| < a$ on $(U \setminus V) \cap \mathrm{cl}(\Omega)$. Choose a smooth bump function $\chi \in \mathcal{C}_0^{\infty}(U)$ such that $\chi = 1$ on V and $0 \leq \chi \leq 1$ on U. Furthermore let $\varphi(z) = (2n + 4) \log |z - p|$.

Taking p to be sufficiently close to 0, there is no loss of generality to assume that $q \in V$.

Let $f \in A^2(\Omega \cap U)$ be such that

$$f(p) = 0, \quad df(p) = 0, \quad \sum_{j,\ell=1}^{n} \xi_j \xi_\ell \frac{\partial^2 f}{\partial z_j \partial z_\ell}(p) = 1$$

and $\|f\|^2_{L^2(\Omega \cap U)} = I_2^{\Omega \cap U}(p; \xi)$.

We apply now a theorem of Hörmander (Theorem 4.4.2, p. 94, [Hörmander 1990]). There exists a locally integrable function u on Ω satisfying

$$\overline{\partial} u = \overline{\partial}(\chi f h^k)$$

with the estimate

$$\int_\Omega |u(z)|^2 e^{-\varphi(z)} d\mu \leq C \int_\Omega |\overline{\partial}(\chi(z) f(z) h(z)^k)|^2 e^{-\varphi(z)} d\mu$$

for some constant depending only on Ω. Simplifying the estimate, one obtains

$$\int_\Omega \frac{|u(z)|^2}{|z-p|^{2n+4}} d\mu \leq C \int_{\Omega \cap (U \setminus V)} \frac{|\overline{\partial}\chi(z)|^2 |f(z)|^2 |h(z)|^{2k}}{|z-p|^{2n+4}} d\mu.$$

Since the righthand side is bounded, we see that u vanishes to order 2 at p. Moreover, one deduces that

$$\|u\|_{L^2(\Omega)} \leq C' a^k \|f\|_{L^2(\Omega \cap U)}$$

for some constant $C' > 0$ depending only on Ω and χ. We point out that all the constants here can be taken independent of Ω_ν, as $\{\Omega_\nu\}$ converges to the bounded domain $\widehat{\Omega}$.

Now let $F_k = \chi f h^k - u$. Then, for every $k > 1$, the function $g_k(z) \equiv F_k(z)(h(p))^{-k}$ satisfies the conditions:

$$g_k(p) = 0, \quad dg_k(p) = 0, \quad \text{and} \quad \sum_{j,\ell=1}^{n} \xi_j \xi_\ell \frac{\partial^2 g_k}{\partial z_j \partial z_\ell}(p) = 1.$$

Thus the definition of the minimum integral I_2 implies that

$$\begin{aligned}
I_2^\Omega(p;\xi) &\leq \|g_k\|^2_{L^2(\Omega)} \\
&= \frac{\|F_k\|^2_{L^2(\Omega)}}{|h(p)|^{2k}} \\
&\leq \frac{(\|\chi f h^k\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})^2}{|h(p)|^{2k}} \\
&\leq \frac{(\|f\|_{L^2(\Omega\cap U)} + C' a^k \|f\|_{L^2(\Omega \cap U)})^2}{|h(p)|^{2k}} \\
&= \frac{(1 + C' a^k)^2}{|h(p)|^{2k}} I_2^{\Omega \cap U}(p;\xi).
\end{aligned}$$

Notice that the quotient $\frac{(1+C'a^k)^2}{|h(p)|^{2k}}$ approaches 1, letting p converge to 0 first and then allowing k to tend to $+\infty$. Notice also that this convergence does not depend on the unit vector ξ, or the choice of domain Ω from the family $\{\Omega_\nu : \nu = 1, 2, \ldots\}$. Hence we obtain

$$\limsup_{p \to 0} \frac{I_2^{\Omega_\nu}(p;\xi)}{I_2^{\Omega_\nu \cap U}(p;\xi)} \leq 1,$$

with the uniformity in ξ and Ω_ν as desired. For I_0 and I_1 one needs simply to change the weight φ slightly. This completes the proof. □

Remark 10.1.5. This localization process is of course stable under the perturbation of the boundary, because the choices for h, U, V, C, C', and χ are independent of the index ν: for instance the constant C, that is from Theorem 4.4.2 of [Hörmander 1990], can actually be chosen to be $\max\{(1 + |z|^2)^2 \colon z \in \Omega \cup \bigcup \Omega_\nu\}$.

10.1.5 Conversion of the Problem by Scaling: Proof of the Theorem 10.1.3

We now show how to *convert* the *asymptotic boundary behavior problem* for the holomorphic sectional curvature of the Bergman metric to the *interior stability problem* of the Bergman metric under a perturbation of the boundaries of domains.

In our case the situation is simple, because the point sequence $\{p_\nu\}$ under consideration is located on the $\operatorname{Re} z_1$-axis.

Let δ be a positive number smaller than 1, to be chosen later. Let

$$\mathcal{E}_\delta = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \operatorname{Re} z_1 > (1-\delta)(|z_1|^2 + \cdots + |z_n|^2)\}$$

and

$$\mathcal{S}_\delta = \{z \in \mathbb{C}^n \mid \operatorname{Re} z_1 > (1+\delta)(|z_1|^2 + \cdots + |z_n|^2)\}.$$

Since $\widehat{\Omega}$ is a domain with $\mathcal{C}^2$ smooth, strongly pseudoconvex boundary, there exists an open neighborhood U of the origin in $\mathbb{C}^n$ and a biholomorphism-into $\Psi : U \to \mathbb{C}^n$ such that

$$\Psi(\widehat{\Omega} \cap U) = \{z \in \Psi(U) \mid \operatorname{Re} z_1 > |z_1|^2 + \cdots + |z_n|^2 + R_2(z)\},$$

where $R_2(z) = o(|z_1|^2 + \cdots + |z_n|^2)$. Let $V = \Psi(U)$. Shrinking the neighborhood U if necessary, one obtains that

$$\mathcal{S}_\delta \cap V \subset \Psi(\widehat{\Omega} \cap U) \subset \mathcal{E}_\delta.$$

Due to the $\mathcal{C}^2$ convergence, and by (1′)–(3′), one deduces that there exists $N > 0$ such that

$$\mathcal{S}_\delta \cap V \subset \Psi(\Omega_\nu \cap U) \subset \mathcal{E}_\delta$$

for every $\nu > N$. Now let $\lambda_\nu \equiv |\Psi(p_\nu)|$ for every ν. Consider the dilatation maps

$$\Lambda_\nu(z_1, \ldots, z_n) \equiv \Big(\frac{z_1}{\lambda_\nu}, \frac{z_2}{\sqrt{\lambda_\nu}}, \ldots, \frac{z_n}{\sqrt{\lambda_\nu}}\Big).$$

(Notice here that the point sequence $\Psi(p_\nu)$ approaches the origin nontangentially to the hypersurface defined by $\operatorname{Re} z_1 = 0$, that is, tangent to $\Psi(\partial\widehat{\Omega})$ at the origin.) Finally let

$$\Phi(z_1, \ldots, z_n) = \Big(\frac{z_1 - 1}{z_1 + 1}, \frac{2z_2}{z_1 + 1}, \ldots, \frac{2z_n}{z_1 + 1}\Big).$$

Then set

$$\sigma_\nu = \Phi \circ \Lambda_\nu \circ \Psi$$

for every ν. Now it is simple to check that, with a composition for each ν by a Möbius transformation adjusting $\sigma_\nu(p_\nu)$ to the origin (while preserving the unit ball), $\{\sigma_\nu\}$ yields a sequence of holomorphic maps satisfying the desired conclusion. $\square$

10.1.6 Interior Stability: Proof of Theorem 10.1.4

The proof is based upon the monotonicity of the Bergman kernel function on the diagonal, which is:

> $K_{\Omega_1}(z,z) \geq K_{\Omega_2}(z,z)$ *for every* $z \in \Omega_1$ *whenever two domains* Ω_1 *and* Ω_2 *in* $\mathbb{C}^n$ *satisfy* $\Omega_1 \subset \Omega_2$.

This follows from the "special basis" characterization of the Bergman kernel in Section 3.1. For simplicity, let us use the notation: $K_\nu = K_{D_\nu}$, $K = K_D$, $K_{\epsilon-} = K_{(1-\epsilon)D}$, $K_{\epsilon+} = K_{(1+\epsilon)D}$.

Let $\widetilde{F}$ be a compact subset of D which contains F in its interior. Choose $N > 0$ so that $\widetilde{F} \subset (1-\epsilon)D$ and $(1-\epsilon)D \subset D_\nu \subset (1+\epsilon)D$ for every $\nu > N$.

Fix $\zeta \in \widetilde{F}$ momentarily and allow $z \in \widetilde{F}$ to vary. Then we get the estimate

$$\begin{aligned}
&\|K_\nu(z,\zeta) - K_{\epsilon+}(z,\zeta)\|^2_{L^2(F)} \\
&\quad \leq \int_{D_\nu} \overline{(K_\nu(w,\zeta) - K_{\epsilon+}(w,\zeta))}(K_\nu(w,\zeta) - K_{\epsilon+}(w,\zeta))\, d\mu(w) \\
&\quad = K_\nu(\zeta,\zeta) - K_{\epsilon+}(\zeta,\zeta) - \int_{(1+\epsilon)D\setminus D_\nu} \overline{K_{\epsilon+}(w,\zeta)} K_{\epsilon+}(w,\zeta) d\mu(w) \\
&\quad \leq K_{\epsilon-}(\zeta,\zeta) - K_{\epsilon+}(\zeta,\zeta).
\end{aligned}$$

Notice that the calculation above has used the reproducing property and the monotonicity on the diagonal of the Bergman kernel function.

Recall that the Bergman kernel function is holomorphic in the first n variables z_j and conjugate-holomorphic in the last n variables ζ_j. Thus $K_\nu(z,\overline{\zeta})$ is holomorphic in both the z and ζ variables. Notice also that the above estimate yields the L^2 convergence of $K_\nu(z,\overline{\zeta})$ to $K(z,\overline{\zeta})$ on $\widetilde{F} \times \widetilde{F}$. One may now use the Cauchy estimates to conclude that K_ν converges uniformly to K on $F \times F$. $\square$

10.1.7 The Bergman Metric near Strongly Pseudoconvex Boundary Points

As an additional application of the arguments introduced thus far, we will now deduce the boundary behavior estimate for the Bergman metric of a bounded, C^2 smooth strongly pseudoconvex domain, establishing the completeness of the Bergman metric there.

Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 smooth, strongly pseudoconvex boundary. Let $v \in \mathbb{C} \setminus \{0\}$. Then, for any $p \in \Omega$, the Bergman metric length $\|v\|_{\Omega,p}$ at p has the following representation by minimum integrals:

$$\|v\|^2_{\Omega,p} = \frac{1}{I_0^\Omega(p) I_1^\Omega(p;v)}.$$

Note that this follows from the exposition in Section 3.1, where Bergman's special orthonormal system for $A^2(\Omega)$ was introduced.

Thus the localization arguments in 10.1.4 imply the following:

> For any $\widehat{p} \in \partial\Omega$, any open neighborhood U of $\widehat{p}$ in $\mathbb{C}^n$, and any positive constant $C > 1$, there exists an open set V satisfying $\widehat{p} \in V \subset\subset U$ such that
>
> $$\frac{1}{C}\|v\|^2_{\Omega\cap U,p} \le \|v\|^2_{\Omega,p} \le C\|v\|^2_{\Omega\cap U,p}$$
>
> for any $p \in V$ and any $v \in \mathbb{C}^n$.

This, together with the scaling method arguments similar to 10.1.5 and 10.1.6 implies immediately the following.

Let p be as above. Then let $\widetilde{p} \in \partial\Omega$ be the closest point to p. (Such a $\widetilde{p}$ is uniquely determined if V is chosen sufficiently small.) Write $v \in \mathbb{C}^n$ as

$$v = v' + v''$$

so that v' is complex tangent to $\partial\Omega$ at $\widetilde{p}$ whereas v'' is complex normal. Then there exists a constant $C' > 0$ such that

$$\|v\|^2_{\Omega,p} \ge C'\left(\frac{\|v'\|^2}{\|p-\widetilde{p}\|} + \frac{\|v''\|^2}{\|p-\widetilde{p}\|^2}\right),$$

where the norm $\|\cdot\|$ on the right-hand side is the Euclidean norm, which is in fact the Bergman metric of the unit ball in $\mathbb{C}^n$ at the origin up to a constant multiple.

Notice that this in particular implies the completeness of the Bergman metric of the bounded strongly pseudoconvex domains, which was used in the exposition of Chapter 3.

10.2 Separating Boundary and Interior Points

An important general theme that emerged in the first four chapters was the idea of finding geometric invariants that distinguished interior points of a strongly pseudoconvex domain that was not biholomorphic to the ball from points near the boundary. Such domains have compact automorphism groups by the general result of Wong–Rosay; but, more explicitly, the compactness of their automorphism groups can be explained and indeed established by considering geometric invariants of the Bergman metric.

Recall the exact setting: By Lu Qi-Keng's theorem (Theorem 4.2.2), if Ω is a bounded domain not biholomorphic to the ball, but with complete Bergman metric, then there is a point $p_0 \in \Omega$ and a holomorphic 2-plane P such that the holomorphic sectional curvature $\kappa(P)$ of the Bergman metric of Ω is not equal to $-4/(n+1)$. If Ω is a C^∞ strongly pseudoconvex domain, the Bergman metric is necessarily complete (Theorem 3.4.1; more generally, see Theorem 3.4.2). And for such Ω, there is also a neighborhood of the boundary of Ω such that every holomorphic sectional curvature κ at every point of this neighborhood satisfies $|\kappa + 4/(n+1)| < |\kappa(P) + 4/(n+1)|$. This gives an intrinsic geometric reason why the $\mathrm{Aut}\,(\Omega)$-orbit of p_0 cannot intersect the neighborhood of the boundary. Moreover, all this can be arranged stably with respect to C^∞ small perturbations of Ω (Theorem 3.5.1). From this, a considerable body of results followed by various normal families arguments in Chapters 3 and 4, results about C^∞ domains and C^∞ perturbations thereof.

In the previous section, the asymptotic constancy of the holomorphic sectional curvature of the Bergman metric near the boundary has been established not just for C^∞ strongly pseudoconvex domains but also for C^2 strongly pseudoconvex domains (Theorem 10.1.1). The Bergman metric is also complete in this case (Theorem 3.4.2) or by the remarks of the previous section. Thus one might expect to extend the results of Chapters 3 and 4 that applied to the C^∞ stability to the more general C^2 situation. This expectation is in fact valid.

In this section, some of the extended results will be indicated.

These extensions to the C^2 situation are of considerable interest: C^2 is the natural home of the concept of strong pseudoconvexity, and the fact that all these results apply at the C^2 level shows that they are, as it were, in the nature of strong pseudoconvexity, without higher derivatives being involved.

The starting point is Theorem 10.1.1 together with the completeness of the Bergman metric. From these together, the C^2 version of Corollary 3.4.4 (Wong's theorem) via curvature arises:

Theorem 10.2.1. *If Ω_0 is a C^2 strongly pseudoconvex domain, then either Ω_0 is biholomorphic to the ball or* Aut (Ω_0) *is compact.*

The proof follows the pattern of the curvature proof of Corollary 3.4.4: there are, if Ω_0 is not biholomorphic to the ball, a point p_0 and a holomorphic 2-plane P at p_0 with the holomorphic sectional curvature $\kappa(P)$ of P not equal to $-4/(n+1)$, as discussed in a previous paragraph in this section. Theorem 10.1.1 and the Aut (Ω_0)-invariance of the Bergman metric and its curvatures imply that the Aut (Ω_0)-orbit of p_0 is bounded away from the boundary $\partial\Omega_0$ of Ω_0. Hence by Proposition 1.3.10 and Corollary 1.3.6, Aut (Ω_0) is compact. This is exactly the same proof as for Corollary 3.4.4 except that Theorem 10.1.1 replaces Theorem 3.4.3. □

Of course, more general results hold: the point here is the curvature method. See Chapter 11 for the maximum generality known for results of this type, about orbits of automorphism groups and strong pseudoconvexity (cf. [Gaussier/Kim/Krantz 2002]).

Since Theorem 10.1.1 gives the C^2 version of Theorem 3.5.1, it is to be expected that Theorem 4.1.1 has a C^2 version, since only the stable asymptotic constancy of curvature near the boundary was used.

Theorem 10.2.2. *There is a neighborhood $\mathcal{U}$ of the unit ball in the C^2 topology on domains with C^2 boundary such that: if $\Omega \in \mathcal{U}$, then either*

(1) *Ω is biholomorphic to the ball,*

or

(2) Aut (Ω) *is compact and acts on Ω with a fixed point.*

Proof. Following the pattern of the proof of Theorem 4.1.1, it suffices to show that, for all Ω C^2 close enough to the unit ball, Ω is C^2 strongly convex (obvious), and then the necessarily complete Bergman metric on Ω has negative sectional curvature. This follows by exactly the same combination of interior stability (Theorem 3.5.2 in the C^∞ case) of the Bergman kernel and stable asymptotic constancy of holomorphic sectional curvature near the boundary of Ω, as in the proof of Theorem 4.1.1.

In the present instance, interior stability is disposed of by Theorem 10.1.4, detouring around any considerations of the stability of the Kohn solution of the $\overline{\partial}$-Neumann problem (the latter was the approach used for Theorem 3.5.2): a convex domain automatically satisfies the containment conditions required for Theorem 10.1.4. On the other hand, the boundary stability of curvature needed is exactly provided by Theorem 10.1.1. The remainder of the proof is precisely as for Theorem 4.1.1. □

Of course, as before with Theorem 4.1.1, Theorem 10.2.2 is a special case of Lempert's result Theorem 4.1.2 on convex domains, since $\mathrm{Aut}\,(\Omega)$ in Theorem 10.2.2 is compact if Ω is not biholomorphic to the ball (cf. the discussion after Theorem 4.1.1).

As it happens, many of the results later on in Chapter 4 hold under very general hypotheses: they really depend only on the region near the boundary being nonequivalent to the interior in some sense so that normal families are stably nondegenerate. Recall the notion of normal convergence of sets (as well as the Carathéodory kernel convergence of sets) introduced in Definition 9.2.2 of Section 9.2.2: a sequence Ω_j of bounded domains (i.e., connected open sets) in $\mathbb{C}^n$ converges normally to a bounded domain Ω_0 if (1) for each compact set $K \subset \Omega_0$, $K \subset \Omega_j$ for every j sufficiently large, and (2) when a compact set has a fixed positive distance (independent of j) from $\mathbb{C}^n \setminus \Omega_j$ for every j, then $K \subset \Omega_0$. Next define:

Definition 10.2.3. A domain Ω_0 in $\mathbb{C}^n$ is called *stably interior* if there is a point $p_0 \in \Omega_0$ and an $\epsilon > 0$ such that, if Ω_j converges normally to Ω_0, then for all sufficiently large j the Euclidean distance $\mathrm{dis}\,(\varphi(p_0), \mathbb{C}^n \setminus \Omega_j) > \epsilon$ for all $\varphi \in \mathrm{Aut}\,(\Omega_j)$.

In the case of smooth (C^∞) domains and C^∞ convergence, the distance to the boundary condition was discussed in Chapter 4. Theorem 4.1.4 can be extended to the stably interior case in general.

Theorem 10.2.4. *If Ω_0 is stably interior, if $\Omega_j \to \Omega_0$ in the C^2 topology of domains and if Ω_0 is rigid (i.e., $\mathrm{Aut}\,(\Omega_0) = \{id_{\Omega_0}\}$), then Ω_j is rigid for all sufficiently large j.*

Proof. It follows from Ω_0 being stably interior that, if $\Omega_j \to \Omega_0$ in the C^2 topology and $\varphi_j \in \mathrm{Aut}\,(\Omega_j)$, then some subsequence φ_{j_k} converges uniformly on compact subsets to $\varphi_0 \in \mathrm{Aut}\,(\Omega_0)$. [See Proposition 9.2.4; see also Theorem 1.3.4.] Since Ω_0 is rigid, $\varphi_0 = \mathrm{id}$. The domain Ω_0 being stably interior also implies that there is a C^2 neighborhood $\mathcal{U}_1$ of Ω_0 such that $\mathrm{Aut}\,(\Omega)$ is compact for each $\Omega \in \mathcal{U}_1$. Let g_0 be the Bergman metric of Ω_0. (Actually, g_0 can be taken to be any metric on Ω_0: the use of the Bergman metric is truly incidental here.) For each $\Omega \in \mathcal{U}_1$, let g_Ω be the average of g_0 over the compact group $\mathrm{Aut}\,(\Omega)$. Note that, if $\Omega_j \to \Omega_0$ (in the C^2 topology), then g_{Ω_j} converges to g_{Ω_0} uniformly (in the C^∞ topology) on compact subsets of Ω_0: this is so because the elements of $\mathrm{Aut}\,(\Omega_j)$ will be C^∞ close to the identity uniformly on each compact subset of Ω_0, by the normal families convergence statement given earlier.

As in the proof of Theorem 4.1.4, we now want to show that, if $\mathrm{Aut}\,(\Omega_j) \neq \{\mathrm{id}_{\Omega_j}\}$ for j large, a contradiction is reached. But now the concluding Riemannian geometry part of the proof of Theorem 4.1.2 applies without change if the Bergman metrics of the Ω_j are replaced by the metrics g_{Ω_j}, which are themselves invariant under $\mathrm{Aut}\,(\Omega_j)$. The point here is that the g_{Ω_j} are automatically C^∞ close on compact sets to g_0: no special observations about Bergman metric stability are needed. $\square$

These arguments could have been used to prove Theorem 4.1.4. Since Bergman metric stability was in sight in that case, it was used throughout. But in fact its only essential use was in guaranteeing that Ω_0 was stably interior.

Theorem 4.4.3, the "semicontinuity theorem," which is itself an extension of Theorem 4.1.4, also has an analogue in the stably interior setting.

Theorem 10.2.5. *If Ω_0 is stably interior and $\Omega_j \to \Omega_0$ normally, then for all sufficiently large j, $\mathrm{Aut}\,(\Omega_j)$ is isomorphic to a subgroup of $\mathrm{Aut}\,(\Omega_0)$.*

Proof. Normal families arguments show that if $\varphi_j \in \mathrm{Aut}\,(\Omega_j)$, then some subsequence $\{\varphi_{j_k}\}$ converges uniformly C^∞ on compact subsets to an element $\varphi_0 \in \mathrm{Aut}\,(\Omega_0)$. It follows that, if $\tau\colon \Omega_0 \to \mathbb{R}$ is a C^∞ exhaustion function that is $\mathrm{Aut}\,(\Omega_0)$-invariant (which always exists by averaging over $\mathrm{Aut}\,(\Omega_0)$), then the average over $\mathrm{Aut}\,(\Omega_j)$ of $\tau|_{\Omega_j \cap \Omega_0}$ converges C^∞ uniformly on compact subsets of Ω_0 to τ itself. Now choose $\alpha \in \mathbb{R}$ a noncritical value of τ, $\alpha \in \mathrm{Range}(\tau)$, then $\tau^{-1}\big((-\infty, \alpha]\big)$ is a C^∞ manifold-with-boundary, i.e., a C^∞ domain with smooth boundary which is $\mathrm{Aut}\,(\Omega_0)$-invariant. Moreover, if $\tau_j =$ the $\mathrm{Aut}\,(\Omega_j)$-average of τ, then, for j large, the set $\tau_j^{-1}\big((-\infty, \alpha]\big)$ is a C^∞ domain with smooth boundary which is C^∞ close to $\tau^{-1}\big((-\infty, \alpha]\big)$. Finally, let g_0 be an $\mathrm{Aut}\,(\Omega_0)$-invariant metric, either the Bergman metric of Ω_0 or the average over $\mathrm{Aut}\,(\Omega_0)$ of an arbitrary metric on Ω_0. Then the quantity given by the $\mathrm{Aut}\,(\Omega_j)$-average of g_0 is $\mathrm{Aut}\,(\Omega_j)$-invariant and, on the $\mathrm{Aut}\,(\Omega_j)$-invariant domain $\tau_j^{-1}\big((-\infty, \alpha]\big)$ is diffeomorphism-conjugate to a subgroup of $\mathrm{Isom}(g_0)$ via a diffeomorphism C^∞ close to the identity (for j large). Call this subgroup G_j. Note that, for j large, G_j lies in a C^∞ neighborhood in $\mathrm{Isom}(g_0)$ of $\mathrm{Aut}\,(\Omega_j)$ restricted to $\tau^{-1}\big((-\infty, \alpha]\big)$. By the standard Lie group theory result used before ([Montgomery/Zippin 1942]), it follows that G_j, and hence also $\mathrm{Aut}\,(\Omega_j)$, for j large, are isomorphic to a subgroup of $\mathrm{Aut}\,(\Omega_0)$. $\square$

From this viewpoint, the curvature invariants of the Bergman metric in the C^∞ case function were used in good part simply to guarantee the stably interior condition: the Bergman metric separates stably some interior points from points near the boundary by automorphism-invariant curvature invariants, in the case of C^∞ strongly pseudoconvex domains. This was discussed in detail in Chapters 3 and 4 and summarized in this section earlier. Theorem 10.1.1 gives the boundary behavior needed to extend this to the C^2 case. When interior stability of the Bergman metric is also assured, e.g., as in Theorem 10.1.3, then the stably interior condition is generated so that the rigidity and semicontinuity theorems, Theorems 10.2.4 and 10.2.5, apply. But in fact, the interior stability of the Bergman metric follows by the L^2 $\overline{\partial}$ technique with weights (almost identical to, and in fact easier than, the techniques demonstrated in Section 10.1.4, based upon [Hörmander 1965] and [Hörmander 1990]). (We shall not include any further details as they would be repetitious.) From this, we obtain the semicontinuity result in the C^2 case.

Theorem 10.2.6. *If Ω_0 is a C^2 strongly pseudoconvex domain, not biholomorphic to the ball, then there is a neighborhood $\mathcal{U}$ of Ω_0 in the C^2 topology of domains with C^2 boundary, such that, if $\Omega \in \mathcal{U}$, then* $\operatorname{Aut}(\Omega)$ *is compact and is Lie-group isomorphic to a subgroup of the compact group* $\operatorname{Aut}(\Omega_0)$.

In connection with Theorems 10.2.4 and 10.2.5, it should be noted that the condition of being stably interior or some related condition is indeed required. Normal convergence of a sequence $\{\Omega_j\}$ of bounded domains to a bounded domain Ω_0 is not enough as such to guarantee that $\operatorname{Aut}(\Omega_j)$ is isomorphic to a subgroup of $\operatorname{Aut}(\Omega_0)$ for all j sufficiently large. It is, for example, possible that Ω_0 could be rigid even though none of the Ω_j are rigid.

A (pseudoconvex) example of this behavior can be obtained as follows: Let S be the boundary sphere of the unit ball in $\mathbb{C}^2$. Make a C^2 small C^∞ perturbation of S in a small neighborhood of the point $(0,1)$ to obtain a hypersurface $\widehat{S}$ such that the interior $\widehat{B}$ of $\widehat{S}$ is C^∞ strongly convex (hence strongly pseudoconvex) and $\widehat{B}$ is rigid. This can be arranged by choosing a suitable perturbation: indeed, generically any C^2 small perturbation will have this property. Now let $\varphi\colon B \to B$ be the holomorphic map in $\operatorname{Aut}(B)$ defined by $(z_1, z_2) \longrightarrow \left(\frac{z_1+\frac{1}{2}}{1+\frac{1}{2}z_1}, \frac{\sqrt{3}z_2}{2(1+\frac{z_1}{2})}\right)$ (cf. Section 1.4). As usual, let $\varphi^0 = \mathrm{id}$, $\varphi^n = \varphi \circ \varphi^{n-1}$, and $\varphi^{-n} = (\varphi^{-1})^n = \varphi^{-1} \circ (\varphi^{-1})^{n-1}$, for $n = 1, 2, 3, \ldots$. Since φ is holomorphic except where $z_1 = -2$, it is easy to see that φ^n, $n \in \mathbb{Z}$, all act on $\widehat{B}$ (when $\widehat{S}$ is close enough to S) extending smoothly across $\widehat{S}$, indeed holomorphically across $\widehat{S}$. Now for each positive integer k, we define a domain Ω_m to be $\bigcup_{k=-\infty}^{+\infty} \varphi^{km}(\widehat{B})$.

If the initial perturbation $\widehat{S}$ is different from S itself only in a sufficiently small neighborhood of $(0,1)$, then the images of the actually perturbed part of S under φ^n are disjoint for all $n = 1, 2, \ldots$. Hence $\bigcup_{k=-K}^{K} \varphi^{km}(\widehat{B})$ will be pseudoconvex, indeed C^∞ strongly pseudoconvex. Thus the Ω_m, being increasing unions of pseudoconvex domains, are pseudoconvex. The the boundaries of Ω_m are not, however, C^2 at $(1,0)$ and $(-1,0)$.

Clearly, φ^m maps Ω_m biholomorphically to itself. So $\operatorname{Aut}(\Omega_m) \neq \{\mathrm{id}\}$: Ω_m is not rigid. But it is not hard to check that the sequence $\{\Omega_m\}$ converges normally to $\widehat{B}$. The logic here is that, when m is large, powers φ^{km}, $m \in \mathbb{Z}$, $k \neq 0$, all "compress" the perturbation near $(0,1)$ that changed S to $\widehat{S}$ into a very small perturbation near $(1,0)$ if $m > 0$, or $(-1,0)$ if $m < 0$. But $\widehat{B}$, the limit domain, is rigid, while the Ω_m are all nonrigid.

Note that there is no inconsistency with Theorem 10.2.4 or Theorem 10.2.5 since Ω_m are not stably interior. And there is no inconsistency with Theorem 10.2.6 because the boundaries of the Ω_m are not C^2 at $(1,0)$ and $(-1,0)$, as already noted.

This example is related to the fact that semicontinuity for compact Riemannian manifolds (Theorem 4.4.1) does not hold for open (noncompact) manifolds, at least for the most natural kind of convergence. A similar example can be constructed. Make a perturbation of the standard metric of $\mathbb{R}^2$ near $(0,0)$, say within the disc of radius $1/2$ around $(0,0)$, in such a way that the resulting metric on $\mathbb{R}^2$, say $\widehat{g}$, has Isom $(\widehat{g}) = \{\mathrm{id}\}$. That this is possible is easy to check by looking at curvature matters. Let, for $m = 1, 2, 3, \ldots$, $g_m =$ the metric obtained by making the same perturbation around the points $(km, 0)$, $k \in \mathbb{Z}$. Isom $(g_m) \neq \{\mathrm{id}\}$, since g_m is invariant under $(x, y) \to (x + km, y)$, $k \in \mathbb{Z}$. But clearly as $m \to +\infty$, the metrics g_m converge C^∞ uniformly on compact sets to $\widehat{g}$.

The existence of such examples clearly has to do with the possibility of all of the automorphisms or isometries other than the identity moving off to infinity, diverging to infinity in the classical language. When this is ruled out by restriction to connected groups, for example, semicontinuity in some form continues to hold.

10.3 Ian Graham's Theorem by Scaling

The arguments here, based on the scaling method that gave a proof of the theorem of Klembeck with stability (Theorem 10.1.1), have turned out to be rather versatile. Here we will see that a slight modification gives a proof of a well-known theorem of Ian Graham [Graham 1975].

Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 smooth, strongly pseudoconvex boundary. Let $T\Omega \ni (z, \xi) \mapsto F_\Omega(z, \xi) \in \mathbb{R}$ denote the Kobayashi–Royden metric (= the infinitesimal Kobayashi pseudometric) of the domain Ω. The goal of this theorem is to identify the asymptotic boundary behavior of $F_\Omega(z, \xi)$ as z approaches the boundary point.

In order to present the theorem of Graham, we give two standard pieces of notation: for $z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, write

$$\langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n$$

and

$$\|z\|^2 = \langle z, z \rangle.$$

Theorem 10.3.1 ([Graham 1975]). *Let $p \in \partial\Omega$, and let ρ denote a $\mathcal{C}^2$ smooth defining function for Ω satisfying $\|\nabla\rho(p)\| = 1$. Then*

$$\lim_{z \to p} F(z, \xi) d(z, \partial\Omega) = \frac{1}{2} \|\xi_N\|,$$

where $\xi_N = \left\langle \xi, \frac{\nabla\rho(p)}{\|\nabla\rho(p)\|} \right\rangle \frac{\nabla\rho(p)}{\|\nabla\rho(p)\|}$, the normal component of ξ to $\partial\Omega$ at p. If $\xi_N(p) = 0$, then

$$\lim_{\Lambda \ni z \to p} F(z, \xi)^2 d(z, \partial\Omega) = \frac{1}{2} L\rho(p; \xi, \bar{\xi}),$$

where Λ denotes a (truncated) cone of arbitrary aperture with vertex at p, and where $L\rho(p;\xi,\bar{\eta}) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\Big|_p \xi_j \bar{\eta}_k$ is the Levi form.

What this theorem says is that, near a strongly pseudoconvex boundary point, the Kobayashi metric is asymptotically the Poincaré–Bergman metric of the unit ball. This rough-sounding statement can be made more precise as follows.

Let $\{p_\nu\} \subset \Omega$ be a sequence of points that accumulates at p. Theorem 9.3.3 implies that there exist an open neighborhood U of p and a sequence of injective holomorphic mappings $\sigma_\nu : \Omega \cap U \to \mathbb{C}^n$ such that

(1) $\sigma_\nu(p_\nu) = 0$

and

(2) for every $r > 0$, there exists an $N > 0$ such that

$$(1-r)B^n \subset \sigma_\nu(\Omega \cap U) \subset (1+r)B^n$$

for every $\nu > N$.

By the distance-decreasing property of the Kobayashi metric, this immediately shows that

$$F_{\Omega \cap U}(p_\nu, \xi) = F_{\sigma_\nu(\Omega \cap U)}(0, d\sigma_\nu|_{p_\nu}(\xi))$$

and consequently $\frac{F_{\Omega \cap U}(p_\nu,\xi)}{F_{B^n}(0,d\sigma_\nu|_{p_\nu}(\xi))}$ converges to 1 uniformly, regardless of the choice of $\xi \in \mathbb{C}^n \setminus \{0\}$.

Lemma 10.3.2. *We have the limit*

$$\lim_{\nu\to\infty} \sup_{\xi \in \mathbb{C}^n \setminus \{0\}} \left| \frac{F_{\Omega \cap U}(p_\nu, \xi)}{F_\Omega(p_\nu, \xi)} - 1 \right| = 0.$$

This lemma follows by a use of the holomorphic peak function at p for the domain Ω and a basic normal family argument.

Therefore

$$\lim_{p_\nu \to p} \frac{F_\Omega(p_\nu, \xi)}{F_{B^n}(0, d\sigma_\nu|_{p_\nu}(\xi))} = 1.$$

Since the construction of the map σ_ν is explicit, a calculation will yield Graham's theorem (Theorem 10.3.1). In fact, this gives more; one sees that the Kobayashi metric is asymptotically Hermitian. We leave the details as an exercise for the interested reader.

10.4 Proper Mappings Between Bounded Strongly Pseudoconvex Domains

Let $n \geq 2$ in this section. Let Ω and $\widetilde{\Omega}$ be bounded strongly pseudoconvex domains in $\mathbb{C}^n$ with C^2 smooth boundary. Assume that there is a proper holomorphic mapping $f : \Omega \to \widetilde{\Omega}$. Denote by $g(z)$ the Jacobian determinant of the proper mapping f at $z \in \Omega$. Then we first prove:

Proposition 10.4.1 ([Pinchuk 1978]). *The function $g(z)$ does not vanish anywhere.*

Proof. Assume the contrary. Let $V = \{z \in \Omega \mid g(z) = 0\}$. The Hartogs extension theorem implies that either V is an empty set, or there exists a sequence $z_\nu \in V$ such that $\lim_{\nu\to\infty} z_\nu = p$ for some $p \in \partial\Omega$. Assume, expecting a contradiction, the latter case.

Then let $w_\nu = f(z_\nu)$ for every $\nu = 1, 2, \ldots$. Since f is proper, it follows that w_ν converges to $\widetilde{p} \in \partial\widetilde{\Omega}$ as ν tends to ∞.

Now we build two scaling sequences, both of them strongly analogous to the scaling sequence (i.e., the centering maps A_ν and $\widetilde{A}_\nu$ for the domain $\widetilde{\Omega}$) followed by the stretching map Λ_ν (and $\widetilde{\Lambda}_\nu$ for $\widetilde{\Omega}$, respectively), corresponding to the point sequence z_ν (and to w_ν, respectively). Write $\omega_\nu = \Lambda_\nu \circ A_\nu$ and $\widetilde{\omega}_\nu = \widetilde{\Lambda}_\nu \circ \widetilde{A}_\nu$, respectively. Then the sequence of mappings

$$h_\nu := \widetilde{\omega}_\nu \circ f \circ \omega_\nu^{-1}$$

contains a subsequence that converges. Denote by $\widehat{h}$ a subsequential limit. As we saw in the scaling method with strongly pseudoconvex domains above, $\widehat{h}$ maps the Siegel half-space $\mathcal{S} := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \operatorname{Re} z_1 > |z_2|^2 + \cdots + |z_n|^2\}$ into itself. In fact, combining with the linear fractional biholomorphism $\Psi : \mathcal{S} \to B^n$ from the Siegel half-space $\mathcal{S}$ onto the unit ball B^n, the sequence $\Psi \circ h_\nu \circ \Psi^{-1}$ gives rise to a subsequential limit, say Ξ, which maps the open unit ball B^n into itself.[1] Replacing the original sequence by subsequences whenever necessary, we shall assume that the full sequence converges. This assumption will not cause any loss of generality in the proof.

Now we show that $\Xi : B^n \to B^n$ is a proper holomorphic mapping. It is not true that the limit of a convergent sequence of proper holomorphic mappings is automatically proper, even if the limit mapping is nondegenerate. Rather it requires some deep understanding of proper holomorphic maps between domains of the same dimension. A famous theorem of Remmert and Stein [Remmert/Stein 1960] says that the degree (the number of pre-images of a point, generically) of a proper holomorphic mapping is finite. Then it is shown by Klingenberg and Pinchuk [Klingenberg/Pinchuk 1991] (see also [Ourimi 2000]) that the limit of a locally uniformly convergent sequence of proper mappings with bounded degree is either degenerate or proper. In our

[1] Surely Ξ coincides with $\Psi \circ \widehat{h} \circ \Psi^{-1}$.

case the sequence $\Psi \circ \widetilde{\omega}_\nu \circ f \circ \omega_\nu^{-1} \circ \Psi^{-1}$ has the same degree as f, as all the maps involved in this construction except f are biholomorphisms. Then the scaling procedure shows that no point in the interior gets mapped to the boundary. So it follows that Ξ is proper holomorphic. Notice that, by a theorem of H. Alexander ([Alexander 1977]), Ξ is a biholomorphic mapping.

On the other hand, if V were nonempty, then we would see that the holomorphic Jacobian determinant of h_ν vanishes at $\omega_\nu(z_n)$. But then this point sequence converges to $(1, 0, \ldots, 0)$. Therefore the Jacobian determinant of Ξ must vanish at the origin, which is impossible for a biholomorphism. This contradiction proves the proposition. □

Theorem 10.4.2. *Let Ω be a simply connected, bounded domain with C^2 strongly pseudoconvex boundary. If $f : \Omega \to \Omega$ is a proper holomorphic mapping and if Ω is simply connected, then f is a biholomorphism.*

Proof. By the preceding proposition, the Jacobian of f vanishes nowhere. Hence f must be a covering map, since every proper, nonsingular differential map of a manifold to another is a covering (a standard topological result; cf. [Browder 1954]). Since Ω is simply connected, f must be a one-to-one covering, hence biholomorphic. □

This theorem is usually known as a theorem of H. Alexander and S. Pinchuk. In [Bedford 1982], it is shown, more generally, that if $f : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping of one C^2 strongly pseudoconvex bounded domain to another, then in fact f is a finite normal covering, i.e., there is a finite subgroup Γ of Aut (Ω_1) acting without fixed points (only the identity has any fixed point) such that Ω_2 is biholomorphic to the quotient of Ω_1 mod Γ. This of course implies Lemma 10.3.2 in particular.

It is an interesting consequence of the properties of the Cheng–Yau Einstein–Kähler metric that "generically" the simple connectivity hypothesis in Lemma 10.3.2 is not needed. first:

Theorem 10.4.3. *If Ω is a C^∞ strongly pseudoconvex bounded domain in $\mathbb{C}^n$ which is not a covering-space quotient of the ball and if $f\colon \Omega \to \Omega$ is a proper holomorphic map, then f is biholomorphic.*

Proof. As before, f is a finite-to-one covering. In particular, f is a local isometry of the Cheng–Yau metric, by uniqueness of the metric. Now, according to [Cheng/Yau 1980], the holomorphic sectional curvature of the Cheng–Yau metric g is asymptotically a negative constant, say $-c_0$ (c_0 depends only on the normalization of the constant Ricci curvature of the Cheng–Yau metric). Since Ω is not a quotient of the ball by hypothesis, the holomorphic sectional curvature of the (complete) Cheng–Yau metric cannot be identically equal to $-c_0$. If it were, then the universal cover would have a complete Kähler metric of constant negative holomorphic sectional curvature and would hence be biholomorphically isometric to the ball with a suitable multiple of its Bergman metric, by standard Kähler geometry. Combining these two facts—asymptotic constancy but no global constancy exactly—gives: there is an $\epsilon > 0$ such that $\{z \in \Omega\colon$ some holomorphic sectional curvature K at z satisfies $|K+c_0| > \epsilon\}$ is a nonempty open set with compact closure in Ω. Choose such an ϵ and let U_ϵ be the open set indicated. Clearly $f^{-1}(U_\epsilon) = U_\epsilon$ since f is holomorphic and curvature-preserving. Now U_ϵ has finite, nonzero volume (with respect to the Cheng–Yau metric). But if the local isometry f is a k-to-one covering, then $f^{-1}(U_\epsilon) = U_\epsilon$ would imply that *the volume of* $U_\epsilon = k\cdot$ *(the volume of* U_ϵ*)*. This is clearly possible only if $k = 1$ so that f is in fact biholomorphic. □

This result and its proof illustrate well the power and utility of the Cheng–Yau metric. No other metric is both guaranteed smooth and preserved (locally) by coverings: the Kobayashi (and Wu of first kind) metric have coverings locally isometric but are not in general smooth enough for curvature to be defined. The always C^∞ Bergman metric does not have the property that biholomorphic coverings are local isometries. Only the Cheng–Yau metric does both the jobs needed here.

11

Afterword

Many of the results in previous chapters concerned bounded strongly pseudoconvex domains in complex Euclidean spaces. As it happens, almost all of these results can be extended in some form to more general situations. In particular, most of them apply in some suitable form to strongly pseudoconvex domains with compact closure in Stein manifolds. The restriction to the Euclidean space case earlier simplified the statements and made for a clearer exposition of the proof techniques. But it is of course important to realize that generalizations are possible when indeed they are possible. In this final chapter, we shall try to indicate these possibilities in enough detail that interested readers will be able to carry through the detailed statements and proofs for themselves in the more general situations which will be indicated.

If Ω is a connected open subset with a compact closure in a complex manifold M such that Ω has nonempty C^∞ strongly pseudoconvex boundary, then, according to [Grauert 1958], Ω is itself a Stein manifold provided that Ω contains no compact complex subvarieties of positive dimension. In particular, in this case, there is a slightly larger C^∞ strongly pseudoconvex domain $\widehat{\Omega}$ which contains the closure of Ω such that $\widehat{\Omega}$ is also a Stein manifold. Thus we shall lose no real generality if we assume that the C^∞ strongly pseudoconvex connected open sets to be considered lie with compact closure in some Stein manifold M of complex dimension n.

This assumption, which we make from now on, yields a number of important properties for Ω almost immediately. By the famous embedding theorem of Bishop, Narasimhan, and Remmert ([Bishop, E. 1961], [Narasimhan 1960], [Remmert 1956]), M can be properly embedded in some complex Euclidean space $\mathbb{C}^N$. In our case, applying this result to M, we find an embedding of Ω into $\mathbb{C}^N$ which is smooth on the closure of Ω, and indeed on an open neighborhood of the closure, which takes a suitable such neighborhood to a bounded set in $\mathbb{C}^N$.

It follows immediately that Ω admits an abundance of bounded holomorphic functions: every holomorphic function on $\mathbb{C}^N$, when restricted to (the image of) Ω, is bounded. This yields immediately that the Carathéodory

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metric of Ω is positive definite and hence that its Kobayashi metric is. (This latter can also be seen directly by Cauchy estimates.) Moreover, the pullback to Ω of a holomorphic $(n,0)$-form on $\mathbb{C}^N$ will be a holomorphic $(n,0)$-form on M which is necessarily L^2 on Ω. It is easy to check that there are enough such forms to guarantee that the intrinsic Bergman metric of Ω in terms of L^2 holomorphic $(n,0)$-forms is defined and is a positive definite Kähler metric on Ω (see Section 3.2).

In particular, it follows that the automorphism group of Ω is a Lie group (see Section 7.2.3), that its isotropy subgroups I_p, $p \in \Omega$, are compact, and that the map of I_p into linear maps of the tangent space at p defined by $f \mapsto df|_p$, $f \in I_p$, is injective: the direct analogues of Theorem 1.3.1 and Corollary 1.3.3 are valid. Also, the action of $\mathrm{Aut}\,(\Omega)$ on Ω is proper. (cf. Theorem 7.2.10.)

To study Ω and in particular $\mathrm{Aut}\,(\Omega)$ further, it is useful to note that Ω (identified with its image as a submanifold in $\mathbb{C}^N$) can be exhibited as the intersection with M (similarly identified) of a C^∞ strongly pseudoconvex domain in $\mathbb{C}^N$ with certain special properties. We begin by noting from standard Stein manifold theory ([Docquier/Grauert 1960]) that there is a neighborhood U of M in $\mathbb{C}^N$ for which there is a *holomorphic retraction* onto M; i.e., there is a holomorphic map $F : U \to M$ such that $F(z) = z$ for every $z \in M$. (Here we identify M with its image in $\mathbb{C}^N$ as before.) Choose a C^∞ strictly plurisubharmonic function φ_1 defined in a neighborhood of the closure of Ω in M such that $\Omega = \{z \colon \varphi_1(z) < 1\}$ and $d\varphi_1$ is nowhere zero on the boundary of Ω. Set $\widehat{\varphi_1} = \varphi_1 \circ F$. Set $\varphi_{2,\epsilon}(z) = \epsilon^{-2}\ \mathrm{dis}^2(z, M)$. Then, for $\epsilon > 0$ sufficiently small, $\varphi_{2,\epsilon}$ is C^∞ for all z with $\varphi_{2,\epsilon}(z) < 2$ and z close enough to Ω. Now declare $\widehat{\Omega}$ to be the set of $z \in U$ such that $F(z)$ lies in the neighborhood of the closure of Ω on which φ_1 is defined and $\widehat{\varphi}_1(z) + \varphi_{2,\epsilon}(z) < 1$. It is straightforward to check that $\widehat{\varphi}_1 + \varphi_{2,\epsilon}$ is, again for $\epsilon > 0$ sufficiently small, C^∞ strictly plurisubharmonic in a neighborhood of the closure of Ω: the function $\widehat{\varphi}_1$ is strictly plurisubharmonic "parallel to M" and $\varphi_{2,\epsilon}$ is strictly plurisubharmonic "perpendicular to M" (cf. [Greene/Wu 1978] and [Elencwajg 1975]). Thus $\widehat{\Omega}$ is C^∞ strongly pseudoconvex—the nonvanishing of $d(\widehat{\varphi}_1 + \varphi_{2,\epsilon})$ at the boundary of $\widehat{\Omega}$ is also clear, for the C^∞ part. Moreover, $\widehat{\Omega} \cap M = \Omega$ and $F(\widehat{\Omega}) \subset \Omega$, since $\widehat{\varphi}_1 < 1$ on $\widehat{\Omega}$ by definition.

The utility of this somewhat intricate construction is that analysis of the $\overline{\partial}$ problem on Ω can be transferred to $\widehat{\Omega}$, a situation—C^∞ strongly pseudoconvex domains in $\mathbb{C}^N$—that is very familiar. Of course, $\overline{\partial}$ analysis can be carried out directly on domains in Stein manifolds. But the present approach will be advantageous when we wish to consider stability matters.

The construction just given yields immediately that, if p is a point of the boundary of Ω in M, then p is a "peak point" in the following (generalized) sense: there is a holomorphic function $f_p : \Omega \to \mathbb{C}$ such that $|f_p(z)| \to 1$ as $z \to p$ while $\limsup |f_p(z)| < 1$ as $z \to q$, $q \neq p$, $q \in \partial\Omega$. This follows since such "peaking functions" exist for each point of the boundary of a C^∞

bounded, strictly pseudoconvex domain in $\mathbb{C}^N$, so that peaking functions can be obtained for Ω by restricting a peaking function for $\widehat{\Omega}$.

The importance for our purposes of the existence of such peaking functions is that this means that the argument of [Rosay 1979] applies to yield the analogous theorem, not just for domains in $\mathbb{C}^n$ as in Section 9.2.4, but also for domains in Stein manifolds:

> *If Ω is a C^∞ strictly pseudoconvex domain in a Stein manifold and if* $\mathrm{Aut}\,(\Omega)$ *is noncompact, then Ω is biholomorphic to the unit ball in $\mathbb{C}^n$, $n = \dim_{\mathbb{C}} \Omega$.*

Actually, the existence of a global peaking function turns out to be unnecessary: the only hypothesis actually needed is strictly local. In particular, this optimal result is obtained in [Gaussier/Kim/Krantz 2002]:

> *If Ω is a domain in a complex manifold with C^2 boundary in a neighborhood of some boundary point p, if p is a strictly pseudoconvex boundary point, and if* $\mathrm{Aut}\,(\Omega)$ *has an orbit that accumulates at p, then Ω is biholomorphic to the unit ball in $\mathbb{C}^n$, $n = \dim_{\mathbb{C}} \Omega$.*

Returning now to the situation of a C^∞ strictly pseudoconvex domain in a Stein manifold, the embedding of Ω in $\widehat{\Omega}$ opens up, as mentioned earlier, the possibility of doing $\overline{\partial}$ analysis on Ω rather explicitly. first, suppose that ω is a $(0,1)$-form on Ω. By the construction of $\widehat{\Omega}$, there is a holomorphic retraction (projection) $F : \widehat{\Omega} \to \Omega$, which in fact is defined and holomorphic on a neighborhood of the closure of $\widehat{\Omega}$. Since holomorphic pullbacks commute with $\overline{\partial}$, we see that $F^*(\overline{\partial}_M \omega) = \overline{\partial}_{\mathbb{C}^N}(F^*\omega)$. In particular, $F^*\omega$ is $\overline{\partial}$ closed if ω is. Moreover, if $\overline{\partial} u = F^*\omega$, then $u|_\Omega$ satisfies $\overline{\partial}_M(u|_M) = \omega$. This setup means that the full power of the regularity theory for the Kohn solution of $\overline{\partial}$ on strongly pseudoconvex domains is available, even though in our setting there is no *a priori* canonical notion of a Kohn solution (orthogonal to holomorphic functions) on Ω, because Ω does not have a canonically specified metric.

In particular, $\overline{\partial}$ localization at boundary points holds in the form needed to make the *scaling method* apply in the form needed to establish the analogue of Theorem 3.4.3 and its stability under perturbation: Theorem 3.5.1 (and also 3.5.2).

Theorem 11.1 (Theorems 3.5.1 and 3.5.2 Extended). *If Ω_0 is a C^∞ strongly pseudoconvex domain with compact closure in a Stein manifold M, then the Bergman metric of Ω_0 is complete, and its holomorphic sectional curvature is asymptotically constant negative $-4/(n+1)$ in the sense that, given $\epsilon > 0$, there is a $\delta > 0$ such that, if $p \in \Omega_0$ with dis $(p, M \setminus \Omega_0) < \delta$, then $\left|K + \frac{4}{n+1}\right| < \epsilon$ for each holomorphic sectional curvature K at p of the Bergman metric of Ω_0. Moreover, this estimate is stable in the sense that there is a $\delta > 0$ and a neighborhood $\mathcal{U}$ of Ω_0 in the C^∞ topology of domains such that, if $\Omega \in \mathcal{U}$ and $p \in \Omega$ with dis $(p, M \setminus \Omega) < \delta$, then $\left|K + \frac{4}{n+1}\right| < \epsilon$ for each holomorphic sectional curvature K at p of the Bergman metric of Ω.*

Here dis means distance in a fixed Kähler metric on M.

It is actually the case that an asymptotic expansion of Fefferman type holds in a neighborhood of each boundary point. first $K(z,w)$, which is now a *double form* of type (n,n)—type $(n,0)$ in z and type $(0,n)$ in w—is again C^∞ on $\mathrm{cl}(\Omega) \times \mathrm{cl}(\Omega) \setminus \{(p,p) \colon p \in \partial\Omega\}$. And, given a boundary point p of Ω with holomorphic coordinates $(z_1, \ldots, z_n)$ defined in a neighborhood U of p in the Stein manifold M, we can write

$$K(z,w) = f(z,w)\ dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{w_1} \wedge \cdots d\overline{w_n},$$

for z, w in the neighborhood U_p of p and in Ω. Then the function $f(z,w)$ has the same form of asymptotic expansion as does the Bergman kernel function in the Euclidean space case (cf. Section 3.4). This is established by using $\overline{\partial}$ localization of the Bergman kernel form, which implies that its asymptotic behavior near p is the same as that of the Bergman kernel of $U_p \cap \Omega$ (where we can take U_p to be itself strongly pseudoconvex). No essentially new ingredients arise here: after the localization argument, one is in the original Fefferman situation. This also holds in stable form, stable under C^∞ perturbation.

Thus, either from the full Fefferman expansion or from the less detailed but still sufficient information arising from the scaling method (see Section 10.1), one can consider boundary orbit accumulation from the curvature viewpoint. In particular, suppose that Ω is, as before, C^∞ strongly pseudoconvex (or even C^2, since the scaling method still applies in that case). Also, suppose that there is a sequence $\{\varphi_j\} \subset \mathrm{Aut}\,(\Omega)$ such that, for some $q \in \Omega$, the sequence $\{\varphi_j(q)\}$ converges to a point p_0 in the boundary of Ω. Then, as in Section 3.4, the (complete) Bergman metric of Ω has constant holomorphic sectional curvature. As in Corollary 3.4.4, one can then deduce that Ω is biholomorphic to the ball. As in the situation of Corollary 3.4.4, standard Kähler geometry gives that the universal cover of Ω is biholomorphic to the ball. To show that the covering map is injective, or equivalently that Ω is simply connected, any of the several methods used to deal with the question for 3.4.4 can be used here.

In particular, Lu Qi-Keng's theorem (Theorem 4.2.2) applies[1] in this case:

Theorem 11.2 (Lu Qi-Keng's Theorem for Stein Domains). *If Ω is a domain with compact closure in a Stein manifold M, and if the Bergman metric of Ω is complete and of constant (negative) holomorphic sectional curvature, then Ω is biholomorphic to the unit ball in $\mathbb{C}^n$, $n = \dim_{\mathbb{C}} M$.*

The proof of this result is obtained by the same method as for the case of domains in $\mathbb{C}^n$, with the one additional feature that a modified definition of

[1]If we are only concerned with establishing the simple connectivity of Ω, Lu's theorem is not really required, as noted in Section 3.4. But we exploit Lu's theorem here, because this generalization of Lu's theorem for such Stein domains is interesting by itself. Note: A domain is called *Stein* if it admits a strictly plurisubharmonic exhaustion function.

Bergman representative coordinates must be given. In the original definition as given in Section 4.2, the coordinates at $w_0 \in \Omega$ were obtained as $\overline{w}$-derivatives of $\log\big(K(z,w)/K(w,w)\big)$, with the derivatives evaluated at w_0. In the present Stein manifold case, the quotient $K(z,w)/K(w,w)$ is not as such defined, since now $K(z,w)$ and $K(w,w)$ are not functions, but are rather *double forms*, one at $(z,w) \in \Omega \times \Omega$ and the other at $(w,w) \in \Omega \times \Omega$ so that, if $z \neq w$, the quotient is not meaningful.

However, this apparent difficulty can be removed by choosing holomorphic coordinate systems $(z_1, \ldots, z_n)$ around the given $z \in \Omega$ and $(w_1, \cdots, w_n)$ around the given w and then writing

$$K(z,w) = f(z,w)\ dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{w_1} \wedge \cdots d\overline{w_n}$$

and

$$K(w,w) = g(w,w)\ dw_1 \wedge \cdots \wedge dw_n \wedge d\overline{w_1} \wedge \cdots d\overline{w_n}.$$

Then the quotient f/g is well defined up to a product of two factors,[2] one a holomorphic function of z, the other a holomorphic function of w—these factors depending on the choice of z and w coordinate systems (the conjugate factors for the w-coordinates cancel since the same factor occurs in f and g). Thus $\overline{w}$-derivatives of $\log(f/g)$ are well defined even though f/g is not well defined itself. Once it is noted that Bergman representative coordinates can be thus defined, the remainder of the proof given in Section 4.2 (see Theorem 4.2.2) applies to establish Lu Qi-Keng's theorem in this Stein domain situation. □

We return now to the function-theoretic and geometric stability properties of compact-closure C^∞ strongly pseudoconvex domains Ω in a Stein manifold M (which, as before, we suppose to have a fixed proper embedding $E\colon M \to \mathbb{C}^N$). Using the construction for representing Ω as the intersection of $E(M) \subset \mathbb{C}^N$ with a C^∞ strongly pseudoconvex domain in $\mathbb{C}^N$ as already discussed, one obtains stable $\overline{\partial}$ estimates for variation of Ω in M from the stable $\overline{\partial}$ estimates for C^∞ strongly pseudoconvex domains in $\mathbb{C}^N$ ([Greene/Krantz 1982]). This stability is the needed ingredient to establish the extension of Theorems 3.5.1 and 3.5.2, as already stated. This theorem in particular gives stable bounds on the distance of orbits from the boundary, analogous to Theorem 3.5.2; this result comes directly from the stability part of the extension of Theorems 3.5.1 and 3.5.2.

Theorem 11.3 (Theorem 3.5.2 Extended). *Suppose that M is a Stein manifold with a fixed but arbitrary Kähler metric and suppose that Ω_0 is a C^∞ strictly pseudoconvex open subset of M with compact closure in M. If Ω_0 is not biholomorphic to the unit ball in $\mathbb{C}^n$, $n = \dim_{\mathbb{C}} M$, and if $p_o \in \Omega_0$,*

[2] One has to check what happens when we work with other holomorphic coordinate systems; that is what is discussed here.

then there is a $\delta > 0$ *and a neighborhood* $\mathcal{U}$ *of* Ω_0 *in the* C^∞ *topology on* C^∞ *compact-closure domains in* M *such that, if* $\Omega \in \mathcal{U}$*, then:*

(1) $p_0 \in \Omega$.
(2) *The domain* Ω *is real diffeomorphic to* Ω_0 *via a diffeomorphism that is* C^∞ *on the closure of* Ω *and with its inverse* C^∞ *on the closure of* Ω_0.
(3) *For every* $\varphi \in \mathrm{Aut}\,(\Omega)$*, the distance in the Kähler metric on* M *from* $\varphi(p_0)$ *to the boundary of* Ω *is* $\geq \delta$.

The proof here follows the pattern of the proof of Theorem 3.5.2.

This result together with the normal families results already noted make it possible to apply exactly the arguments used to prove Theorem 4.4.3 to prove a similar semicontinuity result for perturbation of a given Ω_0 in a Stein manifold, Ω_0 not biholomorphic to the ball.

Theorem 11.4 (Theorem 4.4.3 Extended). *If* Ω_0 *is a* C^∞ *strongly pseudoconvex domain in a Stein manifold* M *with* Ω_0 *not biholomorphic to the unit ball in* $\mathbb{C}^n$*,* $n = \dim_{\mathbb{C}} M$*, then there is a neighborhood* $\mathcal{U}$ *of* Ω_0 *in the* C^∞ *topology such that, if* $\Omega \in \mathcal{U}$*, then* $\mathrm{Aut}\,(\Omega)$ *is isomorphic to a subgroup of* $\mathrm{Aut}\,(\Omega_0)$ *via an isomorphism obtained by conjugation by a real diffeomorphism of* Ω *to* Ω_0*. first, there is a real diffeomorphism* $F\colon \Omega \to \Omega_0$ *such that the map* $\alpha \mapsto F \circ \alpha \circ F^{-1}$*,* $\alpha \in \mathrm{Aut}\,(\Omega)$*, is an injective homomorphism of* $\mathrm{Aut}\,(\Omega)$ *onto a subgroup of* $\mathrm{Aut}\,(\Omega_0)$.

A result for Stein domains analogous to Theorems 4.3.2 and 4.3.3 holds, and the same basic technique applies, but some additional technical considerations arise. The result itself is what one would perhaps expect.

Theorem 11.5 (Theorems 4.3.2 and 4.3.3 Extended). *Suppose that* Ω_0 *is a* C^∞ *compact-closure strictly pseudoconvex domain in a Stein manifold* M*. Then there is a* C^∞ *neighborhood* $\mathcal{O}$ *of the almost complex structure* J_M *of* M *restricted to the closure of* Ω_0 *within the space of all* C^∞ *almost complex structures on the closure of* Ω_0 *with the following property: for each* $J \in \mathcal{O}$ *with* J *integrable on* Ω_0*, there is a* C^∞ *compact-closure domain* Ω_J *in* M *such that* (Ω_0, J) *is biholomorphic to* $\left(\Omega_J, J_M\big|_{\Omega_J}\right)$*. Moreover, given any* C^∞ *neighborhood* $\mathcal{U}$ *of* Ω_0 *in the* C^∞ *topology on domains, the neighborhood* $\mathcal{O}$ *can be chosen so that, for each* $J \in \mathcal{O}$*, the domain* Ω_J *can be chosen to be in* $\mathcal{U}$.

The essential idea of the proof of this result is the same as that of the proof of Theorem 4.3.2, except that we *correct* not the coordinate functions of a domain in $\mathbb{C}^n$ but the embedding functions for M. Specifically, with $E\colon M \to \mathbb{C}^N$ a holomorphic proper embedding as before, write

$$E = (E_1, \ldots, E_N),$$

where each $E_i\colon M \to \mathbb{C}$ is a holomorphic function; holomorphic here means holomorphic in the J_M complex structure. The functions $E_i\big|_{\Omega_0}$ are of course

C^∞ on the closure of Ω_0. They need not be holomorphic relative to another integrable complex structure on (the closure of) Ω_0, but $\overline{\partial_J}E_i$, $\overline{\partial_J}$ relative to the J-structure, is C^∞ small on the closure of Ω_0. Suppose for the moment that $\overline{\partial_J}$ satisfies stable estimates in the same sense as for domains in the proof of Theorem 4.3.2. Then there are C^∞ functions u_j on Ω_0, which are C^∞ small up to the boundary of Ω_0, which satisfy $\overline{\partial_J}u_j = \overline{\partial_J}E_j$, $j = 1, \ldots, N$. Then the map of Ω_0 into $\mathbb{C}^N$ defined by setting (the j-th coordinate function of) $E_J \colon M \to \mathbb{C}^N = E_j - u_j$, $j = 1, \ldots, N$, is J-holomorphic, and C^∞ close to the map E.

Of course there is no guarantee that the image of E_J lies in $E(M)$. But by [Docquier/Grauert 1960] there is a *tubular neighborhood* of $E(M)$, first, an open set U in $\mathbb{C}^N$ that contains $E(M)$ and for which there is a holomorphic mapping $F \colon U \to E(M)$ with $F\big|_{E(M)} =$ identity. For short, there is a *holomorphic retraction* of U onto $E(M)$.

With F so chosen, it then follows from standard differential topology that, when E_J is sufficiently C^∞ (even C^1) close to E on the closure of Ω_0, the map $F \circ E_J$ is a holomorphic diffeomorphism of Ω_0 with the J-complex structure onto its image in $E(M)$, so that $E^{-1} \circ F \circ E_J$ is its desired biholomorphic realization of (Ω_0, J) as a compact-closure domain in M.

The required $\overline{\partial_J}$ estimates, stable in J, are obtained by working through the solution of the $\overline{\partial}$-Neumann problem for strictly pseudoconvex domains in Stein manifolds directly, and checking the stability of each step, as in [Greene/Krantz 1982]—a tedious and fairly difficult process.

If Ω_0 is a compact-closure domain in a Stein manifold M and if G is a compact subgroup of Aut (Ω_0), then a G-invariant Kähler metric on Ω_0 can be obtained as follows: Let φ be a C^∞ strictly plurisubharmonic function on M. Define $\psi : \Omega \to \mathbb{R}$ as the average of $\varphi\big|_{\Omega_0}$ with respect to the G-action. Then the Levi form of ψ is the desired G-invariant metric. If G on Ω_0 extends to act smoothly on the closure of Ω (as always happens, if Ω_0 is C^∞ strongly pseudoconvex in M), then this G-invariant metric will be C^∞ on the closure of Ω_0. In this case, the Kohn solution of the $\overline{\partial}$ problem (orthogonal to holomorphic functions with respect to the Kähler metric) will be G-invariant in the obvious sense. This in turn implies that a G-invariant abstract perturbation of the complex structure of Ω_0 can be realized G-equivariantly.

Equivalently, if G acting on Ω_0 arises as the restriction to Ω_0 of the action of the group G on all of M, the action preserving Ω_0, then every abstract G-invariant perturbation of the complex structure of Ω_0 that is sufficiently C^∞ close to the complex structure of Ω_0 can be realized in the sense of Theorem 11.5 as a G-invariant domain in M which is a C^∞ perturbation of Ω_0. In this sense, Theorem 11.5 holds equivariantly.

While some additional technical details can be expected to and indeed do arise in these developments, it is, from a certain viewpoint, almost to be expected that so much extends to the Stein manifold situation from the Euclidean space pseudoconvex situation. It is indeed one of the grand and recurrent themes of modern several complex variables, dating at least back to K. Oka, E. Cartan and H. Grauert, and in many aspects even back to E. E. Levi, that what happens for pseudoconvex domains in Euclidean space ought also to happen for Stein manifolds and pseudoconvex domains in these manifolds. In this sense, it is gratifying but not surprising that so many of the results developed in earlier chapters for domains in complex Euclidean space can be extended, and indeed extended by essentially the same arguments, to Stein manifolds.

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