$\boxed{\text{Phenomenological Rate Equations}}$

$$\frac{d|E|^2}{dt} = \underbrace{G(n)|E|^2}_{\text{stimulated emission rate}} - \underbrace{\frac{|E|^2}{\tau_p}}_{\text{cavity loss rate}} \qquad (1)$$

$$\frac{d|E|}{dt} = \frac{1}{2}\left(G(n)|E| - \frac{|E|}{\tau_p}\right)$$

$\boxed{\textit{Linear Gain Approximation}}$

$$G(n) \approx G(n_{th}) + \left.\frac{\partial G}{\partial n}\right|_{n=n_{th}} (n - n_{th})$$

$$G_N \equiv \left.\frac{\partial G}{\partial n}\right|_{n=n_{th}}$$

$$\frac{d|E|}{dt} = \frac{1}{2}\left(G(n_{th}) + G_N(n - n_{th}) - \frac{1}{\tau_p}\right)|E|$$

In the steady state,

$$\frac{d|E|}{dt} = 0$$

Therefore,

$$G(n_{th}) = \frac{1}{\tau_p}$$

$\boxed{\textit{Amplitude Rate Equation}}$

$$\frac{d|E|}{dt} = \frac{1}{2} G_N (n - n_{th}) |E| \qquad (2)$$

$\boxed{\text{Fabry-Perot Cavity Mode Frequency}}$

$$\omega_m = m2\pi\frac{c}{2\mu L}$$

$$d\omega_m = m\pi\frac{c}{L}d\left(\frac{1}{\mu}\right) = -m\pi\frac{c}{L\mu^2}d\mu$$

$\boxed{\textit{Linear Index Approximation}}$

$$\mu(n) \approx \mu_f + bn$$

$$b = \left.\frac{\partial\mu}{\partial n}\right|_{n=0}$$

The envelope phase rate equation is given by

$$\frac{d\phi}{dt} = \Delta\omega$$

$$\Delta\omega = \omega(n) - \omega(n_{th}) = m\pi\frac{c}{\mu_{th}^2 L}(\mu - \mu_{th})$$

Therefore, the mode frequency shift is

$$\Delta\omega = r(n - n_{th})$$

from which follows:

$\boxed{\textit{Phase Rate Equation}}$

$$\frac{d\phi}{dt} = r(n - n_{th}) \tag{3}$$

Since, $E = |E|e^{i\phi}$,

$$\frac{dE}{dt} = e^{i\phi}\left(\frac{d|E|}{dt} + i|E|\frac{d\phi}{dt}\right)$$

Combining the amplitude and phase equations (2) and (3) gives:

$$\frac{dE}{dt} = (\frac{1}{2}G_N + ir)(n - n_{th})E$$

$$\boxed{\textit{Complex Field Rate Equation}}$$

$$\frac{dE}{dt} = \frac{1}{2}(1 + i\alpha)G_N(n - n_{th})E \qquad (4)$$

$$\alpha = \frac{2r}{G_N} \propto \frac{\left(\frac{\partial\mu}{\partial n}\right)\Big|_{n=0}}{\left(\frac{\partial G}{\partial n}\right)\Big|_{n=n_{th}}}$$

$$\boxed{\textit{Carrier Density Rate Equation}}$$

$$\frac{dn}{dt} = \underbrace{\frac{I}{eV_a}}_{\text{injection rate}} - \underbrace{\frac{n}{\tau_c}}_{\text{recombination rate}} - \underbrace{G(n)|E|^2}_{\text{stimulated emission rate}} \qquad (5)$$

$$G(n) \approx \frac{1}{\tau_p} + G_N(n - n_{th})$$